A remark on amenable von Neumann subalgebras in a tracial free product

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Abstract: We give a simple proof of a theorem of C. Houdayer that an amenable von Neumann subalgebra in a tracial free product von Neumann algebra $M = M_1 * M_2$ is contained in M_1 whenever it has a diffuse intersection with M_1.

Key words: Maximal amenable von Neumann subalgebra; free product.

Amenable von Neumann subalgebras provide key tools in the study of the structure of the ambient von Neumann algebras. The study of maximal amenable subalgebras has gained momentum after Popa's breakthrough in 1983 ([5]) and culminated in the following recent theorem of Houdayer ([3]):

Theorem. *Let $M = M_1 * M_2$ be a tracial free product of finite von Neumann algebras. If $A \subset M$ is an amenable subalgebra such that $A \cap M_1$ is diffuse, then $A \subset M_1$.*

Most of the results on maximal amenability so far (including the above) are obtained via the refinements of [5]. See [3] and the references therein. Recently however, Boutonnet and Carderi ([2]) have brought an entirely new method. We adapt it and give a simple proof of the above theorem. Recall that a finite von Neumann algebra A in $\mathbf{B}(\mathcal{H})$ is *amenable* if there is a state φ on $\mathbf{B}(\mathcal{H})$ which is A-*central*: $\varphi(ax) = \varphi(xa)$ for $a \in A$ and $x \in \mathbf{B}(\mathcal{H})$.

Lemma. *Let $A \subset \mathbf{B}(\mathcal{H})$ be a finite amenable von Neumann subalgebra and φ be an A-central state on $\mathbf{B}(\mathcal{H})$. If $x \in \mathbf{B}(\mathcal{H})$ is such that the norm closed convex hull of $\{uxu^* \otimes \bar{u}\bar{x}\bar{u}^* : u \in \mathcal{U}(A)\}$ in $\mathbf{B}(\mathcal{H} \otimes \bar{\mathcal{H}})$ contains 0, then $\varphi(x^*A) = 0$.*

Proof. Let us show $\varphi(x^*a) = 0$ for a given a in $\mathcal{U}(A)$. Approximate φ by a net $(S_i)_i$ of positive trace class operators such that $\|[S_i, b]\|_1 \to 0$ for b in A. Note that $\|[S_i^{1/p}, u]\|_p^p \le \|[S_i, u]\|_1$ for any unitary u and $p = 2, 4$ ([4]). Thus, for any $u_1, \ldots, u_n \in \mathcal{U}(A)$,

$$\begin{aligned}|\varphi(x^*a)|^2 &= \lim_i |\mathrm{Tr}(S_i^{1/4} x^* S_i^{1/4} a S_i^{1/2})|^2 \\ &\le \limsup_i |\mathrm{Tr}(S_i^{1/2} x^* S_i^{1/2} x)|\, \mathrm{Tr}(S_i) \\ &= \limsup_i \Big|\frac{1}{n}\sum\nolimits_k \mathrm{Tr}(u_k x u_k^* S_i^{1/2} u_k x^* u_k^* S_i^{1/2})\Big| \\ &\le \Big\|\frac{1}{n}\sum\nolimits_k u_k x u_k^* \otimes \bar{u}_k \bar{x} \bar{u}_k^*\Big\|_{\mathbf{B}(\mathcal{H}\otimes\bar{\mathcal{H}})}.\end{aligned}$$

Now, the assumption implies that $\varphi(x^*a) = 0$. □

Proof of Theorem. The trace on M extends to an A-central state φ on $\mathbf{B}(L^2M)$ by amenability. It suffices to show $\varphi(x^*A) = 0$ for every x in $M \ominus M_1$. We may assume $x = v_1 \cdots v_l$ for some trace zero v_j in $\mathcal{U}(M_{i(j)})$ with $i(j+1) \neq i(j)$. Being diffuse, $A \cap M_1$ contains a unitary u whose nonzero powers are all trace zero. Thus x^*ux and u are free Haar unitaries, and so $\{x^*u^kxu^{-k}\}_{k\in\mathbf{N}}$ is a free family. Hence,

$$\Big\|\frac{1}{m}\sum_{k=1}^{m} u^k x u^{-k} \otimes \bar{u}^k \bar{x} \bar{u}^{-k}\Big\| = \frac{2\sqrt{m-1}}{m} \to 0$$

by [1]. Therefore $\varphi(x^*A) = 0$ by Lemma. □

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