# An extension of necessary and sufficient conditions for concave functions

By Rintaro OHNO

Research Center for Pure and Applied Mathematics, Graduate School of Information Sciences, Tohoku University, 6-3-09 Aramaki aza Aoba, Aoba-ku, Sendai, Miyagi 980-8579, Japan

(Communicated by Shigefumi MORI, M.J.A., Dec. 12, 2014)

**Abstract:** In the present article we discuss necessary and sufficient conditions for concave functions, i.e. meromorphic functions which map the unit disk conformally on a domain whose complement is convex. The conditions will be given with respect to an arbitrary point $p \in (-1,1)$. We will also look at representation formulas for the related functions as well as an application of the derived formula.

**Key words:** Meromorphic univalent functions; concave functions.

**1. Introduction.** Let $\widehat{\mathbf{C}}$ be the Riemann sphere and $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ the open unit disk in the complex plane $\mathbf{C}$. A meromorphic function $f$ is said to be *concave*, if it maps $\mathbf{D}$ conformally onto a concave domain, i.e. $\widehat{\mathbf{C}} \backslash f(\mathbf{D})$ is convex.

Let $q \in \mathbf{D}$. A meromorphic function $f_q$ is said to be in the class $\mathcal{C}o_q$, if it is concave and has a simple pole at $q$.

In particular, it is commonly known that a function $f_0$ belongs to $\mathcal{C}o_0$ if and only if

$$\operatorname{Re}\left(1 + \frac{z f_0''(z)}{f_0'(z)}\right) < 0$$

for all $z \in \mathbf{D}$. For the class $\mathcal{C}o_q$ the inequality

$$\text{(1)} \qquad \operatorname{Re}\left(1 + \frac{z f_q''(z)}{f_q'(z)} + \frac{z+q}{z-q} - \frac{1+\bar{q}z}{1-\bar{q}z}\right) < 0$$

is a necessary and sufficient condition, provided by Pfaltzgraff and Pinchuk in [7].

For simplicity, in this article we will only consider real $q$, meaning $q \in (-1,1)$.

Concave functions of class $\mathcal{C}o_q$ can be expanded as

$$\text{(2)} \qquad f_q(z) = \frac{\operatorname{Res}_q f_q}{z-q} + c_0(f_q) + c_1(f_q)(z-q) + \cdots.$$

Usually normalization considers the Maclaurin expansion for this class (see e.g. [1,2]). Here $\operatorname{Res}_q f_q = c_{-1}(f_q)$ is the residue of $f_q$ at the point $z = q$. In the special case of $q = 0$ sometimes the functions are normalized by $\operatorname{Res}_0 f_0 = c_{-1}(f_0) = 1$.

2010 Mathematics Subject Classification. Primary 30C45.

In the present article we shall prove the following:

**Theorem 1.** *Let $p, q \in (-1,1)$. A meromorphic function $f_q$ with simple pole at $q$ belongs to the class $\mathcal{C}o_q$ if and only if for all $z \in \mathbf{D}$*

$$\text{(3)} \qquad \operatorname{Re}\Bigg(1 - q^2 + \frac{2p(1-q^2)}{1+p^2} \cdot \frac{1-qz}{z-q} - \left(\frac{z-q}{1-qz} + p\right)\left(1 + p\frac{z-q}{1-qz}\right) \times \left(\frac{2q}{1+p^2} + \frac{1-qz}{1+p^2}\frac{f_q''(z)}{f_q'(z)}\right)\Bigg) < 0.$$

For the case $q = 0$ we actually have

**Corollary 2.** *Let $p \in (-1,1)$. A meromorphic function $f_0$ with a simple pole at the origin belongs to the class $\mathcal{C}o_0$ if and only if for all $z \in \mathbf{D}$*

$$\text{(4)} \qquad \operatorname{Re}\left(1 + \frac{2p}{1+p^2} \cdot \frac{1}{z} + \frac{1}{1+p^2}(z+p)(1+pz)\frac{f_0''(z)}{f_0'(z)}\right) < 0.$$

**Remark 3.** For $q = p$ in (3) we obtain the original inequality (1) after normalization. If we put $p = 0$ in (3), we have

$$\operatorname{Re}\left(1 + q^2 - 2qz + \frac{(z-q)(1-qz)f_q''(z)}{f_q'(z)}\right) < 0.$$

This is the same result as Livingston obtained in [4].

The second section will provide the proofs for the Theorems and in the last section, we take a look at an application of Theorem 1.

doi: 10.3792/pjaa.91.1

**2. Proofs.**

*Proof of Theorem 1.* Let $p, q \in (-1,1)$ and $f_q \in \mathcal{C}o_q$. Then there exist both a function $f_0 \in \mathcal{C}o_0$ and a function $f_p \in \mathcal{C}o_p$ such that $C_q \cdot f_q(\mathbf{D}) = C_0 \cdot f_0(\mathbf{D}) = f_p(\mathbf{D})$ with constants $C_0$, $C_q \in \mathbf{C}\backslash\{0\}$.

Using automorphisms of the unit disk, the function $f_0$ can be described by $f_q$ as

$$C_0 \cdot f_0(z) = C_q \cdot f_q\left(\frac{z+q}{1+qz}\right) \tag{5}$$

and $f_p$ can be written as

$$f_p(z) = C_0 \cdot f_0\left(\frac{z-p}{1-pz}\right). \tag{6}$$

For any function of $\mathcal{C}o_p$ we also know that (1) is valid.

Setting

$$Q_1(z) = 1 + \frac{zf_p''(z)}{f_p'(z)} + \frac{z+p}{z-p} - \frac{1+pz}{1-pz}$$

and using (6), we obtain

$$Q_1(z) = 1 + \frac{2p}{z-p} + \frac{(1-p^2)zf_0''\left(\frac{z-p}{1-pz}\right)}{(1-pz)^2 f_0'\left(\frac{z-p}{1-pz}\right)}$$

with respect to $f_0$.

Since $\operatorname{Re} Q_1(z) < 0$ for all $z \in \mathbf{D}$ is only valid if and only if $\operatorname{Re} Q_1(\frac{z+p}{1+pz}) < 0$ for all $z \in \mathbf{D}$, we obtain by a short calculation that

$$\begin{aligned} Q_1&\left(\frac{z+p}{1+pz}\right) \\ &= \frac{1+p^2}{1-p^2} + \frac{2p}{(1-p^2)z} + \frac{(z+p)(1+pz)f_0''(z)}{(1-p^2)f_0'(z)}. \end{aligned} \tag{7}$$

Normalizing (7) for $z=0$ by multiplication with $\frac{1-p^2}{1+p^2}$ leads to

$$\begin{aligned} &\frac{1-p^2}{1+p^2} \cdot Q_1\left(\frac{z+p}{1+pz}\right) \\ &= 1 + \frac{2p}{1+p^2} \cdot \frac{1}{z} + \frac{1}{1+p^2}(z+p)(1+pz)\frac{f_0''(z)}{f_0'(z)}, \end{aligned} \tag{8}$$

which has also negative real part for all $z \in \mathbf{D}$ since $\frac{1-p^2}{1+p^2} > 0$.

Using (5) with

$$Q_2(z) = 1 + \frac{2p}{1+p^2} \cdot \frac{1}{z} + \frac{1}{1+p^2}(z+p)(1+pz)\frac{f_0''(z)}{f_0'(z)}$$

we obtain

$$\begin{aligned} Q_2(z) = 1 &+ \frac{2p}{1+p^2} \cdot \frac{1}{z} - \frac{2q}{1+q^2}\frac{(z+p)(1+pz)}{1+qz} \\ &+ \frac{1-q^2}{1+p^2}\frac{(z+p)(1+pz)}{(1+qz)^2}\frac{f_q''\left(\frac{z+q}{1+qz}\right)}{f_q'\left(\frac{z+q}{1+qz}\right)}. \end{aligned}$$

Again, we have $\operatorname{Re} Q_2(z) < 0$ for all $z \in \mathbf{D}$ if and only if $\operatorname{Re} Q_2(\frac{z-q}{1-qz}) < 0$ for all $z \in \mathbf{D}$. Therefore we know that

$$\begin{aligned} Q_2\left(\frac{z-q}{1-qz}\right) &= 1 + \frac{2p}{1+p^2} \cdot \frac{1-qz}{z-q} \\ &- \left(\frac{z-q}{1-qz} + p\right)\left(1 + p\frac{z-q}{1-qz}\right) \\ &\times \left(\frac{2q}{(1+p^2)(1-q^2)} - \frac{1-qz}{(1+p^2)(1-q^2)}\frac{f_q''(z)}{f_q'(z)}\right) \end{aligned}$$

has negative real part for all $z \in \mathbf{D}$. Multiplying with $1-q^2 > 0$ results in (3). □

*Proof of Corollary 2.* The case $q=0$ obviously only requires the step from $f_p$ to $f_0$, already discussed in the previous proof. Equation (8) therefore provides the statement of the Corollary. □

**Remark 4.** The constants $\mathbf{C}_q$, $q \in (-1,1)$, of (5) and (6) can be described in terms of an integral representation formula introduced in [5], giving

$$\begin{aligned} C_q &= \frac{1-q^2}{1-p^2} \cdot \frac{\operatorname{Res}_p f_p}{\operatorname{Res}_q f_q} \\ &= \frac{-p^2(1-q^2)}{(1-p^2)^3 \cdot \operatorname{Res}_q f_q} \exp \int_0^p \frac{-2\varphi(\zeta)}{1-\zeta\varphi(\zeta)}\, d\zeta. \end{aligned}$$

Using this fact we obtain an alternative integral representation formula.

A function $f_q$ belongs to $\mathcal{C}o_q$ if and only if there exists a holomorphic function $\varphi : \mathbf{D} \to \mathbf{D}$ with $\varphi(p) = p$ such that $f_q$ can be expressed as

$$\begin{aligned} f_q'(z) = &-\frac{(1-qz+p(z-q))^2}{(z-q)^2(1-qz)^2}\operatorname{Res}_q f_q \\ &\times \exp \int_p^{T(z)} \frac{-2\varphi(\zeta)}{1-\zeta\varphi(\zeta)}\, d\zeta \end{aligned} \tag{9}$$

for $z \in \mathbf{D}$, where $T(z) = \frac{(1-pq)z+p-q}{1-pq+(p-q)z}$ is an automorphism of the unit disk, mapping $T(q) = p$ with $T'(q) = 1-p^2 > 0$.

**3. Application.** For the expansion (2) Bhowmik, Ponnusamy and Wirths proved the following in [3].

**Theorem A** [3]. *Let* $q \in (0, 1-\frac{\sqrt{2}}{2}]$ *and* $f_q \in \mathcal{C}o_q$. *Then the variability region of* $c_1(f_q)$ *is given by*

$$\left|\frac{c_1(f_q)}{a_1(f_q)} + \frac{q^4}{(1+q^2)(1-q^2)^3}\right| \le \frac{q^2}{(1+q^2)(1-q^2)^3}$$

*where equality holds if and only if $f_q$ is some specific function.*

Here the value $a_1(f_q)$ is the first coefficient of the non-normalized Maclaurin expansion of $f_q$. From [6] we also know the explicit description of the coefficient body $\{a_1(f_q), c_{-1}(f_q), c_1(f_q)\}$ for $q \in (0,1)$.

As an application of Theorem 1, we will now take a closer look at $\{c_{-1}(f_q), c_1(f_q)\}$ and $\{c_{-1}(f_q), c_2(f_q)\}$.

First we set for $p, q \in (-1,1)$ and $z \in \mathbf{D}$

$$\begin{aligned} P(z) = &-(1-q^2) - \frac{2p(1-q^2)}{1+p^2} \cdot \frac{1-qz}{z-q} \\ &+ \left(\frac{z-q}{1-qz} + p\right)\left(1 + p\frac{z-q}{1-qz}\right) \\ &\times \left(\frac{2q}{1+p^2} + \frac{1-qz}{1+p^2}\frac{f_q''(z)}{f_q'(z)}\right). \end{aligned}$$

Let $P$ have the expansion of the form

$$P(z) = d_0 + d_1(z-q) + d_2(z-q)^2 + \cdots.$$

We calculate

$$\begin{aligned} P(q) &= 1 - q^2 = d_0, \\ P'(q) &= \frac{2p}{1+p^2}\left(1 + (1-q^2)^2\frac{c_1(f_q)}{c_{-1}(f_q)}\right) = d_1 \end{aligned}$$

and

$$\begin{aligned} \frac{P''(q)}{2} = &\frac{2}{(1+p^2)(1-q^2)} \\ &\times \left(-pq - (1-2pq+p^2)(1-q^2)\frac{c_1(f_q)}{c_{-1}(f_q)}\right. \\ &\left. + 3p(1-q^2)^3\frac{c_2(f_q)}{c_{-1}(f_q)}\right) = d_2. \end{aligned}$$

Then the function $\tilde{P}(z)$ defined by

$$\begin{aligned} P\left(\frac{z+q}{1+qz}\right) &= (1-q^2)(1 + d_1 z \\ &\quad + ((1-q^2)d_2 - qd_1)z^2 + \cdots) \\ &= (1-q^2)\tilde{P}(z) \end{aligned}$$

has positive real part for all $z \in \mathbf{D}$ with $\tilde{P}(0) = 1$ and we can write

$$\tilde{P}(z) = 1 + a_1 z + a_2 z^2 + \cdots.$$

Since $\tilde{P}$ belongs to the Carathéodory class of functions, $|a_n| \le 2$ for all $n \in \mathbf{N}$ and

$$(10) \qquad |a_2 + \lambda a_1| \le 2(1 + |\lambda|)$$

for $\lambda \in \mathbf{C}$. Furthermore, we have $a_1 = d_1$ and $a_2 = (1-q^2)d_2 - qd_1$ by equating the coefficients.

This immediately leads us to

$$(11) \qquad \left|1 + (1-q^2)^2\frac{c_1(f_q)}{c_{-1}(f_q)}\right| \le \frac{1+p^2}{|p|}$$

for $\{c_{-1}(f_q), c_1(f_q)\}$.

Since (11) is valid for all $p \in (-1,1)$, we can minimize the right hand side by taking $p \to 1$. This yields

$$\left|1 + (1-q^2)^2\frac{c_1(f_q)}{c_{-1}(f_q)}\right| \le 2,$$

which is similar to a known result from [6, Theorem 1.1].

In case $q = 0$ we have

$$\left|1 + \frac{c_1(f_0)}{c_{-1}(f_0)}\right| \le 2,$$

which is the same result as we would have obtained, if we used the term of Corollary 2 for the definition of $P(z)$ instead of the term from Theorem 1.

For $\{c_{-1}(f_q), c_2(f_q)\}$ we calculate

$$\begin{aligned} d_2 &+ \frac{1+p^2-2pq}{(1-q^2)p}d_1 \\ &= \frac{2}{(1+p^2)(1-q^2)} \\ &\quad \times \left(1 + p^2 - 3pq + 3p(1-q^2)^3\frac{c_2(f_q)}{c_{-1}(f_q)}\right). \end{aligned}$$

In terms of $a_1$ and $a_2$ we obtain

$$\begin{aligned} a_2 &+ \frac{1-pq+p^2}{p}a_1 \\ &= \frac{2}{(1+p^2)}\left(1 + p^2 - 3pq + 3p(1-q^2)^3\frac{c_2(f_q)}{c_{-1}(f_q)}\right). \end{aligned}$$

Therefore using (10) for all $p \in (-1,1)$

$$\begin{aligned} &\left|\frac{1+p^2}{3p} - q + (1-q^2)^3\frac{c_2(f_q)}{c_{-1}(f_q)}\right| \\ &\qquad \le \frac{1+p^2}{3p^2}(1 + |p| - pq + p^2). \end{aligned}$$

In case $q = 0$ we have

$$\left|\frac{1+p^2}{3p} + \frac{c_2(f_0)}{c_{-1}(f_0)}\right| \le \frac{1+p^2}{3p^2}(1 + |p| + p^2),$$

where the area of the disks are minimized for $|p| \to 1$, leading to $|\frac{2}{3} + \frac{c_2(f_0)}{c_{-1}(f_0)}| \le 2$ and $|-\frac{2}{3} + \frac{c_2(f_0)}{c_{-1}(f_0)}| \le 2$, respectively. The range of $\frac{c_2(f_0)}{c_{-1}(f_0)}$ therefore lies in the intersection of these two disks, giving

$$-\frac{4}{3} \le \operatorname{Re}\frac{c_2(f_0)}{c_{-1}(f_0)} \le \frac{4}{3}.$$

**Acknowledgments.** The author thanks Prof. Toshiyuki Sugawa for his valuable suggestions and support. He also expresses his gratitude to the referee for his helpful comments improving this article. This work was supported by Grant-in-Aid for JSPS Fellows No. 26 · 2855.

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