# The zero-mass limit problem for a relativistic spinless particle in an electromagnetic field

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**Abstract:** It is shown that mass-parameter-dependent solutions of the imaginary-time magnetic relativistic Schrödinger equations converge as functionals of Lévy processes represented by stochastic integrals of stationary Poisson point processes if mass-parameter goes to zero.

**Key words:** Magnetic relativistic Schrödinger operator; imaginary-time relativistic Schrödinger equation; Lévy process; path integral formula; Feynman-Kac-Itô formula.

**1. Introduction and results.** Kasahara-Watanabe [12] discussed limit theorems in the framework of semimartingales represented by stochastic integrals of point processes. In fact, they considered a sequence of point processes and their certain functionals represented by stochastic integrals, and proved their convergence in that context.

In this paper we treat a sequence of a slightly more general functionals of special kind of Lévy processes, which have no Gaussian part stemming from relativistic quantum mechanics, to discuss its convergence. Naturally we have in mind the following relativistic Schrödinger equation which describes a spinless quantum particle of mass $m > 0$ (for example, pions) in $\mathbf{R}^d$ under the influence of the vector and scalar potentials $A(x), V(x)$:

$$(1.1) \qquad i\frac{\partial}{\partial t}\psi(x,t) = [H_A^m - m + V]\psi(x,t) \quad (t > 0),$$

where $x \in \mathbf{R}^d$. In this paper, to see the main idea, we only consider the case that $A \in C_0^\infty(\mathbf{R}^d;\mathbf{R}^d)$ and $V \in C_0(\mathbf{R}^d;\mathbf{R})$. Here then $H_A^m$ is defined by

$$(H_A^m f)(x) := \text{Os-}\frac{1}{(2\pi)^d}\iint_{\mathbf{R}^d\times\mathbf{R}^d} e^{i(x-y)\cdot\xi} \times \sqrt{\left|\xi - A(\tfrac{x+y}{2})\right|^2 + m^2} f(y)\,dy\,d\xi$$

for $f \in C_0^\infty(\mathbf{R}^d)$, where "Os" means *oscillatory integral*. $H_A^m$ is called the *Weyl pseudo-differential operator with mid-point prescription*, corresponding to the classical relativistic Hamiltonian $\sqrt{|\xi - A(x)|^2 + m^2}$. It is essentially selfadjoint in $L^2(\mathbf{R}^d)$ on $C_0^\infty(\mathbf{R}^d)$ and bounded from below by $m$ ([5], [10]). We have $H_0^m = \sqrt{-\Delta + m^2}$ for $A \equiv 0$, where $-\Delta$ is the Laplacian in $\mathbf{R}^d$. The light velocity $c$, electric charge $e$ and Planck's constant $h$ are taken to be 1, 1 and $2\pi$ respectively.

The operator $H_A^m - m + V$ was first studied in [9] by one of the authors of this paper to treat the *pure imaginary-time* relativistic Schrödinger equation

$$(1.2) \qquad \frac{\partial}{\partial t}u(x,t) = -[H_A^m - m + V]u(x,t) \quad (t > 0),$$

where $x \in \mathbf{R}^d$. An imaginary-time path integral formula was given on path space $D_0$ to represent the solution of the Cauchy problem for (1.2). Here $D_0$ is the set of the right-continuous paths $X : [0,\infty) \to \mathbf{R}^d$ with left-hand limits and $X(0) = 0$.

We use the probability space $(D_0, \mathcal{F}, \lambda^m)$ treated in [9] with the natural filtration $\{\mathcal{F}(t)\}_{t\geq 0}$, where $\mathcal{F}(t) := \sigma(X(s); s \leq t) \subset \mathcal{F}$. $\{X(t)\}_{t\geq 0}$ is Lévy process, namely, it has stationary independent increments and is stochastically continuous (cf., [11], [15], [1]). $\lambda^m(X; X(t) \in dy)$ is equal to $k_0^m(y,t)dy$, where $k_0^m(y,t)$ is the integral kernel of the operator $e^{-t(\sqrt{-\Delta+m^2}-m)}$ and has an explicit expression

$$(1.3) \qquad k_0^m(y,t) = \begin{cases} 2\left(\frac{m}{2\pi}\right)^{(d+1)/2} \frac{te^{mt}K_{(d+1)/2}(m(|y|^2+t^2)^{1/2})}{(|y|^2+t^2)^{(d+1)/4}}, & m > 0, \\ \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}}\frac{t}{(|y|^2+t^2)^{(d+1)/2}}, & m = 0. \end{cases}$$

Here $K_\nu$ stands for the modified Bessel function of the third kind of order $\nu$.

The characteristic function of $X(t)$ is

$$(1.4) \qquad E^m[e^{i\xi\cdot X(t)}] = e^{-t(\sqrt{|\xi|^2+m^2}-m)}, \quad \xi \in \mathbf{R}^d,$$

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where $E^m$ denotes the expectation over $D_0$ with respect to $\lambda^m$. By the *Lévy-Khintchine formula*,

$$(1.5)\quad \sqrt{|\xi|^2+m^2}-m = -\int_{|y|>0}(e^{i\xi\cdot y}-1-i\xi\cdot y\mathbf{1}_{|y|<1})n^m(dy).$$

Here $n^m(dy)$ is the *Lévy measure*, that is a $\sigma$-finite measure on $\mathbf{R}^d\setminus\{0\}$ satisfying $\int_{|y|>0}(1\wedge|y|^2)n^m(dy)<\infty$, and having density

$$(1.6)\quad n^m(y)=n^m(|y|) = \begin{cases} 2(\frac{m}{2\pi})^{(d+1)/2}\frac{K_{(d+1)/2}(m(|y|)}{|y|^{(d+1)/2}}, & m>0,\\ \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}}\frac{1}{|y|^{d+1}}, & m=0.\end{cases}$$

As shown in [5], $H_A^m$ has another expression connected with the Lévy measure $n^m(dy)$

$$(H_A^m f)(x)=mf(x)-\lim_{r\downarrow 0}\int_{|y|\geq r}[e^{-iy\cdot A(x+\frac12 y)}f(x+y) - f(x)]n^m(dy).$$

For $X\in D_0$, let $N_X(dsdy)$ be a counting measure on $(0,\infty)\times(\mathbf{R}^d\setminus\{0\})$ defined by

$$N_X(E):=\#\{s>0;(s,X(s)-X(s-))\in E\}$$

for $E\in\mathcal{B}(0,\infty)\times\mathcal{B}(\mathbf{R}^d\setminus\{0\})$, where $\mathcal{B}(\cdots)$ are $\sigma$-algebras of Borel sets. $N_X(dsdy)$ is the stationary Poisson random measure with intensity measure $dsn^m(dy)$ with respect to $\lambda^m$. Let $\widetilde{N_X^m}(dsdy):=N_X(dsdy)-dsn^m(dy)$. By the *Lévy-Itô theorem*,

$$(1.7)\quad X(t)=\int_0^t\int_{|y|\geq 1} yN_X(dsdy) + \int_0^t\int_{0<|y|<1} y\widetilde{N_X^m}(dsdy).$$

Here and below, we should understand $\int_0^t:=\int_{(0,t]}$. It can be proved that the solution of (1.2) with initial data $u^m(x,0)=g(x)$ is given by

$$(1.8)\quad u^m(x,t):=E^m[e^{-S^m(t,x,X)}g(x+X(t))],$$

$$(1.9)\quad S^m(\cdot):=iY^m(t,x,X)+\int_0^t V(x+X(s))ds,$$

$$\begin{aligned} Y^m(\cdot):=&\int_0^t\int_{|y|\geq 1}A(x+X(s-)+\tfrac12 y)\cdot yN_X(dsdy)\\ &+\int_0^t\int_{0<|y|<1}A(x+X(s-)+\tfrac12 y)\cdot y\widetilde{N_X^m}(dsdy)\\ &+\int_0^t ds\int_{0<|y|<1}[A(x+X(s)+\tfrac12 y) - A(x+X(s))]\cdot yn^m(dy).\end{aligned}$$

In (1.7) and (1.9) above, the integration regions $|y|\geq 1$ and $0<|y|<1$ may be replaced by $|y|\geq\delta$ and $0<|y|<\delta$ respectively, for any $\delta>0$.

We note that these relativistic quantities, $H_A^m-m+V$, $\sqrt{|\xi|^2+m^2}-m$, $D_0$, $\lambda^m$, $k_0^m(y,t)$ and $X(t)$, correspond to the nonrelativistic ones $\frac{1}{2m}(-i\nabla-A)^2+V$, $\frac{|\xi|^2}{2m}$, $C_0$, *Wiener measure*, the heat kernel $(\frac{m}{2\pi t})^{d/2}e^{-\frac{m}{2t}|y|^2}$, *Brownian motion* $B(t)$, respectively. Here $C_0$ is the space of continuous paths $B:[0,\infty)\to\mathbf{R}^d$ with $B(0)=0$. Furthermore, (1.8) with (1.9) is what *does* correspond to *Feynman-Kac-Itô formula* ([16]).

The purpose of this paper is to answer the following question:

(Q) *When the mass $m>0$ of the particle becomes sufficiently small, how does its property vary?*

**Theorem 1.** *$\lambda^m$ converges weakly to $\lambda^0$ as $m\downarrow 0$.*

**Theorem 2.** *$u^m(\cdot,t)$ converges to $u^0(\cdot,t)$ on $L^2(\mathbf{R}^d)$ as $m\downarrow 0$, uniformly on $[0,T]$.*

Here and below, $0<T<\infty$ can be taken arbitrary. Theorem 2 implies the strong resolvent convergence of $H_A^m-m+V$ to $H_A^0+V$ ([13, IX, Theorem 2.16]). An immediate consequence is the following result for the solution $\psi^m(x,t)$ of the Cauchy problem for (1.1).

**Corollary 1.** *$\psi^m(\cdot,t)$ converges to $\psi^0(\cdot,t)$ on $L^2(\mathbf{R}^d)$ as $m\downarrow 0$, uniformly on $[0,T]$.*

We will prove Theorem 2 by using following:

**Theorem 3.** *$u^m(\cdot,t)$ converges to $u^0(\cdot,t)$ on $C_\infty(\mathbf{R}^d)$ as $m\downarrow 0$, uniformly on $[0,T]$, where $C_\infty(\mathbf{R}^d)$ is the space of the continuous functions $g:\mathbf{R}^d\to\mathbf{C}$ with $|g(x)|\to 0$ as $|x|\to\infty$ with norm $\|g\|_\infty:=\sup_{x\in\mathbf{R}^d}|g(x)|$.*

The crucial idea of proof is to do a change of variable "path". In Sections 2, 3 and 4, these theorems are shown by probabilistic method, although one can more easily show Theorem 2 by operator-theoretical one [6], and also by pseudo-differential calculus [14]. In this paper, as we mentiond before, we treat the problem under a rather mild assumption on the potentials $A(x),V(x)$. We will come to more general case in a forthcoming paper, together for the other two different magnetic relativistic Schrödinger operators ([7], [8]) corresponding to the same classical relativistic Hamiltonian. Another limit problem when the light velocity $c$ goes to infinity (*nonrelativistic limit*) was studied in [4].

**2. Proof of Theorem 1.** We observe the following three facts which imply Theorem 1 ([2, Theorem 13.5]): (i) The finite dimensional distributions with respect to $\lambda^m$ converge weakly to those with respect to $\lambda^0$ as $m \downarrow 0$. (ii) For any $t > 0$, $\lambda^0(X; X(t) - X(t-\varepsilon) \in dy)$ converges weakly to Dirac measure concentrated at the point $0 \in \mathbf{R}^d$ as $\varepsilon \downarrow 0$. (iii) There exist constants $\alpha > \frac{1}{2}$, $\beta > 0$ and a non-decreasing continuous function $F$ on $[0, \infty)$ such that

$$E^m[|X(s) - X(r)|^\beta |X(t) - X(s)|^\beta] \le [F(t) - F(r)]^{2\alpha}, \quad 0 < m < 1,\ 0 \le r < s < t.$$

*Proof.* (i) follows from (1.4), and (ii) from the stochastic continuity of $\{X(t)\}_{t\ge0}$. (iii) Since $\frac{d}{d\tau}\tau^\nu K_\nu(\tau) = -\tau^\nu K_{\nu-1}(\tau) \quad (\tau > 0, \nu > 0)$ ([3, (21), p. 79]) and $\nu \mapsto K_\nu(\tau)$ is strictly increasing in $(0, \infty)$ ([3, (21), p. 82]), we have $(d/d\tau)(e^\tau \tau^\nu K_\nu(\tau)) = e^\tau \tau^\nu (K_\nu(\tau) - K_{\nu-1}(\tau)) < 0$ if $0 < \nu < \frac{1}{2}$. Therefore $\tau \mapsto e^\tau \tau^\nu K_\nu(\tau)$ is strictly decreasing in $(0, \infty)$ and so [3, (41), (42), (43), p. 10]

$$e^\tau \tau^\nu K_\nu(\tau) \le \lim_{\tau\downarrow0} \tau^\nu K_\nu(\tau) = 2^{\nu-1}\Gamma(\nu). \tag{2.1}$$

Then we have for $0 \le r < s < t$, $\frac{1}{2} < \beta < 1$,

$$\begin{aligned} &E^m[|X(s) - X(r)|^\beta |X(t) - X(s)|^\beta] \\ &= \int |y|^\beta k_0^m(y, s-r)dy \int |y|^\beta k_0^m(y, t-s)dy \\ &= C(d,\beta)^2((s-r)(t-s))^\beta \\ &\quad \times e^{m(s-r)}(m(s-r))^{\frac{1-\beta}{2}} K_{\frac{1-\beta}{2}}(m(s-r)) \\ &\quad \times e^{m(t-s)}(m(t-s))^{\frac{1-\beta}{2}} K_{\frac{1-\beta}{2}}(m(t-s)) \\ &\le C(d,\beta)^2 2^{-(1+2\beta)}\Gamma(\tfrac{1-\beta}{2})^2(t-r)^{2\beta}, \end{aligned}$$

where in the second equality we use [4, Lemma 3.3(ii)] with a constant $C(d,\beta)$ depending on $d$ and $\beta$. Therefore (iii) holds for $\frac{1}{2} < \beta < 1$ and $\alpha = \beta$ and $F(p) := C(d,\beta)^{1/\beta}2^{-(1+2\beta)/2\beta}\Gamma(\frac{1-\beta}{2})^{1/\beta}p$. □

**3. Proof of Theorem 2.** We will prove Theorem 2 by assuming validity of Theorem 3. In this and the next section, we assume $V \ge 0$ without loss of generality, since in the general case, we have only to replace $V$ in (1.8), (1.9) by $V - \inf V \ge 0$.

Step I: Let $g \in C_0^\infty(\mathbf{R}^d)$. For $R > 0$, we have

$$\begin{aligned} \|u^m(\cdot,t) - u^0(\cdot,t)\|_2 &\le \|u^m(\cdot,t) - u^0(\cdot,t)\|_{L^2(|x|<R)} \\ &\quad + \|u^m(\cdot,t) - u^0(\cdot,t)\|_{L^2(|x|\ge R)} \\ &=: I_1(t,m,R) + I_2(t,m,R). \end{aligned}$$

From Theorem 3, $I_1(t,m,R)$ converges to zero as $m \downarrow 0$ uniformly on $t \le T$. From (1.8), we have

$$\begin{aligned} I_2(t,m,R) &\le \|u^m(\cdot,t)\|_{L^2(|x|\ge R)} + \|u^0(\cdot,t)\|_{L^2(|x|\ge R)} \\ &\le \left(\int_{|x|\ge R} dx \int k_0^m(y,t)|g(x+y)|^2 dy\right)^{\frac{1}{2}} \\ &\quad + \left(\int_{|x|\ge R} dx \int k_0^0(y,t)|g(x+y)|^2 dy\right)^{\frac{1}{2}} \\ &=: J(t,m,R) + J(t,0,R). \end{aligned}$$

Let $\chi$ be a nonnegative $C_0^\infty(\mathbf{R}^d)$ function such that $\chi(x) = 1$ if $|x| \le \frac{1}{2}$ and $= 0$ if $|x| \ge 1$. Put $h(x) = |g(x)|^2$. Since $\mathbf{1}_{|x|<R} \ge \chi(\frac{x}{R})$, we have

$$\begin{aligned} &J(t,m,R)^2 \\ &\le \int (1 - \chi(\tfrac{x}{R}))dx \int k_0^m(y,t)h(x+y)dy \\ &= \frac{1}{(2\pi)^d}\left[\widehat{h}(0)\int\left(1 - \exp\left\{-t\left[\sqrt{\tfrac{|\eta|^2}{R^2} + m^2} - m\right]\right\}\right)\right. \\ &\quad \times \overline{\widehat{\chi}(\eta)}d\eta + \int (\widehat{h}(0)) - \widehat{h}(\tfrac{\eta}{R})) \\ &\quad \left.\times \exp\left\{-t\left[\sqrt{\tfrac{|\eta|^2}{R^2} + m^2} - m\right]\right\}\overline{\widehat{\chi}(\eta)}d\eta\right], \end{aligned}$$

which converges to zero as $R \to \infty$ uniformly on $t \le T$ and $0 \le m \le 1$. Here, for $\varphi \in \mathcal{S}(\mathbf{R}^d)$, $\widehat{\varphi}$ is the Fourier transform of $\varphi$ given by $\widehat{\varphi}(\xi) = \int e^{-ix\cdot\xi}\varphi(x)dx$ $(\xi \in \mathbf{R}^d)$.

From (1.3) and (2.1), it follows that $k_0^m(y,t) \to k_0^0(y,t)$ as $m \downarrow 0$, and then $J(t,0,R)^2 \le \liminf_{m\downarrow0} J(t,m,R)^2$ by Fatou's lemma. Therefore we have Theorem 2 for this step.

Step II: Let $g \in L^2(\mathbf{R}^d)$. There is a sequence $\{g_n\} \subset C_0^\infty(\mathbf{R}^d)$ such that $g_n \to g$ in $L^2(\mathbf{R}^d)$ as $n \to \infty$. Put $u_n^m(x,t) := E^m[e^{-S^m(t,x,X)}g_n(x + X(t))]$. Then we have

$$\begin{aligned} &\|u^m(\cdot,t) - u^0(\cdot,t)\|_2 \\ &\quad \le \|u^m(\cdot,t) - u_n^m(\cdot,t)\|_2 + \|u_n^m(\cdot,t) - u_n^0(\cdot,t)\|_2 \\ &\quad\quad + \|u_n^0(\cdot,t) - u^0(\cdot,t)\|_2 \\ &\quad \le 2\|g_n - g\|_2 + \|u_n^m(\cdot,t) - u_n^0(\cdot,t)\|_2. \end{aligned}$$

By Step I, we have

$$\limsup_{m\downarrow0} \sup_{t\le T} \|u^m(\cdot,t) - u^0(\cdot,t)\|_2 \le 2\|g_n - g\|_2,$$

which converges to zero as $n \to \infty$. □

**4. Proof of Theorem 3.** From (1.8), we have to prove that

$$\begin{aligned} u^m(x,t) &= E^m[e^{-S^m(t,x,X)}g(x + X(t))] \\ &\to E^0[e^{-S^0(t,x,X)}g(x + X(t))] = u^0(x,t) \end{aligned}$$

as $m \downarrow 0$ in $C_\infty(\mathbf{R}^d)$. But its direct proof seems difficult since both the integrand $e^{-S^m(t,x,X)}g(x +$

$X(t)$) and the probability measure $\lambda^m$ depend on $m$. So we change $E^m[\cdots]$ to $E^0[\cdots]$ by a *change of variable* (i.e., *change of probability measure*) $\lambda^m = \lambda^0 \Phi_m^{-1}$ with path space transformation $\Phi_m : D_0 \to D_0$. If there is such a $\Phi_m$, we can see by (1.4) and (1.5) that the difference between the path $X(t)$ and the transformed path $\Phi_m(X)(t)$ is expressed in terms of the difference between the two Lévy measures $n^0(dy)$ and $n^m(dy)$, so that it is presumed to hold that $n^m(dy) = n^0\phi_m^{-1}(dy)$ for some map $\phi_m : \mathbf{R}^d \setminus \{0\} \to \mathbf{R}^d \setminus \{0\}$.

We will determine $\phi_m$ in such a way that (1) $n^m(dy) = n^0\phi_m^{-1}(dy)$, (2) $\phi_m \in C^1(\mathbf{R}^d \setminus \{0\}; \mathbf{R}^d \setminus \{0\})$, (3) $\phi_m$ is one to one and onto, (4) $\det D\phi_m(z) \neq 0$ for all $z \in \mathbf{R}^d \setminus \{0\}$, where $D\phi_m(z)$ is the Jacobian matrix of $\phi_m$ at the point $z$.

Let $U := \{y \in \mathbf{R}^d \setminus \{0\}; |y| \in U'\}$ for $U' \in \mathcal{B}(0,\infty)$. Introducing the spherical coordinates by $z = r\omega$, $r > 0$, $\omega \in S^{d-1}$, we have

$$n^m(U) = \int_U n^m(|y|)dy = C(d)\int_{U'} n^m(r) r^{d-1} dr,$$

where $C(d)$ is the surface area of the $d$-dimensional unit ball.

Let us assume that $\phi_m^{-1}(z) = l_m(|z|)\frac{z}{|z|}$ for some non-decreasing $C^1$ function $l_m : (0,\infty) \to (0,\infty)$. Then we have

$$\begin{aligned} n^0\phi_m^{-1}(U) &= \int_U n^0(l_m(|z|))|z|^{-(d-1)} l_m(|z|)^{d-1} l_m'(|z|)dz \\ &= C(d)\int_{U'} n^0(l_m(r)) l_m(r)^{d-1} l_m'(r)dr, \end{aligned}$$

where $l_m'(r) = (d/dr)l_m(r)$. Therefore we have

$$n^m(r) r^{d-1} = n^0(l_m(r)) l_m(r)^{d-1} l_m'(r), \quad \text{a.s. } r > 0.$$

If $m > 0$, from (1.6), we have

$$-\frac{d}{dr} l_m(r)^{-1} = 2^{-\frac{d-1}{2}} \Gamma(\tfrac{d+1}{2})^{-1} m^{\frac{d+1}{2}} r^{\frac{d-3}{2}} K_{\frac{d+1}{2}}(mr).$$

We solve this differential equation under boundary condition $l_m(\infty) = \infty$ to get

$$l_m(r) = \frac{2^{\frac{d-1}{2}}\Gamma(\frac{d+1}{2})}{m^{\frac{d+1}{2}} \int_r^\infty u^{\frac{d-3}{2}} K_{\frac{d+1}{2}}(mu)du}. \tag{4.1}$$

Here we note that $0 < \int_r^\infty u^{\frac{d-3}{2}} K_{\frac{d+1}{2}}(mu)du < \infty$ by $K_{\frac{d+1}{2}}(\tau) > 0$ for $\tau > 0$, and [3, (37), (38), p. 9]

$$K_{\frac{d+1}{2}}(\tau) = \left(\frac{\pi}{2}\right)^{1/2} \tau^{-1/2} e^{-\tau}(1 + o(1)), \ \tau \uparrow \infty.$$

**Proposition 1.** (i) *$l_m(r)$ is a strictly increasing $C^\infty$ function of $r \in (0,\infty)$ and $l_m(+0) = 0$, $l_m(\infty) = \infty$.*
(ii) *For all $r > 0$, $l_m(r)$ converges to $r$, strictly decreasingly, as $m \downarrow 0$.*

*Proof.* (2.1) implies $l_m(+0) = 0$. The other claims of (i) follow from (4.1) and the fact that $K_{(d+1)/2}(\tau)$ is a $C^\infty$ function in $(0,\infty)$. The claim (ii) can be proved by the fact that $\tau^\nu K_\nu(\tau)$ is strictly decreasing in $(0,\infty)$ (cf. Section 2, Proof of (ii)), (2.1) and the monotone convergence theorem. □

If $m = 0$, let $l_0(r) := r$. Let us put $\phi_0(z) := z$ and for $m > 0$,

$$\phi_m(z) := l_m^{-1}(|z|)\frac{z}{|z|}, \quad z \in \mathbf{R}^d \setminus \{0\}.$$

Then we have

$$\phi_m^{-1}(z) = l_m(|z|)\frac{z}{|z|}, \quad z \in \mathbf{R}^d \setminus \{0\}.$$

We note that

$$\phi_m(z) \to z, \ |\phi_m(z)| = l_m^{-1}(|z|) \uparrow |z| \tag{4.2}$$

as $m \downarrow 0$ by Proposition 1 (ii).

Let us define $\Phi_0(X) := X$ and for $m > 0$,

$$\begin{aligned} \Phi_m(X)(t) &:= \int_0^t \int_{|y|\geq 1} y N_X(ds\phi_m^{-1}(dy)) \\ &\quad + \int_0^t \int_{0<|y|<1} y \widetilde{N_X^0}(ds\phi_m^{-1}(dy)) \\ &= \int_0^t \int_{|z|\geq l_m(1)} \phi_m(z) N_X(dsdz) + \int_0^t \int_{0<|z|<l_m(1)} \phi_m(z)\widetilde{N_X^0}(dsdz) \\ &= \int_0^t \int_{|z|\geq 1} \phi_m(z) N_X(dsdz) + \int_0^t \int_{0<|z|<1} \phi_m(z)\widetilde{N_X^0}(dsdz). \end{aligned} \tag{4.3}$$

**Proposition 2.** *For every sequence $\{m\}$ with $m \downarrow 0$, there exists a subsequence $\{m'\}$ such that $\sup_{t\leq T} |\Phi_{m'}(X)(t) - X(t)| \to 0$ as $m' \downarrow 0$, $\lambda^0$-a.s. $X \in D_0$.*

*Proof.* From (1.7) and (4.3), we have

$$\begin{aligned} \sup_{t\leq T} |\Phi_m(X)(t) - X(t)| &\leq \int_0^T \int_{|z|\geq 1} |\phi_m(z) - z| N_X(dsdz) \\ &\quad + \sup_{t\leq T}\left|\int_0^t \int_{0<|z|<1} (\phi_m(z) - z)\widetilde{N_X^0}(dsdz)\right| \\ &=: I_1(m,X) + \sup_{t\leq T}|I_2(t,m,X)|. \end{aligned}$$

We have $I_1(m,X) \to 0$ as $m \downarrow 0$ by (4.2) and $\int_0^T \int_{|z|\geq 1} |z| N_X(dsdz) < \infty$. We note that $I_2(t,m,X)$ is the $L^2(D_0;\lambda^0)$-limit of the right-continuous $\{\mathcal{F}(t)\}_{t\geq 0}$-martingale $\{I_2^\varepsilon(t,m,X)\}_{t\geq 0}$ with

$I_2^\varepsilon(t,m,X) := \int_0^t \int_{\varepsilon<|z|<1} (\phi_m(z) - z)\widetilde{N_X^0}(dsdz)$ as $\varepsilon \downarrow 0$, with convergence being uniform on $t \le T$. By taking a subsequence if necessary, $I_2^\varepsilon(t,m,X)$ converges to $I_2(t,m,X)$ as $\varepsilon \downarrow 0$ uniformly on $t \le T$, $\lambda^0$-a.s., and hence $I_2(t,m,X)$ is right-continuous on $t \le T$, $\lambda^0$-a.s. ([11, p. 73, Proof of Theorem 5.1], [15, p. 128–129, Proofs of Lemmas 20.6, 20.7]). Then we use Doob's martingale inequality [1] to have

$$E^0\left[\sup_{t\le T} |I_2(t,m,X)|^2\right] \le 4E^0\left[|I_2(T,m,X)|^2\right]$$
$$\le 4T \int_{0<|z|<1} |\phi_m(z) - z|^2 n^0(dz),$$

which converges to zero as $m \downarrow 0$ by (4.2) and $\int_{0<|z|<1} |z|^2 n^0(dz) < \infty$. □

By (1.8) and $\lambda^m = \lambda^0\Phi_m^{-1}$, we have

$$u^m(x,t) = E^0[e^{-S^m(t,x,\Phi_m(X))} g(x + \Phi_m(X)(t))],$$

and then

$$(4.4)\quad \sup_{t\le T} \|u^m(\cdot,t) - u^0(\cdot,t)\|_\infty \le \|g\|_\infty \sup_{t\le T, x\in\mathbf{R}^d} E^0[|e^{-S^m(t,x,\Phi_m(X))} - e^{-S^0(t,x,X)}|] + E^0\left[\sup_{t\le T} \|g(\cdot + \Phi_m(X)(t)) - g(\cdot + X(t))\|_\infty\right].$$

Since $g \in C_\infty(\mathbf{R}^d)$ is uniformly continuous and bounded on $\mathbf{R}^d$, the second term on the right of (4.4) converges to zero as $m \downarrow 0$.

Next we consider the first term on the right of (4.4). By $N_{\Phi_m(X)}(dsdy) = N_X(ds\phi_m^{-1}(dy))$, we have

$$S^m(t,x,\Phi_m(X))$$
$$= i\Big(\int_0^t\int_{|z|\ge 1} A(x + \Phi_m(X)(s-) + \tfrac{1}{2}\phi_m(z)) \cdot \phi_m(z) N_X(dsdz)$$
$$+ \int_0^t\int_{0<|z|<1} A(x + \Phi_m(X)(s-) + \tfrac{1}{2}\phi_m(z)) \cdot \phi_m(z) \widetilde{N_X^0}(dsdz)$$
$$+ \int_0^t ds \int_{0<|z|<1} [A(x + \Phi_m(X)(s) + \tfrac{1}{2}\phi_m(z)) - A(x + \Phi_m(X)(s))] \cdot \phi_m(z) n^0(dz)\Big)$$
$$+ \int_0^t V(x + \Phi_m(X)(s))ds$$
$$=: i\big(S_1^m(t,x,X) + S_2^m(t,x,X) + S_3^m(t,x,X)\big) + S_4^m(t,x,X).$$

By the inequality

$$|e^{-(ia+b)} - e^{-(ia'+b')}| \le e^{-b}|e^{-ia} - e^{-ia'}| + |b - b'|$$

for any $a, a' \in \mathbf{R}$, $b, b' \ge 0$, $\sup E^0[\cdots]$ of the first term on the right of (4.4) is less than or equal to

$$(4.5)\quad E^0\left[\sup_{t\le T} \|e^{-iS_1^m(t,\cdot,X)} - e^{-iS_1^0(t,\cdot,X)}\|_\infty\right] + \sup_{x\in\mathbf{R}^d} E^0\left[\sup_{t\le T} |S_2^m(t,x,X) - S_2^0(t,x,X)|\right]$$
$$+ E^0\left[\sup_{t\le T} \|S_3^m(t,\cdot,X) - S_3^0(t,\cdot,X)\|_\infty\right] + E^0\left[\sup_{t\le T} \|S_4^m(t,\cdot,X) - S_4^0(t,\cdot,X)\|_\infty\right].$$

Now, let $\{m\}$ be a sequence with $m \downarrow 0$ and $\{m'\}$ any subsequence of $\{m\}$. By Proposition 2, there exists a subsequence $\{m''\}$ of $\{m'\}$ such that $\sup_{t\le T} |\Phi_{m''}(X)(t) - X(t)| \to 0$ as $m'' \downarrow 0$, $\lambda^0$-a.s.

To prove that each term of (4.5) converges to zero as $m'' \downarrow 0$, we first note that

$$S_1^{m''}(t,x,X) - S_1^0(t,x,X)$$
$$= \int_0^t \int_{|z|\ge 1} (A(x + \Phi_{m''}(X)(s-) + \tfrac{1}{2}\phi_{m''}(z)) - A(x + X(s-) + \tfrac{1}{2}z)) \cdot \phi_{m''}(z) N_X(dsdz)$$
$$+ \int_0^t \int_{|z|\ge 1} A(x + X(s-) + \tfrac{1}{2}z) \cdot (\phi_{m''}(z) - z) N_X(dsdz).$$

Then the integrand of the first term of (4.5) is less than or equal to

$$\int_0^T \int_{|z|\ge 1} \sup_{x\in\mathbf{R}^d} |A(x + \Phi_{m''}(X)(s-) + \tfrac{1}{2}\phi_{m''}(z)) - A(x + X(s-) + \tfrac{1}{2}z)||z| N_X(dsdz)$$
$$+ \sup_{x\in\mathbf{R}^d} |A(x)| \int_0^T \int_{|z|\ge 1} |\phi_{m''}(z) - z| N_X(dsdz),$$

which converges to zero as $m'' \downarrow 0$ since $A \in C_0^\infty(\mathbf{R}^d;\mathbf{R}^d)$ is uniformly continuous on $\mathbf{R}^d$.

Next, since $S_2^m(t,x,X)$ is seen to be right-continuous, by Schwarz's inequality and Doob's martingale inequality, $E^0[\cdots]$ of the second term of (4.5) is less than or equal to

$$2E^0\left[\int_0^T ds \int_{0<|z|<1} |A(x + \Phi_{m''}(X)(s-) + \tfrac{1}{2}\phi_{m''}(z)) \cdot \phi_{m''}(z) - A(x + X(s-) + \tfrac{1}{2}z) \cdot z|^2 n^0(dz)\right]^{\frac{1}{2}}.$$

By the inequality $(a+b)^2 \le 2(a^2+b^2)$ for any $a, b \in \mathbf{R}$, $E^0[\cdots]$ above is less than or equal to

$$2\Big\{E^0\left[\int_0^T ds \int_{0<|z|<1} \sup_{x\in\mathbf{R}^d} |A(x + \Phi_{m''}(X)(s-) + \tfrac{1}{2}\phi_{m''}(z)) - A(x + X(s-) + \tfrac{1}{2}z)|^2 |z|^2 n^0(dz)\right]$$
$$+ T \sup_{x\in\mathbf{R}^d} |A(x)|^2 \int_{0<|z|<1} |\phi_{m''}(z) - z|^2 n^0(dz)\Big\},$$

which converges to zero as $m'' \downarrow 0$. As for the third term of (4.5), by the mean value theorem, we have

$$S_3^{m''}(t,x,X) - S_3^0(t,x,X) = \frac{1}{2}\int_0^t ds \int_{0<|z|<1} n^0(dz)$$
$$\times \int_0^1 [(W_{x,X}^{m''}(s,\theta)\phi_{m''}(z))\cdot\phi_{m''}(z)$$
$$- (W_{x,X}^0(s,\theta)z)\cdot z]d\theta.$$

Here $W_{x,X}^{m''}(s,\theta)$ and $W_{x,X}^0(s,\theta)$ are $d\times d$ matrices defined by

$$W_{x,X}^{m''}(s,\theta) = DA(x+\Phi_{m''}(X)(s) + \tfrac{1}{2}\phi_{m''}(z)\theta),$$
$$W_{x,X}^0(s,\theta) = DA(x + X(s) + \tfrac{1}{2}z\theta),$$

where $DA(\cdot)$ is the Jacobian matrix of $A$. Since

$$(W_{x,X}^{m''}(s,\theta)\phi_{m''}(z))\cdot\phi_{m''}(z) - (W_{x,X}^0(s,\theta)z)\cdot z$$
$$= (W_{x,X}^{m''}(s,\theta)\phi_{m''}(z))\cdot(\phi_{m''}(z)-z)$$
$$+ ((W_{x,X}^{m''}(s,\theta) - W_{x,X}^0(s,\theta))\phi_{m''}(z))\cdot z$$
$$+ (W_{x,X}^0(s,\theta)(\phi_{m''}(z)-z))\cdot z,$$

the integrand of the third term of (4.5) is less than or equal to

$$T\sup_{x\in\mathbf{R}^d}\|DA(x)\|\int_{0<|z|<1}|\phi_{m''}(z)-z||z|n^0(dz)$$
$$+\frac{1}{2}\int_0^T ds\int_{0<|z|<1}|z|^2 n^0(dz)$$
$$\times\int_0^1 \sup_{x\in\mathbf{R}^d}\|W_{x,X}^{m''}(s,\theta) - W_{x,X}^0(s,\theta)\|d\theta,$$

where $\|\cdot\|$ is the norm of matrices. This is less than or equal to $3T\sup_{x\in\mathbf{R}^d}\|DA(x)\|\int_{0<|z|<1}|z|^2 n^0(dz) < \infty$, and converges to zero as $m''\downarrow 0$ because each component of $DA$ is uniformly continuous on $\mathbf{R}^d$.

Finally, the fourth term of (4.5) is less than or equal to $E^0[\int_0^T \|V(\cdot+\Phi_{m''}(X)(s)) - V(\cdot + X(s))\|_\infty ds]$, which converges to zero as $m''\downarrow 0$ since $V\in C_0(\mathbf{R}^d;\mathbf{R})$ is uniformly continuous on $\mathbf{R}^d$. Thus we have $\sup_{t\le T}\|u^{m''}(\cdot,t) - u^0(\cdot,t)\|_\infty \to 0$ as $m''\downarrow 0$, and hence $\sup_{t\le T}\|u^m(\cdot,t)-u^0(\cdot,t)\|_\infty\to 0$ as $m\downarrow 0$. □

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