On the vanishing of the holomorphic invariants for Kähler-Ricci solitons

By Shunsuke SAITO

Department of Mathematics, Tokyo Institute of Technology,
2-12-1 O-okayama, Meguro-ku, Tokyo 152-8551, Japan

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Abstract: We prove the vanishing of the Futaki-type invariant defined by Tian and Zhu, which is an obstruction to the existence of Kähler-Ricci solitons.

Key words: Kähler-Ricci solitons; holomorphic invariants.

1. Introduction. In Kähler geometry, one of the main problems is to find canonical metrics on Fano manifolds. Our interest is the existence problem of Kähler-Ricci solitons, which have the following importance: Kähler-Ricci solitons are the self-similar solutions of Kähler-Ricci solitons, and are also generalization of Kähler-Einstein metrics different from extremal Kähler metrics.

Let M be an n-dimensional Fano manifold, and ω a Kähler form on M which represents the first Chern class $c_1(M)$ of M. Since both the Ricci form $\mathrm{Ric}(\omega)$ of ω and ω represent $c_1(M)$, there exists a unique smooth real-valued function f_ω on M such that

$$\mathrm{Ric}(\omega) - \omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial} f_\omega, \quad \int_M (e^{f_\omega} - 1)\,\omega^n = 0.$$

Let $\mathfrak{h}(M)$ be the Lie algebra consisting of all holomorphic vector fields on M, and X an element of $\mathfrak{h}(M)$. Since M is Fano, there exists a unique smooth complex-valued function $\theta_X(\omega)$ on M such that

$$(1) \quad i_X\omega = \frac{\sqrt{-1}}{2\pi}\,\bar{\partial}\theta_X(\omega), \quad \int_M \theta_X(\omega) e^{f_\omega}\,\omega^n = 0.$$

Throughout this paper, we fix the normalization of the Hamiltonian function of X as above. Using these functions, in [5], Tian and Zhu defined a functional $F_X\colon \mathfrak{h}(M) \to \mathbf{C}$ by

$$F_X(v) := \int_M v(f_\omega - \theta_X(\omega)) e^{\theta_X(\omega)}\omega^n, \quad v \in \mathfrak{h}(M).$$

They proved that F_X is independent of the choice of ω and that if M admits a Kähler-Ricci soliton with respect to X, then F_X vanishes. Note that if X is identically zero, then F_X coincides with the original Futaki invariant defined in [1].

Next, let us look for a candidate for the holomorphic vector fields of Kähler-Ricci solitons. Let $\mathrm{Aut}^0(M)$ be the identity component of the holomorphic automorphism group of M, K its maximal compact subgroup. Then the Chevalley decomposition gives us the semi-direct product

$$\mathrm{Aut}^0(M) = \mathrm{Aut}_r(M) \ltimes R_u,$$

where $\mathrm{Aut}_r(M)$ is a reductive Lie subgroup of $\mathrm{Aut}^0(M)$ and the complexification of K, and R_u is the unipotent radical of $\mathrm{Aut}^0(M)$. Let $\mathfrak{h}_r(M)$, $\mathfrak{h}_u(M)$ be the Lie algebras of $\mathrm{Aut}_r(M)$, and R_u, respectively. From the decomposition above, we have

$$\mathfrak{h}(M) = \mathfrak{h}_r(M) + \mathfrak{h}_u(M).$$

In [5], Tian and Zhu found a prospective holomorphic vector field: there exists a unique holomorphic vector field X in the reductive part $\mathfrak{h}_r(M)$ such that its imaginary part generates a compact one-parameter subgroup of $\mathrm{Aut}_r(M)$ and F_X vanishes on $\mathfrak{h}_r(M)$. Note that if M has a Kähler-Ricci soliton with respect to some holomorphic vector field X, then X must be this X. Meanwhile, they asked whether, for this vector field X, the obstruction identically vanishes or not ([5, Problem 2.1]). Except for the toric case [6], there has been no investigation for this. Hence, their question has been still open.

On the other hand, in [3], Mabuchi proved that the Futaki invariant vanishes on the unipotent part

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$\mathfrak{h}_u(M)$. We extend this result to F_X as follows:

Theorem 1.1. *For any holomorphic vector field X in the reductive part $\mathfrak{h}_r(M)$ whose imaginary part generates a compact one-parameter subgroup of* $\mathrm{Aut}_r(M)$*, the holomorphic invariant F_X vanishes on the unipotent part $\mathfrak{h}_u(M)$. In particular, if we take X as above, then F_X vanishes on $\mathfrak{h}(M)$.*

This gives an affirmative answer to the Tian-Zhu problem.

2. Proof of the main theorem. In this section, we complete the proof of Theorem 1.1. Our approach is similar to that of Mabuchi [3] for the Futaki invariant.

Before giving the proof, we note that our normalization convention (1) is equivalent to

$$\theta_X(\omega) + \Delta_\omega \theta_X(\omega) + X f_\omega = 0$$

and using this identity, we can rewrite F_X as

$$F_X(v) = -\int_M \theta_v(\omega) e^{\theta_X(\omega)} \omega^n.$$

For the details, see [5, Section 2]. This form is convenient for our purpose.

Now, we prove our main theorem.

Proof of Theorem 1.1. At first, take a sufficiently large integer m, so that K_M^{-m} is very ample. Let Y be a holomorphic vector field in the unipotent part $\mathfrak{h}_u(M)$. Since the infinitesimal action of Y on $H^0(M, K_M^{-m})$ is nilpotent, there exists a basis $\{\sigma_0, \dots, \sigma_N\}$ for $H^0(M, K_M^{-m})$ such that

$$Y \cdot \sigma_0 = 0, \quad Y \cdot \sigma_i = e_{i-1}\sigma_{i-1} \quad (i = 1, \dots, N),$$

where each e_i is 0 or 1. For any positive number ε, we define a fiber metric h_ε on K_M^{-1} by

$$h_\varepsilon := \left\{ \sum_{i=0}^{N} (\varepsilon^i \sigma_i) \otimes (\varepsilon^i \bar{\sigma}_i) \right\}^{-1/m}.$$

Let ω_ε be the first Chern form of h_ε. By the definition of h_ε, ω_ε is proportional to a pullback of the Fubini-Study metric ω_{FS} of $\mathbf{P}H^0(M, K_M^{-m})^*$:

$$\omega_\varepsilon = \frac{1}{m} \Phi_\varepsilon^* \omega_{FS},$$

where $\Phi_\varepsilon : M \to \mathbf{P}H^0(M, K_M^{-m})^*$ is a projective embedding defined by a basis $\{\sigma_0, \varepsilon\sigma_1, \dots, \varepsilon^N \sigma_N\}$. Hence, ω_ε is a Kähler form in $c_1(M)$.

By the Calabi-Yau theorem, for ω_ε, there exists a Kähler form η_ε in $c_1(M)$ such that $\mathrm{Ric}(\eta_\varepsilon) = \omega_\varepsilon$. Then, h_ε coincides with the volume form η_ε^n. The infinitesimal action of Y on h_ε is written as

$$\begin{aligned} Y \cdot h_\varepsilon &= -\frac{1}{m} \left\{ \sum_{i=0}^{N} (\varepsilon^i \sigma_i) \otimes (\varepsilon^i \bar{\sigma}_i) \right\}^{-1} \\ &\quad \times \left\{ \sum_{i=0}^{N} \varepsilon^i (Y \cdot \sigma_i) \otimes (\varepsilon^i \bar{\sigma}_i) \right\} h_\varepsilon \\ &= -\frac{\varepsilon}{m} \left\{ \sum_{i=0}^{N} (\varepsilon^i \sigma_i) \otimes (\varepsilon^i \bar{\sigma}_i) \right\}^{-1} \\ &\quad \times \left\{ \sum_{i=1}^{N} (\varepsilon^{i-1} e_{i-1} \sigma_{i-1}) \otimes (\varepsilon^i \bar{\sigma}_i) \right\} h_\varepsilon. \end{aligned}$$

Since the action of Y on η_ε^n is just a Lie derivative, we have

$$\begin{aligned} \mathrm{div}_{\eta_\varepsilon} Y &= -\frac{\varepsilon}{m} \left\{ \sum_{i=0}^{N} (\varepsilon^i \sigma_i) \otimes (\varepsilon^i \bar{\sigma}_i) \right\}^{-1} \\ &\quad \times \left\{ \sum_{i=1}^{N} (\varepsilon^{i-1} e_{i-1} \sigma_{i-1}) \otimes (\varepsilon^i \bar{\sigma}_i) \right\}. \end{aligned}$$

On the other hand, by the Theorem 7.1 in [2],

$$\theta_Y(\omega_\varepsilon) = -\Delta_{\eta_\varepsilon} \theta_Y(\eta_\varepsilon) = -\mathrm{div}_{\eta_\varepsilon} Y.$$

(For the proof, see [4, p.25].) Hence, we obtain the explicit description of the Hamiltonian function of Y with respect to ω_ε:

$$\begin{aligned} \theta_Y(\omega_\varepsilon) &= \frac{\varepsilon}{m} \left\{ \sum_{i=0}^{N} (\varepsilon^i \sigma_i) \otimes (\varepsilon^i \bar{\sigma}_i) \right\}^{-1} \\ &\quad \times \left\{ \sum_{i=1}^{N} (\varepsilon^{i-1} e_{i-1} \sigma_{i-1}) \otimes (\varepsilon^i \bar{\sigma}_i) \right\}. \end{aligned}$$

Put $v_i := \varepsilon^{i-1} e_{i-1} \sigma_{i-1}$ $(i = 1, \dots, N)$ and $w_i := \varepsilon^i \sigma_i$ $(i = 0, \dots, N)$. Using the Cauchy-Schwarz inequality, we get the following estimate:

$$\begin{aligned} |\theta_Y(\omega_\varepsilon)|^2 &= \frac{\varepsilon^2}{m^2} \frac{\left| \sum_{i=1}^{N} v_i \otimes \bar{w}_i \right|^2}{\left\{ \sum_{i=0}^{N} w_i \otimes \bar{w}_i \right\}^2} \\ &\le \frac{\varepsilon^2}{m^2} \frac{\left| \sum_{i=1}^{N} v_i \otimes \bar{w}_i \right|^2}{\left\{ \sum_{i=1}^{N} w_i \otimes \bar{w}_i \right\} \left\{ \sum_{i=1}^{N} v_i \otimes \bar{v}_i \right\}} \le \frac{\varepsilon^2}{m^2}. \end{aligned}$$

Therefore, for any positive number ε, we have

$$
\begin{aligned}
(2) \qquad |F_X(Y)| &= \left| \int_M \theta_Y(\omega_\varepsilon) e^{\theta_X(\omega_\varepsilon)}\, \omega_\varepsilon^n \right| \\
&\leq \frac{\varepsilon}{m} \| e^{\theta_X(\omega_\varepsilon)} \|_{C^0} \int_M \omega_\varepsilon^n \\
&= \frac{\varepsilon}{m} \| e^{\theta_X(\omega_\varepsilon)} \|_{C^0} \langle [M], c_1(M)^n \rangle,
\end{aligned}
$$

where $[M]$ is the fundamental class of M.

To finish the proof, we need the following lemma due to Zhu:

Lemma 2.1 ([7, Corollary 5.3]). *Let (M, ω_g) be a Kähler manifold with nontrivial holomorphic vector field X. Suppose that ϕ is a smooth function on M such that $\omega_g + \sqrt{-1}\partial\bar{\partial}\phi$ is a Kähler form and $X\phi$ is a real-valued function. Then, there is a uniform constant C independent of ϕ such that $|X\phi| < C$.*

If we fix some reference Kähler form ω_0 in $c_1(M)$, then ω_ε can be written as $\omega_\varepsilon = \omega_0 + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_\varepsilon$, where φ_ε is a smooth real-valued function on M. Then, $\theta_X(\omega_\varepsilon) = \theta_X(\omega_0) + X\varphi_\varepsilon$. From the assumption on X, $X\varphi_\varepsilon$ is real-valued. By the lemma above, there exists a positive constant C independent of ε such that

$$
(3) \quad \|\theta_X(\omega_\varepsilon)\|_{C^0} \leq \|\theta_X(\omega_0)\|_{C^0} + \|X\phi_\varepsilon\|_{C^0} < C.
$$

Combining (2) and (3), we get $F_X(Y) = 0$. □

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