Toroidal Seifert fibered surgeries on alternating knots

Dedicated to Professor Yoshihiko Marumoto for his 60th birthday

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Abstract: We give a complete classification of toroidal Seifert fibered surgeries on alternating knots. Precisely, we show that if an alternating knot K admits a toroidal Seifert fibered surgery, then K is either the trefoil knot and the surgery slope is zero, or the connected sum of a $(2,p)$-torus knot and a $(2,q)$-torus knot and the surgery slope is $2(p+q)$ with $|p|, |q| \geq 3$.

Key words: Seifert fibered surgery; toroidal surgery; alternating knot.

1. Introduction. The hyperbolic Dehn surgery theorem, due to Thurston [18, Theorem 5.8.2], states that all but finitely many Dehn surgeries on a hyperbolic knot yield hyperbolic manifolds. Here a knot is called *hyperbolic* if its complement admits a complete hyperbolic structure of finite volume. In view of this, a Dehn surgery on a hyperbolic knot yielding a non-hyperbolic manifold is called *exceptional*. As a consequence of the Geometrization Conjecture, raised by Thurston [19, Section 6, question 1], and established by celebrated Perelman's works [14–16], exceptional surgeries are classified into *Seifert fibered surgeries*, *toroidal surgeries* or *reducible surgeries*. We refer the reader to [1] for a survey.

Here we note that the classification is not exclusive, for there exist Seifert fibered 3-manifolds which are toroidal or reducible. However, a hyperbolic knot in the 3-sphere S^3 is conjectured to admit no reducible surgery. This is the Cabling Conjecture [4] which is well known but still open. Thus, we consider in this paper a Dehn surgery on a knot in S^3 yielding a 3-manifold which is toroidal and Seifert fibered, called a *toroidal Seifert fibered surgery*.

It was shown that there exist infinitely many hyperbolic knots in S^3 each of which admits a toroidal Seifert fibered surgery by Eudave–Muñoz [3, Proposition 4.5 (1) and (3)], and Gordon and Luecke [5] independently. On the other hand, Motegi [11] studied toroidal Seifert fibered surgeries on symmetric knots, and gave several restrictions on the existence of such surgeries. In particular, he showed that just the trefoil knot admits a toroidal Seifert fibered surgery among two-bridge knots [11, Corollary 1.6]. Furthermore the authors showed that if a Montesinos knot admits a toroidal Seifert fibered surgery, then the knot is the trefoil knot and the surgery slope is zero [6].

In this paper, we show the following.

Theorem 1. *If an alternating knot K admits a toroidal Seifert fibered surgery, then K is either the trefoil knot and the surgery slope is zero, or the connected sum of a $(2,p)$-torus knot and a $(2,q)$-torus knot and the surgery slope is $2(p+q)$. Here p and q are odd integers with $|p|, |q| \geq 3$.*

We note that Theorem 1 for hyperbolic alternating knots also follows from a complete classification of exceptional surgeries on hyperbolic alternating knots recently achieved by the first author and Masai [7]. While the classification is established by heavy computer-aided calculations, the proof given in this paper is quite simpler and direct.

2. Proof. We start with recalling definitions and basic facts.

A knot in the 3-sphere S^3 is called *alternating* if it admits a diagram with alternately arranged over-crossings and under-crossings running along it. Menasco [9] showed that an alternating knot is hyperbolic unless it is the connected sum of them or a $(2,p)$-torus knot.

Let K be a knot in S^3 and $E(K)$ the exterior of K. A *slope* on the boundary torus $\partial E(K)$ is an isotopy class of a non-trivial simple closed curve on

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$\partial E(K)$. For a slope γ on $\partial E(K)$, we denote by $K(\gamma)$ the 3-manifold obtained by Dehn surgery on K along the slope γ, i.e., $K(\gamma)$ is obtained by gluing a solid torus V to $E(K)$ so that a simple closed curve representing γ bounds a meridian disk in V. We call such a slope γ the *surgery slope*. It is well known that a slope on $\partial E(K)$ is parameterized by an element of $\mathbf{Q} \cup \{1/0\}$ by using the standard meridian-longitude system for K. Thus, when a slope γ corresponds to $r \in \mathbf{Q} \cup \{1/0\}$, we call the Dehn surgery along γ the *r-surgery* for brevity, and denote $K(\gamma)$ by $K(r)$. See [17] for basic references.

Proof of Theorem 1. Now we start the proof of Theorem 1 which will be achieved by the following two claims.

Claim 1. If a prime alternating knot K admits a toroidal Seifert fibered surgery, then K is the trefoil knot and the surgery slope is zero.

Proof. Let K be a prime alternating knot such that $K(r)$ is a toroidal Seifert fibered 3-manifold. Then K is either a two-bridge knot or an alternating pretzel knot of length three, see [2, Lemma 3.1], [13, p13].

If K is a two-bridge knot, then K must be the trefoil knot and $r = 0$ [11, Corollary 1.6].

Assume that K is an alternating pretzel knot of length three $P(a, b, c)$ which is not a two-bridge knot. Then K is a small knot [12]. Therefore K must be fibered and $r = 0$ [8, Proposition 1]. If the integers a, b, and c are odd, then $P(a, b, c)$ is a genus one knot. This contradicts to the assumption that K is not a two-bridge knot since the genus one fibered knots are just the trefoil knot and the figure-eight knot. If one of the integers a, b, and c is even and the others are odd, then the surgery slope r is a boundary slope of a non-orientable surface with crosscap number two [2], [13]. Without loss of generality, we may assume that a is even and b, c are odd. Then we have $r = 2(b + c)$. Since K is alternating, the sign of b coincides with that of c. Therefore we have $r = 2(b + c) \neq 0$. This contradicts the condition $r = 0$. □

Claim 2. If a composite alternating knot K admits a toroidal Seifert fibered surgery, then K is the connected sum of a $(2, p)$-torus knot and a $(2, q)$-torus knot, and the surgery slope is $2(p + q)$. Here p and q are integers with $|p|, |q| \geq 3$.

Proof. According to the classification of non-simple Seifert fibered surgeries on non-hyperbolic knots [10, Theorem 1.2], a composite alternating knot admitting a toroidal Seifert fibered surgery is just the connected sum of a $(2, p)$-torus knot and a $(2, q)$-torus knot, and the surgery slope is $2(p + q)$ with $|p|, |q| \geq 3$. □

By Claims 1 and 2, the proof of Theorem 1 has completed. □

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