Fundamental groups of join-type curves —achievements and perspectives

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Abstract: The aim of this survey article is to bring together recent advances concerning the fundamental groups of join-type curves. Though the paper is of purely expository nature, we do also announce a new result.

Key words: Join-type curves; fundamental groups.

1. Introduction. Fundamental groups of plane curve complements play an important role in the study of branched coverings. They may also be useful to distinguish the connected components of equisingular moduli spaces. The systematic study of these groups goes back to the 1930s with the founding works of O. Zariski and E. R. van Kampen. See [9] and [6]. These papers provide a powerful method to find a presentation of the fundamental group of (the complement of) any algebraic curve C in $\mathbf{CP}^2$. The generators are loops, in a generic line L, around the intersection points of L with C. To find the relations, one considers a generic pencil containing L and one identifies each generator with its transforms by monodromy around the *special* lines of this pencil. (By 'special' line, we mean a line that is tangent to the curve or that crosses a singularity.) In short, $\pi_1(\mathbf{CP}^2 \setminus C)$ is the quotient of $\pi_1(L \cap (\mathbf{CP}^2 \setminus C))$ by the monodromy relations. Note that, in practice, it may be extremely difficult to find the monodromy relations, especially when the special lines of the pencil are not over the real numbers.

The fundamental groups of curves of degree ≤ 5 are well known, and there is an abundant literature dealing with curves of degree 6. In higher degrees, the systematic study of the group $\pi_1(\mathbf{CP}^2 \setminus C)$ is not an easy task. Among the pioneer and most remarkable results, we should certainly mention the famous theorem of O. Zariski [9], W. Fulton [5] and P. Deligne [1]: *if C is a curve having only nodes as singularities, then the fundamental group $\pi_1(\mathbf{CP}^2 \setminus C)$ is abelian.* We should also quote the following theorem due to M. V. Nori [7]: *if C is an irreducible curve of degree d having only nodes and cusps as singularities (say, n nodes and c cusps) such that $2n + 6c < d^2$, then $\pi_1(\mathbf{CP}^2 \setminus C)$ is abelian.* (Note that when the inequality is not satisfied, the group $\pi_1(\mathbf{CP}^2 \setminus C)$ may be non-abelian, as shown by the famous Zariski's three-cuspidal quartic [9].)

In the series of papers [2–4,8], the authors investigated another general family of curves called *join-type* curves. These curves are defined as follows:

Definition 1.1. Let $\nu_1, \dots, \nu_\ell, \lambda_1, \dots, \lambda_m$ be positive integers with $\sum_{j=1}^{\ell} \nu_j = \sum_{i=1}^{m} \lambda_i$. A curve C in $\mathbf{CP}^2$ is called a *join-type* curve with *exponents* $(\nu_1, \dots, \nu_\ell; \lambda_1, \dots, \lambda_m)$ if it is defined by an equation of the form

$$a \cdot \prod_{j=1}^{\ell} (Y - \beta_j Z)^{\nu_j} = b \cdot \prod_{i=1}^{m} (X - \alpha_i Z)^{\lambda_i},$$

where $a, b \in \mathbf{C} \setminus \{0\}$, and $\beta_1, \dots, \beta_\ell$ (respectively, $\alpha_1, \dots, \alpha_m$) are mutually distinct complex numbers. (Here, X, Y, Z are homogeneous coordinates in $\mathbf{CP}^2$.)

In the chart $\mathbf{C}^2 := \mathbf{CP}^2 \setminus \{Z = 0\}$, with coordinates $x = X/Z$ and $y = Y/Z$, the curve C is defined by the equation $f(y) = g(x)$, where

$$f(y) := a \cdot \prod_{j=1}^{\ell} (y - \beta_j)^{\nu_j} \quad \text{and}$$

$$g(x) := b \cdot \prod_{i=1}^{m} (x - \alpha_i)^{\lambda_i}.$$

The aim of the present paper is to give a short survey and discuss the future perspectives concerning the fundamental groups of these curves. Though

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the paper is of purely expository nature, we do also announce a new result (Theorem 8.3 and Corollary 8.4 — details of the proof will be given in [4]).

2. Singular points of join-type curves. The singular points of a join-type curve C (i.e., the points (x,y) satisfying $f(y)=g(x)$ and $f'(y)=g'(x)=0$) divide into two categories: the points (x,y) which also satisfy $f(y)=g(x)=0$, and those for which $f(y)\neq 0$ and $g(x)\neq 0$. Clearly, the singular points contained in the intersection of lines $f(y)=g(x)=0$ are the points (α_i,β_j) with $\lambda_i,\nu_j\geq 2$. Hereafter, such singular points will be called *inner* singularities, while the singular points (x,y) with $f(y)\neq 0$ and $g(x)\neq 0$ will be called *outer* or *exceptional* singularities. It is easy to see that the singular points of a join-type curve are Brieskorn–Pham singularities $\mathbf{B}_{\nu,\lambda}$ (normal form $y^\nu - x^\lambda$). For instance, inner singularities are of type $\mathbf{B}_{\nu_j,\lambda_i}$. Clearly, for generic values of a and b, under any fixed choice of $\alpha_1,\ldots,\alpha_m,\beta_1,\ldots,\beta_\ell$, the curve C has only inner singularities. Hereafter, we shall say that C is *generic* if it has only inner singularities.

We say that C is an $\mathbf{R}$-join-type curve if a, b, α_i $(1\leq i\leq m)$ and β_j $(1\leq j\leq \ell)$ are *real* numbers. It is easy to see that exceptional singularities of $\mathbf{R}$-join-type curves can be only *node* singularities (i.e., Brieskorn–Pham singularities of type $\mathbf{B}_{2,2}$). Note that a *generic* curve C with *non-real* coefficients can always be deformed to an $\mathbf{R}$-join-type curve C_1 by a deformation $\{C_t\}_{0\leq t\leq 1}$ such that $C_0=C$ and C_t is generic with the same exponents as C. (In particular, the topological type of C_t (respectively, of $\mathbf{CP}^2\setminus C_t$) is independent of t.) In general, this is no longer true for curves having exceptional singularities.

3. The groups $G(p;q)$ and $G(p;q;r)$. Let p,q,r be positive integers. In this section, we recall the definitions of the groups $G(p;q)$ and $G(p;q;r)$ introduced in [8] and which appear as the fundamental groups of the curves studied in this paper.

The group $G(p;q)$ is defined by the presentation

$$\begin{aligned}(1)\qquad &\langle \omega,\ a_k\ (k\in\mathbf{Z})\mid \omega=a_{p-1}a_{p-2}\ldots a_0,\\ &\quad a_{k+q}=a_k,\ a_{k+p}=\omega a_k\omega^{-1}\ (k\in\mathbf{Z})\,\rangle.\end{aligned}$$

It is abelian if and only if $q=1$ or $p=1$ or $p=q=2$. More precisely,

$$G(p;q)\simeq\begin{cases}\mathbf{Z} & \text{if } q=1 \text{ or } p=1;\\ \mathbf{Z}^2 & \text{if } p=q=2.\end{cases}$$

From a purely algebraic point of view, it is not obvious that the groups $G(p;q)$ and $G(q;p)$ are isomorphic. However, this is an immediate corollary of Theorem 4.1 below.

The group $G(p;q;r)$ is defined to be the quotient of $G(p;q)$ by the normal subgroup generated by ω^r. It is abelian if and only if one of the following conditions is satisfied:

(a) $\gcd(p,q)=\gcd(q,r)=1$;
(b) $p=1$;
(c) $\gcd(p,q)=2$, $\gcd(q/2,r)=1$ and $p=2$.

More precisely,

$$G(p;q;r)\simeq\begin{cases}\mathbf{Z}_{pr} & \text{if (a) holds;}\\ \mathbf{Z}_r & \text{if (b) holds;}\\ \mathbf{Z}\times\mathbf{Z}_r & \text{if (c) holds.}\end{cases}$$

As a consequence of Corollary 4.2 below, if k is any integer divisible by both p and q, then we have $G(p;q;k/p)\simeq G(q;p;k/q)$.

4. Generic curves. We use the same notation as in Section 1. Furthermore, we shall denote by ν_0 (respectively, by λ_0) the greatest common divisor of $\nu_1,\ldots,\nu_\ell$ (respectively, of $\lambda_1,\ldots,\lambda_m$).

The fundamental groups of generic join-type curves are given as follows:

Theorem 4.1 (cf. [8]). *Suppose that C is a generic join-type curve. Then, the fundamental group $\pi_1(\mathbf{C}^2\setminus C)$ is isomorphic to $G(\nu_0;\lambda_0)$.*

Corollary 4.2 (cf. [8]). *With the same hypotheses as in Theorem 4.1, the group $\pi_1(\mathbf{CP}^2\setminus C)$ is isomorphic to $G(\nu_0;\lambda_0;d/\nu_0)$, where d is the degree of the curve.*

Example 4.3. If C is generic and if λ_0 or ν_0 is equal to 1, then $\pi_1(\mathbf{C}^2\setminus C)\simeq\mathbf{Z}$ and $\pi_1(\mathbf{CP}^2\setminus C)\simeq\mathbf{Z}_d$.

For non-generic join-type curves (i.e., when exceptional singularities do occur), the classification of the fundamental groups is far from being complete, even for curves having only real coefficients. However, recently, some progress have been achieved. So far, if we restrict ourselves to the class of $\mathbf{R}$-join-type curves, then all the groups are known up to degree 7 (cf. Sections 5 and 6). (We recall that in degrees ≤ 5 all the fundamental groups are well known regardless of the nature of the curve.) If, furthermore, we restrict our study to so-called 'semi-generic' curves (see Definition 8.1 below), then, regardless of the degree, as in Theorem 4.1 and Corollary 4.2, the fundamental groups $\pi_1(\mathbf{C}^2\setminus C)$ and $\pi_1(\mathbf{CP}^2\setminus C)$ are given by the groups $G(\nu_0;\lambda_0)$ and $G(\nu_0;\lambda_0;d/\nu_0)$ respectively (cf. Theorem 8.3, Corollary 8.4 and the comment after Corollary 8.4).

5. R-join-type sextics. Let us start with the fundamental groups of **R**-join-type curves of degree 6 for which the classification is complete.

If $C=\bigcup_{i=1}^r C_i$ is the irreducible decomposition of C, then the r-ple $\mathscr{T}:=\{\deg(C_1),\ldots,\deg(C_r)\}$ is called the *component type* of C. (Here, $\deg(C_i)$ is the degree of C_i.) The next theorem gives the fundamental groups of **R**-join-type sextics according to their component types.

Theorem 5.1 (cf. [2]). *Suppose that C is an **R**-join-type sextic with component type $\mathscr{T}$. Again, let $\nu_0:=\gcd(\nu_1,\ldots,\nu_\ell)$ and $\lambda_0:=\gcd(\lambda_1,\ldots,\lambda_m)$.*

(a) *If $\lambda_0=1$ or $\nu_0=1$, then $\mathscr{T}$ is one of the sets $\{6\}$, $\{5,1\}$, $\{4,2\}$, $\{3,3\}$, $\{4,1,1\}$, $\{3,2,1\}$, $\{2,2,2\}$ or $\{2,2,1,1\}$; as for the fundamental group, we have:*

$$\pi_1(\mathbf{CP}^2\setminus C)\simeq\begin{cases}\mathbf{Z}_6 & \text{if } \mathscr{T}=\{6\},\\ \mathbf{Z} & \text{if } \mathscr{T}=\{5,1\},\\ \mathbf{Z}\times\mathbf{Z}_2 & \text{if } \mathscr{T}=\{4,2\},\\ \mathbf{Z}\times\mathbf{Z}_3 & \text{if } \mathscr{T}=\{3,3\},\\ \mathbf{Z}^2 & \text{if } \mathscr{T}=\{4,1,1\},\\ \mathbf{Z}^2 & \text{if } \mathscr{T}=\{3,2,1\},\\ \mathbf{Z}^2\times\mathbf{Z}_2 & \text{if } \mathscr{T}=\{2,2,2\},\\ \mathbf{Z}^3 & \text{if } \mathscr{T}=\{2,2,1,1\}.\end{cases}$$

(b) *If $2\le\lambda_0,\nu_0<6$ and $\gcd(\lambda_0,\nu_0)=1$, then C is irreducible (i.e., $\mathscr{T}=\{6\}$) and*

$$\pi_1(\mathbf{CP}^2\setminus C)\simeq\mathbf{Z}_2*\mathbf{Z}_3,$$

where $\mathbf{Z}_2*\mathbf{Z}_3$ is the free product of $\mathbf{Z}_2$ and $\mathbf{Z}_3$.

(c) *If $2\le\lambda_0,\nu_0<6$ and $\gcd(\lambda_0,\nu_0)=2$, then $\mathscr{T}$ is one of the sets $\{3,3\}$, $\{3,2,1\}$ or $\{2,2,1,1\}$, and*

$$\pi_1(\mathbf{CP}^2\setminus C)\simeq\begin{cases}\mathbf{Z}\times\mathbf{Z}_3 & \text{if } \mathscr{T}=\{3,3\},\\ \mathbf{Z}^2 & \text{if } \mathscr{T}=\{3,2,1\},\\ \mathbf{Z}^3 & \text{if } \mathscr{T}=\{2,2,1,1\}.\end{cases}$$

(d) *If $2\le\lambda_0,\nu_0<6$ and $\gcd(\lambda_0,\nu_0)=3$, then $\mathscr{T}$ is either $\{2,2,2\}$ or $\{2,2,1,1\}$, and*

$$\pi_1(\mathbf{CP}^2\setminus C)\simeq\begin{cases}\mathbf{Z}_2\times\mathbf{F}(2) & \text{if } \mathscr{T}=\{2,2,2\},\\ \mathbf{Z}\times\mathbf{F}(2) & \text{if } \mathscr{T}=\{2,2,1,1\},\end{cases}$$

where $\mathbf{F}(2)$ is a free group of rank 2.

(e) *Finally, if $\lambda_0=6$ or $\nu_0=6$, then*

$$\pi_1(\mathbf{CP}^2\setminus C)\simeq G(\nu_0;\lambda_0;6/\nu_0).$$

In the latter case, exceptional singularities do not occur, and therefore the fundamental group is also given by Corollary 4.2.

Remark 5.2. Note that weak Zariski pairs made of curves with the same exponents (but with different component type) may occur. A weak Zariski pair means a pair of curves with the same degree, the same singularities but not the same embedded topology. In [2], we found an example of two **R**-join-type sextics, C and C', both with 11 nodes and with exponents $(2,2,2;2,2,2)$, such that $\pi_1(\mathbf{CP}^2\setminus C)\simeq\mathbf{Z}\times\mathbf{Z}_3$ and $\pi_1(\mathbf{CP}^2\setminus C')\simeq\mathbf{Z}^2$.

6. R-join-type septics. In degree 7, the classification of the fundamental groups of **R**-join-type curves is also completed.

Theorem 6.1 (cf. [3]). *Suppose C is an **R**-join-type septic, and let $\mathscr{E}:=(\nu_1,\ldots,\nu_\ell;\lambda_1,\ldots,\lambda_m)$ be its set of exponents.* (*In degree 7, we always have $\lambda_0=1$ or $\nu_0=1$ except when $\mathscr{E}$ is the set $(7;7)$.*)

(a) *If $\mathscr{E}$ is not the set $(7;7)$, then the fundamental group $\pi_1(\mathbf{CP}^2\setminus C)$ is abelian. When, in addition, $\mathscr{E}$ is neither the set $(2,2,2,1;2,2,2,1)$ nor the set $(1,\ldots,1;1,\ldots,1)$, the group $\pi_1(\mathbf{CP}^2\setminus C)$ is isomorphic to $\mathbf{Z}_7$ or $\mathbf{Z}$ depending on whether the curve is irreducible or has two irreducible components. When $\mathscr{E}$ is the set $(2,2,2,1;2,2,2,1)$ or the set $(1,\ldots,1;1,\ldots,1)$, the group $\pi_1(\mathbf{CP}^2\setminus C)$ is isomorphic to $\mathbf{Z}_7$, $\mathbf{Z}$ or $\mathbf{Z}^3$ depending on whether the curve has one, two or four irreducible components.*

(b) *If $\mathscr{E}=(7;7)$, then $\pi_1(\mathbf{CP}^2\setminus C)$ is non-abelian, isomorphic to the free group of rank 6.*

Actually, when $\mathscr{E}$ is the set $(2,2,2,1;2,2,2,1)$ or the set $(1,\ldots,1;1,\ldots,1)$, the curve C has only node singularities. The number n of nodes is ≤ 15 except in two special cases where it is equal to 18. Whenever $n\le 15$, the group $\pi_1(\mathbf{CP}^2\setminus C)$ is isomorphic to $\mathbf{Z}_7$ or $\mathbf{Z}$. When $n=18$, $\pi_1(\mathbf{CP}^2\setminus C)$ is isomorphic to $\mathbf{Z}^3$.

When $\mathscr{E}=(7;7)$, exceptional singularities do not occur, and therefore the fundamental group is also given by Corollary 4.2. Actually, in this special case, we can also observe that the curve is a union of seven concurrent lines, and hence its complement is $\mathbf{C}\times(\mathbf{C}\setminus\{6\text{ points}\})$. By Corollary 4.2, when $\mathscr{E}$ is not the set $(7;7)$ and the curve C does not have any exceptional singularity, the group $\pi_1(\mathbf{CP}^2\setminus C)$ is always isomorphic to $\mathbf{Z}_7$. (In particular, this is the case when $\mathscr{E}$ is of the form $(7;\lambda_1,\ldots,\lambda_m)$ with $m\ge 2$.)

The next two sections concern our recent work on 'semi-generic' join-type curves. Theorem 8.3 and Corollary 8.4 are new. (We shall not give the proofs here; details will be published in [4].) In fact, these two results can be stated in the more general setting of 'generalized' join-type curves. We introduce this new class of curves in the next section.

7. Generalized join-type curves. Let again $\nu_1, \dots, \nu_\ell, \lambda_1, \dots, \lambda_m$ be positive integers, $\nu_0 := \gcd(\nu_1, \dots, \nu_\ell)$ and $\lambda_0 := \gcd(\lambda_1, \dots, \lambda_m)$. Set $n := \sum_{j=1}^{\ell} \nu_j$ and $n' := \sum_{i=1}^{m} \lambda_i$. Here, we *no longer* assume $n = n'$. A curve C is called a *generalized* join-type curve with exponents $(\nu_1, \dots, \nu_\ell; \lambda_1, \dots, \lambda_m)$ if it is defined by an equation of the form

$$a \cdot Z^{d-n} \cdot \prod_{j=1}^{\ell} (Y - \beta_j Z)^{\nu_j} = b \cdot Z^{d-n'} \cdot \prod_{i=1}^{m} (X - \alpha_i Z)^{\lambda_i},$$

where, as above, $a, b \in \mathbf{C} \setminus \{0\}$, $\beta_1, \dots, \beta_\ell$ (respectively, $\alpha_1, \dots, \alpha_m$) are mutually distinct complex numbers, and $d := \max\{n, n'\}$. (When $n = n'$, this definition coincides with Definition 1.1.) If, furthermore, all these coefficients are real, then we say that C is a generalized $\mathbf{R}$-join-type curve. In the chart $\mathbf{C}^2 := \mathbf{CP}^2 \setminus \{Z = 0\}$, the curve C is given by the equation $f(y) = g(x)$, where

$$f(y) := a \cdot \prod_{j=1}^{\ell} (y - \beta_j)^{\nu_j} \text{ and } g(x) := b \cdot \prod_{i=1}^{m} (x - \alpha_i)^{\lambda_i}.$$

8. Semi-generic curves. In this section, we assume that C is a generalized $\mathbf{R}$-join-type curve. Without loss of generality, we can assume that the real numbers α_i $(1 \le i \le m)$ and β_j $(1 \le j \le \ell)$ satisfy the inequalities $\alpha_1 < \dots < \alpha_m$ and $\beta_1 < \dots < \beta_\ell$. By considering the restriction of the function $g(x)$ to the real numbers, it is easy to see that the equation $g'(x) = 0$ has at least one real root γ_i in the open interval (α_i, α_{i+1}) for each $i = 1, \dots, m-1$. Since the degree of

$$g'(x) \Big/ \prod_{i=1}^{m} (x - \alpha_i)^{\lambda_i - 1}$$

is $m - 1$, it follows that the roots of $g'(x) = 0$ are exactly $\gamma_1, \dots, \gamma_{m-1}$ and the α_i's with $\lambda_i \ge 2$. In particular, this shows that $\gamma_1, \dots, \gamma_{m-1}$ are *simple* roots of $g'(x) = 0$. Similarly, the equation $f'(y) = 0$ has $\ell - 1$ simple roots $\delta_1, \dots, \delta_{\ell-1}$ such that $\beta_j < \delta_j < \beta_{j+1}$ for $1 \le j \le \ell - 1$. Of course, the β_j's with $\nu_j \ge 2$ are also roots of $f'(y) = 0$. (They are simple for $\nu_j = 2$.)

Definition 8.1. Suppose that C is a generalized $\mathbf{R}$-join-type curve.

(a) We say that C is *generic* if, for any $1 \le i \le m - 1$, $g(\gamma_i)$ is a regular value for f (i.e., $g(\gamma_i) \ne f(\delta_j)$ for any $1 \le j \le \ell - 1$). Of course, this is equivalent to the condition that, for any $1 \le j \le \ell - 1$, $f(\delta_j)$ is a regular value for g. When $n = n'$, this definition of the genericity coincides with the one given in Section 2.

(b) We say that C is *semi-generic with respect to* g if there exists an integer i_0 $(1 \le i_0 \le m)$ such that $g(\gamma_{i_0-1})$ and $g(\gamma_{i_0})$ are regular values for f. (When $i_0 = 1$, the condition for $g(\gamma_{i_0-1})$ is empty; when $i_0 = m$, the condition for $g(\gamma_{i_0})$ is empty.) The semi-genericity *with respect to* f is defined similarly by exchanging the roles of f and g.

Observe that a generic curve is always semi-generic with respect to both g and f, while the converse is not true. Also, note that C can be semi-generic with respect to g without being semi-generic with respect to f.

The proof of Theorem 4.1 above, which is given in [8], also works in the case $n \ne n'$. More precisely, we have the following theorem.

Theorem 8.2 (cf. [8]). *If C is a generalized generic join-type curve, then $\pi_1(\mathbf{C}^2 \setminus C) \simeq G(\nu_0; \lambda_0)$.*

The following two results extend Theorems 8.2 (and 4.1) and Corollary 4.2 to semi-generic generalized $\mathbf{R}$-join-type curves. (Details of the proof will be given in [4].)

Theorem 8.3. *Let C be a generalized $\mathbf{R}$-join-type curve. If C is semi-generic with respect to g, then $\pi_1(\mathbf{C}^2 \setminus C)$ is isomorphic to $G(\nu_0; \lambda_0)$.*

Corollary 8.4. *With the same hypotheses as in Theorem 8.3, we have:*

$$\pi_1(\mathbf{CP}^2 \setminus C) \simeq \begin{cases} G(\nu_0; \lambda_0; n/\nu_0) & \text{if } n \ge n', \\ G(\lambda_0; \nu_0; n'/\lambda_0) & \text{if } n' \ge n. \end{cases}$$

In particular, as announced at the end of Section 4, if C is an 'ordinary' $\mathbf{R}$-join-type curve (i.e., $n = n'$) which is semi-generic with respect to g, then

$$\pi_1(\mathbf{CP}^2 \setminus C) \simeq G(\nu_0; \lambda_0; d/\nu_0) \simeq G(\lambda_0; \nu_0; d/\lambda_0),$$

where d is the degree of the curve (cf. Section 3).

Note that the conclusions of Theorem 8.3 and Corollary 8.4 are still valid if we suppose that C is semi-generic with respect to f.

Example 8.5. With the same hypotheses as in Theorem 8.3, if n is a prime number and $\ell \geq 2$, then $\nu_0 = 1$, and hence $\pi_1(\mathbf{C}^2 \setminus C)$ is isomorphic to $G(1; \lambda_0) \simeq \mathbf{Z}$ while $\pi_1(\mathbf{CP}^2 \setminus C)$ is isomorphic to $\mathbf{Z}_n$ or $\mathbf{Z}_{n'}$ depending on whether $n \geq n'$ or $n' \geq n$. (Of course, if n' is a prime number and $m \geq 2$, then $\lambda_0 = 1$, and we get the same conclusions.)

Example 8.6. Consider the generalized **R**-join-type curve C defined by the (affine) equation $f(y) = g(x)$, where $f(y) = c(y+1)^2 y^3 (y-1)$, with $c > 0$, and $g(x) = (x+2)^2 x^4 (x-2)$. Clearly, f has four critical points $\beta_1 = -1 < \delta_1 < \beta_2 = 0 < \delta_2$. The function g also has four critical points $\alpha_1 = -2 < \gamma_1 < \alpha_2 = 0 < \gamma_2$. We choose the coefficient c so that $f(\delta_2) = g(\gamma_2)$. Then, the curve C is not generic. However, as $g(\gamma_1)$ is a regular value for f, the curve is semi-generic with respect to g. Then, by Theorem 8.3 and Corollary 8.4, we have $\pi_1(\mathbf{C}^2 \setminus C) \simeq G(1;1) \simeq \mathbf{Z}$ and $\pi_1(\mathbf{CP}^2 \setminus C) \simeq G(1;1;7) \simeq \mathbf{Z}_7$.

9. Future perspectives. We conclude this paper with a conjecture that extends to generalized **R**-join-type curves a former conjecture made in [2]. (The statement given in [2] (which includes join-type curves with non-real coefficients) is incorrect. It should be replaced by the statement given here.)

Conjecture 9.1. Let C and C' be two generalized **R**-join-type curves with the same set of exponents $(\nu_1, \ldots, \nu_\ell; \lambda_1, \ldots, \lambda_m)$ and the same component type (see Section 5 for the definition). We suppose that at least one of these two curves is generic. Then, we have the isomorphisms:

- $\pi_1(\mathbf{C}^2 \setminus C) \simeq \pi_1(\mathbf{C}^2 \setminus C') \simeq G(\nu_0; \lambda_0)$;
- $\pi_1(\mathbf{CP}^2 \setminus C) \simeq \begin{cases} G(\nu_0; \lambda_0; n/\nu_0) & \text{if } n \geq n', \\ G(\lambda_0; \nu_0; n'/\lambda_0) & \text{if } n' \geq n. \end{cases}$

The conjecture is true for (ordinary) **R**-join-type curves of degree 6 or 7 as well as for generalized semi-generic **R**-join-type curves.

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