A matrix equation on triangulated Riemann surfaces

By Daisuke YAMAKI

Department of Mathematics, Tokyo Institute of Technology,
Oh-okayama, Meguro-ku, Tokyo 152-8551, Japan

(Communicated by Masaki KASHIWARA, M.J.A., Jan. 14, 2014)

Abstract: In [1], Wilson defined holomorphic 1-cochains and combinatrial period matrices of triangulated Riemann surfaces by using the combinatorial Hodge star operator, introduced in [2]. In this paper, we define a matrix and call this matrix the associate matrix. Then, we prove that among the three matrices, which are a period matrix, a combinatorial period matrix which is introduced by Wilson [2] and an associate matrix, there exists a matrix equation. Then we also show that an associate matrix is an element of the Siegel upper half space, so this means that a trianguted Riemann surface gives three elements of the Siegel upper half space.

Key words: Triangulated Riemann surface; combinatorial Hodge theory; associate matrix.

1. Introduction. In [2], Wilson defined the combinatorial Hodge star operator to define holomorphic 1-cochains of a triangulated Riemann surface whose simplicial 1-cochains are equipped with a non-degenerate inner product. In [1], Wilson studied the periods of holomorphic 1-cochains and defined the combinatorial period matrix. Using a particularly nice inner product which is called the Whitney inner product and written by $\langle\cdot,\cdot\rangle_C$, introduced in [8], Wilson proved that for a triangulated Riemann surface, the combinatorial period matrix converges to the (conformal) period matrix as the mesh of the triangulation tends to zero.

In this paper, we will introduce a new matrix and call this matrix the associate matrix. Then we will show that among three matrices which are a (conformal) period matrix, a combinatorial period matrix and an associate matrix, there exists an equation which is the main result. From this matrix equation, we will show that an associate matrix is an element of the Siegel upper half space as well as a period matrix and a combinatorial period matrix. Thus a triangulated Riemann surface gives three elements of the Siegel upper half space.

In this paper, we define triangulated Riemann surfaces as follow. Let M be a closed Riemann surface of genus g, $\{a,b\} := \{a_1,\cdots,a_g,b_1,\cdots,b_g\}$ a homology basis which satisfies the single relation $a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1} = 1$ and K a triangulation of M. Then, we define a triangulated Riemann surface by a triple $(M,\{a,b\},K)$. By the Riemann's bi-linear relations of forms (see [3]), we obtain the canonical basis $\{\theta_1,\cdots,\theta_g\}$ for holomorphic 1-forms which satisfies $\int_{a_j}\theta_j = 1$ and $\int_{a_k}\theta_j = 0$ for $j \neq k$. The canonical basis $\{\theta_1,\cdots,\theta_g\}$ of holomorphic 1-forms gives the (conformal) period matrix Π: $\Pi = (\pi_{jk})_{1\le j,k\le g}$, where $\pi_{jk} = \int_{b_k}\theta_j$.

In [1], Wilson defined combinatorial period matrices by using the Whitney embedding W of cochains into piecewise-linear differential forms, introduced in [8]. By the Riemann's bi-linear relation of cochains, showed in [1], we obtain the canonical basis $\{\sigma_1,\cdots,\sigma_g\}$ for holomorphic 1-cochains which satisfies $\int_{a_j}W\sigma_j = 1$ and $\int_{a_k}W\sigma_j = 0$ for $j \neq k$. The canonical basis $\{\sigma_1,\cdots,\sigma_g\}$ of holomorphic 1-cochains gives the combinatorial period matrix Π_K: $\Pi_K = (\pi_{jk}^K)_{1\le j,k\le g}$, where $\pi_{jk}^K = \int_{b_k}W\sigma_j$.

Then, we will introduce the following matrix by using the canonical basis $\{\theta_1,\cdots,\theta_g\}$ for holomorphic 1-forms and the canonical basis $\{\sigma_1,\cdots,\sigma_g\}$ for holomorphic 1-cochains. We define a matrix Λ_K by

$$\Lambda_K = (\langle W\sigma_j, \star\theta_k\rangle_\Omega)_{1\le j,k\le g},$$

and call this matrix Λ_K the associate matrix.

We will prove the following result:

Theorem 4.2. *Let $(M,\{a,b\},K)$ be a triangulated Riemann surface with the period matrix Π, the combinatorial period matrix Π_K and the associate matrix Λ_K. Then, the following matrix equation holds:*

2010 Mathematics Subject Classification. Primary 30F99.

doi: 10.3792/pjaa.90.37

$$\Pi = \overline{\Pi_K} - \overline{\Lambda_K}.$$

2. Period matrices of closed Riemann surfaces. In this section, we review the construction of period matrices of closed Riemann surfaces.

Let M be a closed Riemann surface. The Hodge star operator $\star$ on the complex valued 1-forms of M may be defined in local coordinates $(U, x+iy)$ by $\star dx = dy$ and $\star dy = -dx$ and extended over $\mathbf{C}$ linearly. This is well defined using the Cauchy-Riemann equations for the coordinate interchanges. The Hodge star operator restricts to an orthogonal automorphism of complex valued 1-forms that squares to $-Id$.

2.1. Holomorphic forms on closed Riemann surfaces. Let $\Omega^j(M)$ denote the set of smooth complex valued differential forms of degree j on M. Then, we define an inner product on $\Omega(M) = \oplus_{j\in\{0,1,2\}}\Omega^j(M)$ as follows:

Definition 2.1. For $\omega, \eta \in \Omega(M)$, we define an inner product $\langle\cdot,\cdot\rangle_\Omega$ on $\Omega(M)$ by

$$\langle\omega,\eta\rangle_\Omega = \int_M \omega\wedge\star\overline{\eta}.$$

Definition 2.2. The adjoint operator of an exterior derivative d, denoted by d^*, is defined by $\langle d^*\omega,\eta\rangle_\Omega = \langle\omega, d\eta\rangle_\Omega$.

These operators give rise to harmonic forms:

Definition 2.3. The space $\mathcal{H}\Omega^j(M)$ of harmonic j-forms on M is defined to be

$$\mathcal{H}\Omega^j(M) = \left\{\omega\in\Omega^j(M) | d\omega = d^*\omega = 0\right\}.$$

The following theorem holds (see [6]):

Theorem 2.4 ([6]). *There is an orthogonal direct sum decomposition*

$$\Omega^j(M) = d\Omega^{j-1}(M)\oplus\mathcal{H}\Omega^j(M)\oplus d^*\Omega^{j+1}(M).$$

The harmonic 1-forms split into an orthogonal sum of holomorophic and anti-holomorphic 1-forms corresponding to the $-i$ and $+i$ eigenspaces of the Hodge star operator.

Definition 2.5. The space of holomorphic 1-forms on M is defined to be

$$\mathcal{H}\Omega^{1,0}(M) = \left\{\omega\in\mathcal{H}\Omega^1(M) | \star\omega = -i\omega\right\},$$

and the space of anti-holomorphic 1-forms on M is defined to be

$$\mathcal{H}\Omega^{0,1}(M) = \left\{\omega\in\mathcal{H}\Omega^1(M) | \star\omega = +i\omega\right\}.$$

Theorem 2.6 ([3]). *The following hold:*

(1) *There exists the following orthogonal decomposition:*

$$\mathcal{H}\Omega^1(M) = \mathcal{H}\Omega^{1,0}(M)\oplus\mathcal{H}\Omega^{0,1}(M),$$

(2) $\dim\mathcal{H}\Omega^{1,0}(M) = \dim\mathcal{H}\Omega^{0,1}(M) = g$,

(3) *Complex conjugation maps $\mathcal{H}\Omega^{1,0}(M)$ to $\mathcal{H}\Omega^{0,1}(M)$ and vise versa.*

2.2. Period matrices. For a closed Riemann surface M of genus g, we choose a point $p\in M$ and denote by $\pi_1(M,p)$ the fundamental group formed by the homotopy classes of closed curves from p.

The group can be generated by $2g$ generators $a_1, b_1, \cdots, a_g, b_g$ which satisfy the single relation $a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1} = 1$. Any such ordered system of generators is called a canonical homology basis.

Given M and $\{a,b\} = \{a_1,\cdots,a_g,b_1,\cdots,b_g\}$ there exists a unique holomorphic 1-form θ_j with period 1 along a_k and periods 0 along all $a_m, m\neq k$. The period of θ_j along b_j is denoted by π_{jk}. These numbers are elements of the period matrix Π associated with M and $\{a,b\}$.

Definition 2.7. For $\omega\in\mathcal{H}\Omega^1(M)$, we define the A-periods $\omega(a_j)$ and B-periods $\omega(b_j)$ by

$$\omega(a_j) := \int_{a_j}\omega,\quad \omega(b_j) := \int_{b_j}\omega,$$

where $1\le j\le g$.

Riemann showed that for any fixed canonical homology basis these periods satisfy the so-called Riemann's bi-linear relations. See [3] for detail.

Lemma 2.8 ([3]). *For ω_1, $\omega_2\in\mathcal{H}\Omega^1(M)$, the following holds:*

$$\langle\star\omega_1,\omega_2\rangle_\Omega = \sum_{j=1}^g(\omega_1(a_j)\overline{\omega_2(b_j)} - \omega_1(b_j)\overline{\omega_2(a_j)}).$$

Theorem 2.9 ([3] [Riemann's bi-linear relation of forms]). *For $\omega_1,\omega_2\in\mathcal{H}\Omega^{1,0}(M)$, the following holds:*

$$\sum_{j=1}^g(\omega_1(a_j)\omega_2(b_j) - \omega_1(b_j)\omega_2(a_j)) = 0.$$

The Riemann's bi-linear relations yields the following properties:

Corollary 2.10 ([3]). *Let ω be a holomorphic 1-form.*

(1) *If all the $\omega(a_j)$ or all the $\omega(b_j)$ vanish, then $\omega = 0$.*

(2) *If all the $\omega(a_j)$ and all the $\omega(b_j)$ are real, then $\omega = 0$.*

By Corollary 2.10 (1), a basis $\{\theta_1, \cdots, \theta_g\}$ is uniquely determined by a pair $(M, \{a, b\})$. We call this basis the canonical basis for holomorphic 1-forms.

Definition 2.11. Let M be a closed Riemann surface of genus g and $\{a, b\}$ a canonical homology basis. Let $\{\theta_1, \cdots, \theta_g\}$ be the canonical basis for holomorphic 1-forms, so $\theta_j(a_k) = \delta_{jk}$. We define the period matrix $\Pi = (\pi_{jk})_{1\leq j,k\leq g}$ to be the $(g \times g)$ matrix of B-periods:

$$\pi_{jk} := \theta_k(b_j).$$

Note that period matrices lie in the Siegel upper half space. See [3] for detail.

3. Combinatorial Hodge theory on triangulated Riemann surfaces. Let K be a C^∞ triangulation of M whose ordering of the vertices are fixed. Also, we assume that all simplices in K have nice shape. This means that the shapes do not too thin. Let $C^j(K)$ the set of the simplicial cochains of degree j of K with values in $\mathbf{C}$. We denote the i-th vertex of K by p_i. Since M is compact, we can identify the cochains and chains of K. For $c \in C^j(K)$ write $c = \sum_\tau c_\tau \cdot \tau$ where $c_\tau \in \mathbf{C}$ and the sum is over all j-simplices τ of K. We write $\tau = [p_{i_0}, p_{i_1}, \cdots, p_{i_j}]$ of K with the vertices in an increasing sequence with respect to the ordering of vertices in K. We assume our triangulation K is a subdivision of the cellular decomposition given by the canonical homology basis. Each element of the canonical basis is represented as a sum of 1-simplices in K (see [1]).

Definition 3.1. Under the above settings, we call a triple $(M, \{a, b\}, K)$ a triangulated Riemann surface.

Definition 3.2. We define the mesh η of K by

$$\eta(K) = \sup r(p, q),$$

where r means the geodesic distance in M and the supremum is taken over all pairs of vertices p, q of a 1-simplex in K.

3.1. Whitney forms. Now we review some definitions and results of Whitney forms. Given the ordering of the vertices of K, we have a coboundary operator $\delta : C^j \to C^{j+1}$.

Let μ_i define the barycentric coordinate corresponding to the i-th vartex p_i of K as follows:

Definition 3.3. The barycentric coordinates μ_i corresponding to each p_i are defined by the following properties: for each simplex τ in K,
(1) $\mu_i : \tau \to [0, 1]$,
(2) $\sum_i \mu_i(p) = 1$,
(3) $p = \sum_i \mu_i(p)p_i$ for $p \in \tau$.

We write $c = \sum_\tau c_\tau \cdot \tau$ where $c_\tau \in \mathbf{C}$ and the sum is over all j-simplices τ of K. We now define the Whitney embedding of cochains into piecewise-linear differential forms.

Definition 3.4. For τ as above, we define

$$W\tau = j! \sum_{k=0}^{j} (-1)^k \mu_k d\mu_0 \wedge \cdots \wedge d\mu_{k-1} \wedge d\mu_{k+1} \wedge \cdots \wedge d\mu_j.$$

W is defined on all of C^j by extending linearly.

Note that the coordinates μ_k are not even of class C^1, but they are C^∞ on the interior of any n-simplex of K. Hence, $d\mu_k$ is defined and $W\tau$ is well defined. By the same consideration, dW is also well defined, where d denotes exterior derivative.

Several properties of the map W are given below.

Proposition 3.5 ([5]). *The following hold:*
(1) $W\tau = 0$ *on* $M\backslash\overline{St(\tau)}$,
(2) $dW = W\delta$,
where St *denotes the open star and* $\overline{St(\tau)}$ *is the closure of* $St(\tau)$.

Now we define a particularly nice inner product on $C(K) = \oplus_j C^j(K)$:

Definition 3.6. An inner product $\langle\cdot,\cdot\rangle_C$ on $C(K)$ is defined as follows:

$$\langle\sigma, \tau\rangle_C = \langle W\sigma, W\tau\rangle_\Omega \quad for \quad \sigma, \tau \in C(K).$$

This inner product $\langle\cdot,\cdot\rangle_C$ is called the Whitney inner product.

3.2. Holomorphic cochains. Now suppose that $C(K)$ are equipped with the Whitney inner product. Then one can define further structures on the cochains.

Definition 3.7. The adjoint of δ, denoted by δ^*, is defined by $\langle\delta^*\sigma, \tau\rangle_C = \langle\sigma, \delta\tau\rangle_C$.

These operators δ, δ^* give rise to harmonic cochains:

Definition 3.8. Harmonic j-cochains of K are defined to be

$$\mathcal{H}C^1(K) = \{\sigma \in C^j(K) | \delta\sigma = \delta^*\sigma = 0\}.$$

The following theorem is due to Eckmann [7]:

Theorem 3.9 ([7]). *For* $C(K)$ *equipped with the Whitney inner product, there is an orthogonal direct sum decomposition*

$$C^j(K) = \delta C^{j-1}(K) \oplus \mathcal{H}C^j(K) \oplus \delta^* C^{j+1}(K).$$

Next, we recall the definition of the combinatorial Hodge star operator $\bigstar$, defined by Wilson [2]:

Definition 3.10. For $\sigma \in C^j(K)$, we define $\bigstar\sigma \in C^{2-j}(K)$ by

$$\langle \bigstar\sigma, \tau \rangle_C = \int_M W\sigma \wedge \overline{W\tau} \quad \textit{for } \tau \in C^{2-j}(K).$$

By $\bigstar$, a harmonic 1-cochains can be splited into two part. The one is called the holomorphic 1-cochain and the another is called anti-holomorphic 1-cochains (see [1]).

Definition 3.11. The space of holomorophic 1-cochains $\mathcal{H}C^{1,0}(K)$ is defined by the span of the eigenvectors for non-positive imaginary eigenvalues of $\bigstar$ and the space of anti-holomorphic 1-cochains $\mathcal{H}C^{0,1}(K)$ is defined by the span of the eigenvectors for non-negative imaginary eigenvalues of $\bigstar$.

The following theorem is due to [1]:

Theorem 3.12 ([1]). *The following hold:*

(1) *There exists the following orthogonal decomposition:*

$$\mathcal{H}C^1(K) = \mathcal{H}C^{1,0}(K) \oplus \mathcal{H}C^{0,1}(K),$$

(2) $\dim \mathcal{H}C^{1,0}(K) = \dim \mathcal{H}C^{0,1}(K) = g,$

(3) *Complex conjugation maps* $\mathcal{H}C^{1,0}(K)$ *to* $\mathcal{H}C^{0,1}(K)$ *and vise verse.*

3.3. Combinatorial period matrices. Now we review the construction of combinatorial period matrices, introduced by Wilson [1].

In [1], Wilson showed that there exists the Riemann's bi-linear relation of cochains and one can take the canonical basis for holomorphic 1-cochains, which is uniquely determined by a triangulated Riemann surface. A combinatrial period matrix is defined by the canonical basis for holomorphic 1-cochains of a triangulated Riemann surface. Wilson also showed that the combinatrial period matrix of a triangulated Riemann surface converges to the period matrix, as the mesh of the triangulation tends to zero.

Definition 3.13. For $\sigma \in \mathcal{H}C^1(K)$, we define the combinatorial A-periods $\sigma(a_j)$ and combinatorial B-periods $\sigma(b_j)$ by

$$\sigma(a_j) := \int_{a_j} W\sigma, \quad \sigma(b_j) := \int_{b_j} W\sigma,$$

where $1 \leq j \leq g$.

Wilson [1] showed that the Riemann's bi-linear relation holds for harmonic 1-cochains with respect to the Whitney inner product:

Theorem 3.14 ([1] [Riemann's bi-linear relations of cochains]). *For* $\sigma_1, \ \sigma_2 \in \mathcal{H}C^{1,0}(K)$, *the following holds:*

$$\sum_{j=1}^{g} (\sigma_1(a_j)\sigma_2(b_j) - \sigma_1(b_j)\sigma_2(a_j)) = 0.$$

One can define the canonical basis for holomorphic 1-cochains (see [1]):

Definition 3.15. For a triangulated Riemann surface $(M, \{a, b\}, K)$ of genus g, the canonical basis $\{\sigma_1, \cdots \sigma_g\}$ for holomorphic 1-cochains is defined as follows: $\sigma_j(a_k) = \delta_{jk}$.

By Riemann's bi-linear relation of cochains, the canonical basis for holomorphic 1-cochains is uniquely determined by a triangulated Riemann surface. See [1] for detail.

Next, we define combinatorial period matrices introduced in [1]:

Definition 3.16. Let $(M, \{a, b\}, K)$ be a triangulated Riemann surface of genus g, and let $\{\sigma_1, \cdots, \sigma_g\}$ be the canonical basis for holomorphic 1-cochains of $(M, \{a, b\}, K)$. The combinatorial period matrix Π_K is defined by

$$\Pi_K := (\pi_{ij}^K)_{1 \leq i,j \leq g} \quad \textit{where} \quad \pi_{ij}^K := \sigma_i(b_j).$$

A combinatorial period matrix satisfies some following properties.

Theorem 3.17 ([1]). *Let* Π_K *be the combinatorial period matrix of a triangulated Riemann surface* $(M, \{a, b\}, K)$. *Then,* Π_K *is an element of the Siegel upper half space.*

Wilson showed that if complex valued simplicial cochains of a triangulated Riemann surface are equipped with the Whitney inner product, the all of these structures provide a good approximation to the analogues (see [4,5]). In particular, the holomorphic and anti-holomorphic 1-cochains converge to the holomorphic and anti-holomorphic 1-forms, and the combinatorial period matrix converges to the period matrix of the associated Riemann surface, as the mesh of the triangulation tends to zero. Hence, a period matrix is a limit point of a sequence of combinatorial period matrices.

Theorem 3.18 ([1]). *Let* M *be a closed Riemann surface and* K *a triangulation of* M. *We set a sequence* $\{K_n\}_{n \in \mathbf{N}}$ *of subdivisions of* K *which satisfies*

$$\lim_{n\to\infty} \eta(K_n) = 0,$$

where $\eta(K_n)$ is the mesh of K_n.
Then,

$$\lim_{n\to\infty} \Pi_{K_n} = \Pi,$$

where Π is the period matrix and Π_{K_n} is the combinatorial period matrix of a triangulated Riemann surface $(M, \{a,b\}, K_n)$.

4. Main results. For a triangulated Riemann surface, we checked the definitions of period matrices and the combinatorial period matrices. In general, it is unclear whether or not the two matrices coincide.

In this section, we will introduce a new matrix which is uniquely determined by a triangulated Riemann surface as well as a period matrix and a combinatorial period matrix and call this new matrix the associate matrix of a triangulated Riemann surface. Then we will show that a matrix equation is holding among the period matrix, the combinatorial period matrix and the associate matrix of a triangulated Riemann surface.

Definition 4.1. Let $(M, \{a,b\}, K)$ be a triangulated Riemann surface of genus g. Let $\{\theta_1, \cdots, \theta_g\}$ be the canonical basis for holomorphic 1-forms and $\{\sigma_1, \cdots, \sigma_g\}$ the canonical basis for holomorphic 1-cochains of $(M, \{a,b\}, K)$. We define the associate matrix Λ_K of $(M, \{a,b\}, K)$ by

$$\Lambda_K := (\langle W\sigma_j, \star\theta_k\rangle_\Omega)_{1\le j,k\le g}.$$

Note that the associate matrix is well defined by choosing any Riemannian metric in the conformal class of the Riemann surface. Since $\{\theta_1, \cdots, \theta_g\}$ and $\{\sigma_1, \cdots, \sigma_g\}$ are uniquely determined by a triangulated Riemann surface $(M, \{a,b\}, K)$, so Λ_K is uniquely determined.

Next theorem is the main result of this paper:

Theorem 4.2. *Let $(M, \{a,b\}, K)$ be a triangulated Riemann surface with the period matrix Π, the combinatorial period matrix Π_K and the associate matrix Λ_K. Then, the following matrix equation holds:*

$$\Pi = \overline{\Pi_K} - \overline{\Lambda_K}.$$

Proof. Set

$$\widetilde{C}_K := \frac{1}{2i}(\Pi - \Pi_K)(\operatorname{Im}\Pi)^{-1},$$

and

$$C_K := E - \widetilde{C}_K,$$

where E is the $(g \times g)$ identity matrix.
Note that since $\operatorname{Im}\Pi$ is positive definite, there exists $(\operatorname{Im}\Pi)^{-1}$.
We compute

$$\begin{aligned}\Pi_K &= \Pi - 2i\widetilde{C}_K \operatorname{Im}\Pi \\ &= (C_K + \widetilde{C}_K)\Pi - 2i\widetilde{C}_K \operatorname{Im}\Pi \\ &= C_K\Pi + \widetilde{C}_K\overline{\Pi}.\end{aligned}$$

Let c_{jk} be the (j,k)-entry of C_K and $\widetilde{c}_{jk}$ the (j,k)-entry of $\widetilde{C}_K$.
Then, for each j, k, we have

$$\int_{b_k} W\sigma_j = \sum_{m=1}^{g} c_{jm}\int_{b_k}\theta_m + \sum_{m=1}^{g}\widetilde{c}_{jm}\int_{b_k}\overline{\theta_m},$$

and

$$\int_{b_k}\left(W\sigma_j - \sum_{m=1}^{g} c_{jm}\theta_m - \sum_{m=1}^{g}\widetilde{c}_{jm}\overline{\theta_m}\right) = 0.$$

On the other hand, we compute

$$\begin{aligned}&\int_{a_k}\left(W\sigma_j - \sum_{m=1}^{g} c_{jm}\theta_m - \sum_{m=1}^{g}\widetilde{c}_{jm}\overline{\theta_m}\right) \\ &= \int_{a_k} W\sigma_j - \sum_{m=1}^{g} c_{jm}\int_{a_k}\theta_m - \sum_{m=1}^{g}\widetilde{c}_{jm}\int_{a_k}\overline{\theta_m} \\ &= \delta_{jk} - \sum_{m=1}^{g}(c_{jm} + \widetilde{c}_{jm})\delta_{km} \\ &= \delta_{jk} - \sum_{m=1}^{g}\delta_{jm}\delta_{km} \\ &= 0.\end{aligned}$$

Namely, all A-periods and B-periods of $W\sigma_j - \sum_{m=1}^{g} c_{jm}\theta_m - \sum_{m=1}^{g}\widetilde{c}_{jm}\overline{\theta_m}$ are zero. By Proposition 3.5 and $\delta\sigma_j = 0$, we have $dW\sigma_j = W\delta\sigma_j = 0$ on the interior τ^i of any n-simplex τ in K, where $n = 1, 2$. This implies that $W\sigma_j - \sum_{m=1}^{g} c_{jm}\theta_m - \sum_{m=1}^{g}\widetilde{c}_{jm}\overline{\theta_m}$ is closed on τ^i. By de Rham's theorem, the closed form $W\sigma_j - \sum_{m=1}^{g} c_{jm}\theta_m - \sum_{m=1}^{g}\widetilde{c}_{jm}\overline{\theta_m}$ is exact: there exists df_j such that

$$W\sigma_j - \sum_{m=1}^{g} c_{jm}\theta_m - \sum_{m=1}^{g}\widetilde{c}_{jm}\overline{\theta_m} = df_j,$$

on M$\backslash\{p \in M | p : \textit{vertex in } K\}$.
Since $\{p \in M | p : \textit{vertex in } K\}$ is a null set, we have

$$\langle df_j, \theta_k\rangle_\Omega = \left\langle W\sigma_j - \sum_{m=1}^{g} c_{jm}\theta_m - \sum_{m=1}^{g}\widetilde{c}_{jm}\overline{\theta_m}, \theta_k\right\rangle_\Omega$$

$$= \langle W\sigma_j, \theta_k\rangle_\Omega - \sum_{m=1}^{g} c_{jm}\langle \theta_m, \theta_k\rangle_\Omega - \sum_{m=1}^{g} \widetilde{c}_{jm}\langle \overline{\theta_j}, \theta_k\rangle_\Omega.$$

By Theorem 2.6 and $d^*\theta_k = 0$, we obtain $\langle \overline{\theta_m}, \theta_k\rangle_\Omega = 0$ and $\langle \theta_m, df_j\rangle_\Omega = \langle d^*\theta_m, f_j\rangle_\Omega = 0$. So, we have

$$\langle W\sigma_j, \theta_k\rangle_\Omega = \sum_{m=1}^{g} c_{jm}\langle \theta_m, \theta_k\rangle_\Omega.$$

By Riemann's bi-linear relation of forms (Lemma 2.8), we obtain

$$\begin{aligned}\langle \theta_m, \theta_k\rangle_\Omega &= i\langle -i\theta_m, \theta_k\rangle_\Omega \\ &= i\langle \star\theta_m, \theta_k\rangle_\Omega \\ &= i\sum_{s=1}^{g}\left(\int_{a_s}\theta_m \int_{b_s}\overline{\theta_k} - \int_{b_s}\theta_m\int_{a_s}\overline{\theta_k}\right) \\ &= i(\overline{\pi_{km}} - \pi_{mk}) \\ &= i(\overline{\pi_{mk}} - \pi_{mk}) \\ &= 2\operatorname{Im}\pi_{mk}.\end{aligned}$$

Note that since the period matrix Π is lie in the Siegel upper half space, the period matrix Π is symmetric and we obtain $\pi_{km} = \pi_{mk}$. Thus, we have

$$\begin{aligned}\Lambda_K &= (\langle W\sigma_j, \star\theta_k\rangle_\Omega)_{1\le j,k\le g} \\ &= i(\langle W\sigma_j, \theta_k\rangle_\Omega)_{1\le j,k\le g} \\ &= i\left(2\sum_{m=1}^{g} c_{jm}\operatorname{Im}\pi_{mk}\right)_{1\le j,k\le g} \\ &= 2i(c_{jm})_{1\le j,m\le g}(\operatorname{Im}\pi_{mk})_{1\le m,k\le g} \\ &= 2iC_K\operatorname{Im}\Pi.\end{aligned}$$

So we conclude

$$\begin{aligned}\Pi_K &= \Pi - 2i\widetilde{C}_K\operatorname{Im}\Pi \\ &= \Pi - 2i(E - C_K)\operatorname{Im}\Pi \\ &= \Pi - 2i\operatorname{Im}\Pi + 2iC_K\operatorname{Im}\Pi \\ &= \overline{\Pi} + \Lambda_K.\end{aligned}$$

□

By Theorem 4.2, we see that an associate matrix is an element of the Siegel upper half space. This implies that a triangulated Riemann surface gives three elements of the Siegel upper half space.

Corollary 4.3. *Let $(M, \{a, b\}, K)$ be a triangulated Riemann surface, and let Π be the period matrix, Π_K the combinatorial period matrix and Λ_K the associate matrix of $(M, \{a, b\}, K)$. Then, Λ_K is an element of the Siegel upper half space. Also Λ_K is not equal to Π nor Π_K.*

Proof. By Theorem 4.2, we have

$$\begin{aligned}\Lambda_K &= \Pi_K - \overline{\Pi} \\ &= (\operatorname{Re}\Pi_K - \operatorname{Re}\Pi) + i(\operatorname{Im}\Pi_K + \operatorname{Im}\Pi).\end{aligned}$$

For any $x \in \mathbf{R}^g$, we see that

$$\begin{aligned}{}^t x(\operatorname{Im}\Lambda_K)x &= {}^t x(\operatorname{Im}\Pi_K + \operatorname{Im}\Pi)x \\ &= {}^t x(\operatorname{Im}\Pi_K)x + {}^t x(\operatorname{Im}\Pi)x > 0.\end{aligned}$$

This implies that Λ_K is symmetric and $\operatorname{Im}\Lambda_K$ is positive definite, so Λ_K is an element of the Siegel upper half space.

Next, we assume that Λ_K is equal to Π. Then, by Theorem 4.2, we have

$$\Pi_K = \overline{\Pi} + \Lambda_K = \overline{\Pi} + \Pi = 2\operatorname{Re}\Pi.$$

This is a contradiction, because the imaginary part of a combinatorial period matrix is not equal to zero matrix. In a similar way, one can check that Λ_K is not equal to Π_K as well. □

Acknowledgments. The author would like to thank Profs. Hiroshige Shiga and Masaharu Tanabe for many comments, suggestions and support on my work. The author also thank Dr. Keiji Tagami for his help and comments.

References

[1] S. O. Wilson, Conformal cochains, Trans. Amer. Math. Soc. **360** (2008), no. 10, 5247–5264.
[2] S. O. Wilson, Cochain algebra on manifolds and convergence under refinement, Topology Appl. **154** (2007), no. 9, 1898–1920.
[3] H. M. Farkas and I. Kra, *Riemann surfaces*, 2nd ed., Graduate Texts in Mathematics, 71, Springer, New York, 1992.
[4] J. Dodziuk, Finite-difference approach to the Hodge theory of harmonic forms, Amer. J. Math. **98** (1976), no. 1, 79–104.
[5] J. Dodziuk and V. K. Patodi, Riemannian structures and triangulations of manifolds, J. Indian Math. Soc. (N.S.) **40** (1976), nos. 1–4, 1–52.
[6] M. Spivak, *A comprehensive introduction to differential geometry. Vol. IV*, Publish or Perish, Boston, MA, 1975.
[7] B. Eckmann, Harmonische Funktionen und Randwertaufgaben in einem Komplex, Comment. Math. Helv. **17** (1945), 240–255.
[8] H. Whitney, *Geometric integration theory*, Princeton Univ. Press, Princeton, NJ, 1957.