The structure of Deitmar schemes, I

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Abstract: We explain how one can naturally associate a Deitmar scheme (which is a scheme defined over the field with one element, $\mathbf{F}_1$) to a so-called "loose graph" (which is a generalization of a graph). Several properties of the Deitmar scheme can be proven easily from the combinatorics of the (loose) graph, and it also appears that known realizations of objects over $\mathbf{F}_1$ (such as combinatorial $\mathbf{F}_1$-projective and $\mathbf{F}_1$-affine spaces) exactly depict the loose graph which corresponds to the associated Deitmar scheme. This idea is then conjecturally generalized so as to describe all Deitmar schemes in a similar synthetic manner.

Key words: Field with one element; Deitmar scheme; loose graph; automorphism group.

1. Introduction. One of the earliest realizations of $\mathbf{F}_1$ was through the scheme theory developed by Deitmar [1], which is based on the observation that commutative rings over $\mathbf{F}_1$ could be imagined as commutative multiplicative monoids (with an absorbing element). For these "rings", one can naturally define (prime) ideals, localization, a Spec-construction, etc., and Deitmar has proven that a natural base extension to $\mathbf{Z}$ of varieties in this context leads to toric varieties [2]. Also, there is a promising theory for zeta functions of Deitmar schemes [2] which agrees with Kurokawa's theory for zeta functions of schemes "defined over $\mathbf{F}_1$" [6].

More general scheme theories over $\mathbf{F}_1$ have seen the light of day since Deitmar's (see for instance [7,8] for an account), but in one way or the other, Deitmar schemes always appear to be the core of such schemes. So Deitmar schemes are one of the very basic objects in $\mathbf{F}_1$-theory, and one needs to understand them as well as possible. (We refer to the monograph [12] for background on the object $\mathbf{F}_1$.)

Besides the Algebraic Geometry side of $\mathbf{F}_1$-theory, there is also the combinatorial-synthetic side (cf. [10]): in an old paper [13], Tits already described symmetric groups as Chevalley groups over $\mathbf{F}_1$, seen as limit objects of projective general linear groups over finite fields, and the corresponding geometric modules (buildings), which are just projective spaces over the same fields in this case, become complete graphs with the full subgraph structure. (Tits also describes several other spherical buildings over $\mathbf{F}_1$ in *loc. cit.*; their automorphism groups are the Weyl groups of the thick buildings of the same type; see also [10].)

In this note, we want to glue these two theories together. We will start with a loose graph, which is a more general object than a graph, and we will construct a Deitmar scheme from it of which the closed points correspond to the vertices of the loose graph. Several fundamental properties of the Deitmar scheme can then be obtained easily from the combinatorics of the loose graph, such as connectedness, the automorphism group, etc. And very interestingly, it appears that a number of combinatorial $\mathbf{F}_1$-objects (such as the combinatorial $\mathbf{F}_1$-projective space of above) are just loose graphs, and moreover, the associated Deitmar schemes will precisely be the scheme versions in Deitmar's theory of these objects. As a model example, the projective space scheme $\mathrm{Proj}(\mathbf{F}_1[X_0,\ldots,X_n])$ arises from the complete graph C_{n+1} on $n+1$ vertices, and the latter is precisely how a projective space of dimension n over $\mathbf{F}_1$ should look like (Tits). For further reference, we call a complete graph on $n+1$ vertices (n any cardinal number) with the full complete subgraph structure a *projective space* of dimension n *over* $\mathbf{F}_1$. An edge with its two vertices is a combinatorial projective line over $\mathbf{F}_1$, and more generally, any subset of vertices together with the induced graph structure defines a linear subspace.

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Then if $\mathrm{Aut}_{\mathrm{sch}}(\cdot)$, respectively $\mathrm{Aut}_{\mathrm{synth}}(\cdot)$, denotes the scheme theoretic automorphism group, respectively the combinatorial ("synthetic") automorphism group, and Γ is a loose graph while $S(\Gamma)$ is the corresponding Deitmar scheme, we have

$$\mathrm{Aut}_{\mathrm{sch}}(S(\Gamma)) \cong \mathrm{Aut}_{\mathrm{synth}}(\Gamma). \tag{1}$$

As a corollary, we will obtain that any group can occur as the full automorphism group of some Deitmar scheme.

So we will bring combinatorial $\mathbf{F}_1$-objects and Deitmar schemes together through loose graphs (and a more general construction).

2. Deitmar schemes and examples. Define an $\mathbf{F}_1$*-ring* to be a commutative monoid with an absorbing element 0.

For the definition of Deitmar scheme, we refer to [1] or [11]. Let us just mention that one defines an *affine Deitmar scheme* $\mathrm{Spec}(A)$ similarly as an affine Grothendieck scheme (by building a Zariski-type topology and structure sheaf of $\mathbf{F}_1$-rings on the set of monoidal prime ideals of the commutative monoid with zero A). A general *Deitmar scheme* then is a *monoidal space* (a topological space endowed with a sheaf of $\mathbf{F}_1$-rings), locally isomorphic to affine Deitmar schemes. (Deitmar schemes are sometimes also called $\mathcal{D}_0$*-schemes* or $\mathcal{M}_0$*-schemes* in some papers.)

2.1. Polynomial rings. Define

$$\mathbf{F}_1[X_1,\ldots,X_n] := \{0\} \cup \{X_1^{u_1}\ldots X_n^{u_n} | u_j \in \mathbf{N}\}, \tag{2}$$

that is, the union of $\{0\}$ and the (abelian) monoid generated by the X_j.

2.2. Affine space. Let $A = \mathbf{F}_1[X_1,\ldots,X_n]$. Denote $\mathrm{Spec}(\mathbf{F}_1[X_1,\ldots,X_n])$ by $\mathbf{A}^n_{\mathbf{F}_1}$ and call it the *n-dimensional affine space over* $\mathbf{F}_1$. The $\neq (0)$ prime ideals of A are of the form

$$\mathfrak{p}_I = \bigcup_{i\in I}(X_i), \tag{3}$$

where I is a subset of $\{1,\ldots,n\}$ and $(X_i) = X_iA = \{X_ia \mid a \in A\}$. The stalk of the structure sheaf at $\mathfrak{p}_I$ is the localization of A at the multiplicative set S that contains all products of elements X_j where $j \notin I$.

2.3. Proj-schemes. In [9] we introduced the Proj-scheme construction for Deitmar schemes (see also [11]). We quickly repeat this procedure.

Monoid quotients. Let M be a commutative unital monoid (with 0), and I an ideal of M. We define the monoidal quotient M/I to be the set $\{[m] \in M | m \in M\}/([m] = [0]$ if $m \in I)$.

The Proj-construction. Consider the $\mathbf{F}_1$-ring $\mathbf{F}_1[X_0,X_1,\ldots,X_m]$, where $m \in \mathbf{N}$. Since any polynomial is homogeneous in this ring, we have a natural grading

$$\mathbf{F}_1[X_0,\ldots,X_m] = \bigoplus_{i\geq 0} R_i = \coprod_{i\geq 0} R_i, \tag{4}$$

where R_i consists of the elements of $\mathbf{F}_1[X_0,X_1,\ldots,X_m]$ of total degree i, for $i \in \mathbf{N}$. The *irrelevant ideal* is defined as

$$\mathrm{Irr} = \{0\} \cup \coprod_{i\geq 1} R_i. \tag{5}$$

(It is just the monoid minus the element 1.) Now $\mathrm{Proj}(\mathbf{F}_1[X_0,\ldots,X_m]) =: \mathrm{Proj}(\mathbf{F}_1[\mathbf{X}])$ consists, as a set, of the prime ideals of $\mathbf{F}_1[X_0,X_1,\ldots,X_m]$ which do not contain Irr (so only Irr is left out of the complete set of prime ideals). The closed sets of the (Zariski) topology on this set are defined as usual: for any ideal I of $\mathbf{F}_1[X_0,X_1,\ldots,X_m]$, we define

$$V(I) := \{\mathfrak{p}|\mathfrak{p} \in \mathrm{Proj}(\mathbf{F}_1[\mathbf{X}]), \quad I \subseteq \mathfrak{p}\}, \tag{6}$$

where $V(I) = \emptyset$ if $I = \mathrm{Irr}$ and $V(\{0\}) = \mathrm{Proj}(\mathbf{F}_1(\mathbf{X}))$, the open sets then being of the form

$$D(I) := \{\mathfrak{p}|\mathfrak{p} \in \mathrm{Proj}(\mathbf{F}_1[\mathbf{X}]), \quad I \not\subseteq \mathfrak{p}\}. \tag{7}$$

It is obvious that $\mathrm{Proj}(\mathbf{F}_1[\mathbf{X}])$ is a Deitmar scheme. (The structure sheaf is described below in a more general setting.) Each ideal (X_i) defines an open set $D((X_i))$ such that the restriction of the scheme to this set is isomorphic to $\mathrm{Spec}(\mathbf{F}_1[\mathbf{X}_{(i)}])$, where $\mathbf{X}_{(i)}$ is $X_0,X_1,\ldots,X_m$ with X_i left out.

Remark 2.1. Note that the closed points and projective sublines of $\mathrm{Proj}(\mathbf{F}_1[X_0,\ldots,X_m])$ form a complete graph on $m+1$ vertices, so we can easily switch between combinatorial $\mathbf{F}_1$-projective spaces and $\mathrm{Proj}(\mathbf{F}_1[\mathbf{X}])$-schemes.

In general, suppose M is a commutative unital monoid (with 0) with a grading

$$M = \coprod_{i\geq 0} M_i, \tag{8}$$

where the M_i are the sets with elements of total degree i (for $i \in \mathbf{N}$), and let, as above, the *irrelevant ideal* be $\mathrm{Irr} = \{0\} \cup \coprod_{i\geq 1} M_i$. Define the topology $\mathrm{Proj}(M)$ as before (noting that homogeneous (prime) ideals are the same as ordinary monoidal (prime) ideals here). For an open U, define $\mathcal{O}_M(U)$ as consisting of all functions

$$f : U \longrightarrow \coprod_{\mathfrak{p}\in U} M_{(\mathfrak{p})}, \tag{9}$$

where $M_{(\mathfrak{p})}$ is the subset of $M_\mathfrak{p}$ of fractions of elements with the same degree, for which $f(\mathfrak{p}) \in M_{(\mathfrak{p})}$ for each $\mathfrak{p} \in U$, and such that there exists a neighborhood V of $\mathfrak{p}$ in U, and elements $u, v \in M$, for which $v \notin \mathfrak{q}$ for every $\mathfrak{q} \in V$, and $f(\mathfrak{q}) = \frac{u}{v}$ in $M_{(\mathfrak{q})}$.

In this way we obtain a sheaf of $\mathbf{F}_1$-rings on $\mathrm{Proj}(M)$ making it a Deitmar scheme.

Now recall the following results.

Theorem 2.2 ([1]). *Let A be an $\mathbf{F}_1$-ring.*

(i) *For each $\mathfrak{p} \in \mathrm{Spec}(A)$, the stalk $\mathcal{O}_\mathfrak{p}$ of the structure sheaf is isomorphic to the localization of A at $\mathfrak{p}$.*

(ii) *For global sections, we have $\Gamma(\mathrm{Spec}(A), \mathcal{O}) := \mathcal{O}(\mathrm{Spec}(A)) \cong A$.*

Theorem 2.3 ([1]).

(i) *For an $\mathbf{F}_1$-ring A, we have that $(\mathrm{Spec}(A), \mathcal{O}_A)$ is a monoidal space.*

(ii) *If $\alpha : A \longrightarrow B$ is a morphism of monoids, then α induces a morphism of monoidal spaces*

$$(f, f^{\#}) : \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A), \tag{10}$$

yielding a functorial bijection

$$\mathrm{Hom}(A, B) \cong \mathrm{Hom}_{\mathrm{loc}}(\mathrm{Spec}(B), \mathrm{Spec}(A)), \tag{11}$$

where on the right hand side we only consider local morphisms.

For the sake of convenience, we will make the surjectivity part of Theorem 2.3(ii) explicit.

Theorem 2.4. *Any local morphism of monoidal spaces $(f, f^{\#})$: $\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ is induced by a monoidal morphism $\alpha = \alpha_{(f,f^{\#})}$ as in* Theorem 2.3(ii).

Proof. Let $(f, f^{\#})$ be as in the statement of the theorem; then taking global sections, $f^{\#}$ induces a morphism $\phi : \Gamma(\mathrm{Spec}(A), \mathcal{O}) \longrightarrow \Gamma(\mathrm{Spec}(B), \mathcal{O})$, which by Theorem 2.2 is a morphism $\phi : A \longrightarrow B$. For any $\mathfrak{p} \in \mathrm{Spec}(B)$, we have a local morphism $f^{\#}_\mathfrak{p} : A_{f(\mathfrak{p})} \longrightarrow B_\mathfrak{p}$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & & \downarrow \\ A_{f(\mathfrak{p})} & \xrightarrow{f^{\#}_\mathfrak{p}} & B_\mathfrak{p}. \end{array} \tag{12}$$

As $f^{\#}$ is a local homomorphism, we have that $\phi^{-1}(\mathfrak{p}) = f(\mathfrak{p})$, so that f coincides with the map $\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ induced by ϕ. It follows readily that the monoid homomorphism $\phi = \alpha_{(f,f^{\#})}$ induces $(f, f^{\#})$. □

Since any automorphism is local, we have the following implication.

Corollary 2.5. *If $(f, f^{\#}) \in \mathrm{Aut}(\mathrm{Spec}(A))$ is such that $f = \mathbf{1}$ implies that $\alpha_{(f,f^{\#})} = \mathbf{1}$, then $f^{\#}$ also is trivial.* □

So if the topology of $\mathrm{Spec}(A)$ is sufficiently fine, the only element in $\mathrm{Aut}(\mathrm{Spec}(A))$ with trivial component f, is the trivial one.

3. Loose graphs.

Define a *loose graph* ("L-graph") to be a triple $(V, E, \mathbf{I})$, where V is a set of "vertices", E is a set of "edges" ($V \cap E = \emptyset$), and $\mathbf{I}$ is a symmetric relation on $(V \times E) \cup (E \times V)$ (which indicates when a vertex and an edge are incident), with the additional property that *each edge is incident with at most two distinct vertices.* (In other words, it relaxes the definition of graphs, in that an edge can now also have one, or even no, endpoint(s). Also, since we introduce loose graphs as incidence geometries, we do not allow loops, and the geometry is undirected.)

3.1. Embedding theorem.

Let $\Gamma = (V, E, \mathbf{I})$ be a loose graph. We define a projective space $\mathbf{P}(\Gamma)$ over $\mathbf{F}_1$ as follows. Let $E' \subseteq E$ be the set of "loose edges" — edges with only a single endpoint. On each of these edges, we add a new endpoint, as such creating a point set V' which is in bijective correspondence with E'. Now $\mathbf{P}(\Gamma)$ is the complete graph on the vertex set $V \cup V'$. As such, we have an embedding of geometries

$$\psi : \Gamma \hookrightarrow \mathbf{P}(\Gamma) = \mathbf{P}, \tag{13}$$

where $\mathbf{P}$ is the combinatorial projective space over $\mathbf{F}_1$ of dimension $|V| + |V'| - 1$. If Γ is a graph, then $E' = \emptyset$ and the dimension of $\mathbf{P}$ is $|V| - 1$.

Theorem 3.1. *The following properties clearly hold.*

DIM $\mathbf{P}$ *has minimal dimenson $|V| + |V'| - 1$ with respect to the embedding property (that is, there is no combinatorial projective space over $\mathbf{F}_1$ of smaller dimension in which Γ embeds).*

AUT *Each automorphism of Γ is faithfully induced by an automorphism of $\mathbf{P}$. (So that in particular, only the identity automorphism of $\mathbf{P}$ can fix any element of $E \cup V$.)* □

3.2. Example. Projective completion. Note that if one starts with a combinatorial affine space $\mathbf{A}$ over $\mathbf{F}_1$, considered as a loose graph, $\mathbf{P}(\mathbf{A})$ is precisely the projective completion of $\mathbf{A}$.

4. From loose graphs to Deitmar schemes.

4.1. Patching and the functor Θ. Now let $\Gamma = (V, E, \mathbf{I})$ be a not necessarily finite graph. We will give a "patching" argument as follows.

Consider $\mathbf{P} = \mathbf{P}(\Gamma)$, and note that since Γ is a graph, $\mathbf{P} \setminus \Gamma$ — when $\mathbf{P}$ is considered as a graph — is just a set S of edges. Let μ be arbitrary in S, and let z be one of the two (closed) points on μ in $\mathbf{P} = \mathrm{Proj}(\mathbf{F}_1[X_i]_{i\in V})$ (recall Remark 2.1). Suppose that in the projective space $\mathbf{P}$, z is defined by the ideal generated by the polynomials

$$X_i, \quad i \in V, i \neq j = j(z). \tag{14}$$

Let $\mathbf{P}(z)$ be the complement in $\mathbf{P}$ of z; it is a hyperplane defined by $X_j = 0$ (and it forms a complete graph on all the points but z). Denote the corresponding closed subset of $\mathrm{Proj}(\mathbf{F}_1[X_i]_{i\in V})$ by $C(z)$. Let $z' \neq z$ be the other point of the edge μ corresponding to the index $j' = j(z') \in V$. Define the subset $\mathbf{P}(z') = \mathbf{P} \setminus \{z'\}$ of V, and denote the corresponding closed subset by $C(z')$. Finally, define

$$C(\mu) = C(z) \cup C(z'). \tag{15}$$

It is also closed in $\mathrm{Proj}(\mathbf{F}_1[X_i]_{i\in V})$, and the corresponding closed subscheme is the projective space $\mathbf{P}$ "without the edge μ"; the coordinate ring is $\mathbf{F}_1[X_i]_{i\in V}/I_\mu$ (where $(X_jX_l) =: I_\mu$) and its scheme is the Proj-scheme defined by this ring. Now introduce the closed subset

$$C(\Gamma) = \cap_{\mu\in S} C(\mu). \tag{16}$$

Then $C(\Gamma)$ defines a closed subscheme $S(\Gamma)$ which corresponds to the graph Γ. We have

$$S(\Gamma) = \mathrm{Proj}(\mathbf{F}_1[X_i]_{i\in V}/ \cup_{\mu\in S} I_\mu). \tag{17}$$

In this presentation, an edge corresponds to a relation, and we construct a coordinate ring for $\Theta(\Gamma) = S(\Gamma)$ by deleting all relations of the ambient space $\mathbf{P}(\Gamma)$ which are defined by edges in the complement of Γ. We call a Deitmar scheme $S(\Gamma)$ constructed from a graph Γ a *G-scheme.*

A similar construction can be done for loose graphs, cf. §§4.3.

4.2. Automorphism groups. The next theorem shows that the automorphism group of projective spaces from the incidence geometrical point of view, which we denote by $\mathrm{Aut}_{\mathrm{synth}}(.)$, coincides with the automorphism group from the point of view of $\mathbf{F}_1$-schemes, denoted $\mathrm{Aut}_{\mathrm{sch}}(.)$.

Theorem 4.1. *Let $\mathbf{P}$ be a projective space over $\mathbf{F}_1$, and let $\mathrm{Proj}(\mathbf{F}_1[X_i]_{i\in\mathbf{P}})$ be the corresponding projective scheme. Then we have*

$$\mathrm{Aut}_{\mathrm{synth}}(\mathbf{P}) \cong \mathrm{Aut}_{\mathrm{sch}}(\mathrm{Proj}(\mathbf{F}_1[X_i]_{i\in\mathbf{P}})). \tag{18}$$

Proof. It is clear that any element of $\mathrm{Aut}_{\mathrm{inc}}(\mathbf{P}) \cong \mathrm{Sym}(\mathbf{P})$ induces an automorphism of $\mathrm{Proj}(\mathbf{F}_1[X_i]_{i\in\mathbf{P}})$, by permuting the set $\{X_i | i \in \mathbf{P}\}$. (This is just the action of the symmetric group on the closed points or hyperplanes.) So if $\mathrm{Aut}_{\mathrm{sch}}(\mathrm{Proj}(\mathbf{F}_1[X_i]_{i\in\mathbf{P}}))$ strictly contains $\mathrm{Aut}_{\mathrm{synth}}(\mathbf{P})$, there are elements in $\mathrm{Aut}_{\mathrm{sch}}(\mathrm{Proj}(\mathbf{F}_1[X_i]_{i\in\mathbf{P}}))$ fixing all closed points. Now if some scheme automorphism of $\mathrm{Proj}(\mathbf{F}_1[X_i]_{i\in\mathbf{P}})$ enjoys this feature, then it is easy to see that all prime ideals are also fixed (since prime ideals are of the form $\cup_J RX_i$, with $R = \mathbf{F}_1[X_i]_{i\in\mathbf{P}}$), and hence the topology is fixed elementwise. By Corollary 2.5 this induces a trivial sheaf isomorphism, which is what we wanted to prove. □

In fact, this is only a special case of a more general result which we will encounter later.

A similar proof (considering the action on the ideals that correspond to the "directions" instead of the closed points) leads to the same theorem for affine spaces:

Theorem 4.2. *Let $\mathbf{A}$ be an affine space over $\mathbf{F}_1$, and let $\mathrm{Spec}(\mathbf{F}_1[X_i]_{i\in\mathbf{A}})$ be the corresponding Deitmar scheme. Then we have*

$$\mathrm{Aut}_{\mathrm{synth}}(\mathbf{A}) \cong \mathrm{Aut}_{\mathrm{sch}}(\mathrm{Spec}(\mathbf{F}_1[X_i]_{i\in\mathbf{A}})). \tag{19}$$

□

At the level of $\mathbf{F}_1$ there are no translations, nor exotic scheme automorphisms (each automorphism of an affine Deitmar scheme is linear, i.e., linearly extends to an automorphism of the projective completion, as is the case for its combinatorial version). (The corresponding $\mathbf{F}_1$-ring automorphisms are the ones given by permuting indices — no proper polynomial automorphisms occur.)

The following theorem, using the notation of the introductory paragraph of this section, is easy to obtain. We will denote the category of (undirected, loopless) graphs and natural morphisms by $\mathbf{G}$.

Theorem 4.3. *For any element* $\Gamma \in \mathbf{G}$*, we have that*

$$\mathrm{Aut}(\Gamma)_{\mathrm{synth}} \cong \mathrm{Aut}(S(\Gamma))_{\mathrm{sch}}. \tag{20}$$

Proof. Obviously any graph automorphism of Γ induces naturally a scheme automorphism of $S(\Gamma)$. Suppose $\alpha \in \mathrm{Aut}(S(\Gamma)) \setminus \mathrm{Aut}(\Gamma)$. As any such α induces a homeomorphism of the topology, an automorphism is naturally induced on Γ, so we may as well suppose that $\alpha \equiv \mathbf{1}$ on Γ, so also on the topology of $S(\Gamma)$. Write $S(\Gamma) = \cup_i \mathrm{Spec}(A_i)$, the $\mathrm{Spec}(A_i)$ being affine Deitmar schemes. Then each such scheme is isomorphic to an affine space scheme (by the simple form of the G-schemes), and the theorem easily follows from Theorem 4.2. □

As a corollary, we can show easily that any (finite or infinite) group can occur as the automorphism group of some Deitmar scheme.

Corollary 4.4. *Each group* H *is the full automorphism group of some Deitmar scheme.*

Proof. By combined work of Frucht [4] (for the finite case), de Groot [3] and Izbicki [5] (for the infinite case), any group H is the full automorphism group of some graph. Now apply the previous theorem. □

4.3. Extension to loose graphs. Let $\Gamma = (V, E)$ be a connected loose graph. We distinguish three types:

- type I graphs;
- type II complements of graphs $\Delta \subseteq C$ (where C is some complete graph in which Δ is embedded);
- type III loose graphs not of type I nor II.

If Γ is of type I, we have seen how to associate a closed $\mathcal{D}_0$-subscheme $S(\Gamma)$ of $\mathbf{P}(\Gamma)$ to Γ. If Γ is of type II, then we define the Deitmar scheme $S(\Gamma)$ naturally on the open set of $\mathbf{P}(\Gamma)$ which is the complement of the (closed) point set of the graph Γ^c (the complement of Γ in $\mathbf{P}(\Gamma)$). If Γ is of type III, $S(\Gamma)$ is the Deitmar scheme defined by the intersection of the closed subscheme defined on its graph theoretical completion $\overline{\Gamma} \neq \Gamma$, and the open set which is the complement of the complete graph defined on the vertices of $\overline{\Gamma} \setminus \Gamma$. As such we have:

Proposition 4.5. *Each loose graph* Γ *defines a Deitmar scheme* $S(\Gamma)$. □

We call such a scheme a *loose scheme*.

Denote the category of loose (undirected, loopless) graphs and natural morphisms by **LG**. The following theorem is obtained in a similar way as Theorem 4.3.

Theorem 4.6. *For any element* $\Gamma \in \mathbf{LG}$*, we have that*

$$\mathrm{Aut}(\Gamma)_{\mathrm{synth}} \cong \mathrm{Aut}(S(\Gamma))_{\mathrm{sch}}. \tag{21}$$

□

4.4. Connectedness. Elements of the category of loose schemes have many important properties which can easily be read from the corresponding loose graph — recall for instance Theorem 4.3. Another one is:

Theorem 4.7. *A loose scheme* $S(\Gamma)$ *is connected if and only if the loose graph* Γ *is connected.*

Proof. Suppose $\Gamma = (V, E)$ is not connected. It is clearly sufficient to consider the case where we have two connected components, say Γ_1 and Γ_2. As $\mathbf{P}(\Gamma) = \mathbf{P}(\mathbf{P}(\Gamma_1) \coprod \mathbf{P}(\Gamma_2))$, the fact that there are no relations between generators in different components readily implies that $S(\Gamma)$ is also not connected as a Deitmar scheme.

The converse is similar. □

5. Weighted incidence geometries and Deitmar schemes. It is clear that the functor

$$\mathcal{L} : \mathbf{LG} \longrightarrow \mathbf{D} : \Gamma \longrightarrow S(\Gamma) \tag{22}$$

from the category of loose graphs to the category of Deitmar schemes is not surjective at all. Still, it is possible to adapt the ideas of above to make the functor surjective. We propose to associate a Deitmar scheme to a *weighted incidence geometry* (that is, an incidence geometry coming with a weight function on the point set) in a related way as one does for loose graphs. As such, all Deitmar schemes could be constructed from a combinatorial geometry, and they could be studied through these geometries.

Let $\Gamma = (\mathcal{P}, \mathcal{B}, \mathbf{I})$ be a rank 2 incidence geometry, that is, $\mathcal{P}$ is a set of which the elements are "points", $\mathcal{B}$ consists of "lines" ($\mathcal{P} \cap \mathcal{B} = \emptyset$), and $\mathbf{I}$ is a symmetric relation on $(\mathcal{P} \times \mathcal{B}) \cup (\mathcal{B} \times \mathcal{P})$ called "incidence". Let $\omega : \mathcal{P} \longrightarrow \mathbf{N}^{\times}$ be a *weight function* which assigns a strictly positive integer to each point. Assume now that

- $(\#)_1$ any line has only a finite number of points;
- $(\#)_2$ this number is at least two.

Define a Deitmar scheme as follows. For a line L, let $\mathcal{P}_L$ be the points incident with L. Now define the Deitmar scheme $\mathcal{S}(\Gamma)$ as

$$\mathcal{S}(\Gamma) := \mathrm{Spec}(\mathbf{F}_1[X_u]_{u\in\mathcal{P}}/I_\Gamma), \tag{23}$$

where I_Γ is the ideal

$$I_\Gamma := \bigcup_{L\in\mathcal{B}}\left(\prod_{\ell\in\mathcal{P}_L} X_\ell^{\omega(\ell)}\right), \tag{24}$$

and define $\widehat{\mathcal{S}(\Gamma)}$ as

$$\widehat{\mathcal{S}(\Gamma)} := \mathrm{Proj}(\mathbf{F}_1[X_u]_{u\in\mathcal{P}}/I_\Gamma). \tag{25}$$

So each line of Γ defines an ideal in $\mathbf{F}_1[X_u]_{u\in\mathcal{P}}$. (It is now clear to the reader why we need $(\#)_1$ and $(\#)_2$.)

Remark 5.1. Note that the constructions of §4 and §5 can be adapted to **Z**-schemes.

In a future paper, we will study the functors $\mathcal{S}(\cdot)$ and $\widehat{\mathcal{S}(\cdot)}$ in this setting, and explain the connection with the loose graph functor. If everything works out well, we will then be able to describe and study (e.g.) toric varieties through rank 2 incidence geometries.

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