Dunkl transform of (β,γ)-Dunkl Lipschitz functions

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Abstract: In this paper, we obtain an analog of Younis's Theorem 5.2 in [7] for the Dunkl transform on the real line for functions satisfying the (β,γ)-Dunkl Lipschitz condition in the space $\mathrm{L}^p(\mathbf{R},|x|^{2\alpha+1}dx)$, where $\alpha\geq-\frac{1}{2}$.

Key words: Dunkl operator; Dunkl transform; generalized translation operator.

1. Introduction and preliminaries. Dunkl operators provide an essential tool to extend Fourier analysis on Euclidean spaces. These operators have been introduced in 1989, by C. Dunkl in [2], the Dunkl kernel e_α is used to define the Dunkl transform $\mathcal{F}_\alpha$ which was introduced by C. Dunkl in [3]. The Dunkl transform on the real line, which enjoys properties similar to those of Fourier transform, is generalization of the Fourier transform.

Younis ([7], Theorem 5.2) characterized the set of functions in $\mathrm{L}^2(\mathbf{R})$ satisfying the Dini-Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transforms, namely we have

Theorem 1.1 ([7], Theorem 5.2). *Let $f\in\mathrm{L}^2(\mathbf{R})$. Then the following are equivalents*:

1. $\|f(.+h)-f(.)\|_{\mathrm{L}^2(\mathbf{R})}=O\big(\frac{h^\alpha}{(\log\frac{1}{h})^\beta}\big)$ *as* $h\longrightarrow 0$, $0<\alpha<1$, $\beta>0$,
2. $\int_{|\lambda|\geq r}|\mathcal{F}(f)(\lambda)|^2d\lambda=O(r^{-2\alpha}(\log r)^{-2\beta})$ *as* $r\longrightarrow+\infty$,

where $\mathcal{F}$ stands for the Fourier transform of f.

In this paper, we obtain an analog of Theorem 1.1 for the Dunkl transform on the real line. For this purpose, we use a generalized translation operator.

Let $\mathrm{L}_{p,\alpha}=\mathrm{L}^p(\mathbf{R},|x|^{2\alpha+1}dx)$, where $\alpha\geq-\frac{1}{2}$, denote the L^p space of functions f defined on $\mathbf{R}$ endowed with the following finite norm

$$\|f\|_{p,\alpha}=\left(\frac{1}{2^{\alpha+1}\Gamma(\alpha+1)}\int_{\mathbf{R}}|f(x)|^p|x|^{2\alpha+1}dx\right)^{1/p},$$

where $1<p\leq 2$.

Let $j_\alpha(z)$ denote the normalized Bessel function of the first kind of order α given by

$$j_\alpha(z)=\Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+\alpha+1)}\left(\frac{z}{2}\right)^{2n},\ z\in\mathbf{C},$$

where $\mathbf{C}$ denotes the complex plane. Define the Dunkl kernel e_α by

$$e_\alpha(x)=j_\alpha(x)+ic_\alpha xj_{\alpha+1}(x),$$

where $c_\alpha=(2\alpha+2)^{-1}$, $i=\sqrt{-1}$.

We define a differential-difference Dunkl operator

$$\mathrm{D}_\alpha f(x)=\frac{df}{dx}(x)+\left(\alpha+\frac{1}{2}\right)\frac{f(x)-f(-x)}{x},\quad f\in\mathrm{C}^1(\mathbf{R}).$$

So the function $y=e_\alpha(x)$ satisfies the equation $\mathrm{D}_\alpha y-iy=0$ with the initial condition $y(0)=1$ and it is the unique solution (see [4]). In the limit case with $\alpha=-\frac{1}{2}$ the Dunkl kernel coincides with the usual exponential function e^{ix}.

Lemma 1.2. *For $x\in\mathbf{R}$ the following inequalities are fulfilled*

1. $|e_\alpha(x)|\leq 1$, *and the equality is attained only with* $x=0$,
2. $|1-e_\alpha(x)|\leq 2|x|$,
3. $|1-e_\alpha(x)|\geq c$, *with* $|x|\geq 1$, *where c is a certain constant which depends only on α.*

Proof. See Lemma 2.9 in [1]. □

The Dunkl transform of order α for $f\in\mathrm{L}_{p,\alpha}$ is defined by

$$\mathcal{F}_\alpha(f)(\lambda)=\frac{1}{2^{\alpha+1}\Gamma(\alpha+1)}\int_{\mathbf{R}}f(x)e_\alpha(\lambda x)|x|^{2\alpha+1}dx.$$

For $\alpha=-\frac{1}{2}$, $\mathcal{F}_\alpha$ coincides with the classical Fourier transform.

The inverse Dunkl transform is defined by the formula

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$$f(x) = \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)} \int_{\mathbf{R}} \mathcal{F}_\alpha(f)(\lambda) e_\alpha(-\lambda x)|\lambda|^{2\alpha+1} d\lambda.$$

By Plancherel's theorem and Marcinkiewics interpolation theorem (see [5]) we get for $f \in \mathrm{L}_{p,\alpha}$ with $1 < p \leq 2$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|\mathcal{F}_\alpha(f)\|_{q,\alpha} \leq C_1 \|f\|_{p,\alpha}, \tag{1}$$

where C_1 is a positive constant.

K. Trimèche introduced in [6] the generalized translation operator T_h, on $\mathrm{L}_{p,\alpha}$. For $f \in \mathrm{L}_{p,\alpha}$ we have

$$\mathcal{F}_\alpha(\mathrm{T}_h f)(\lambda) = e_\alpha(\lambda h)\mathcal{F}_\alpha(f)(\lambda). \tag{2}$$

2. Main results. In this section we give the main results of this paper. We need first to define (β, γ)-Dunkl Lipschitz class.

Definition 2.1. A function $f \in \mathrm{L}_{p,\alpha}$ is said to be in the (β, γ)-Dunkl Lipschitz class, denoted by $DLip(p, \beta, \gamma)$; if

$$\|\mathrm{T}_h f(.) - f(.)\|_{p,\alpha} = O\left(\frac{h^\beta}{(\log \frac{1}{h})^\gamma}\right) \text{ as } h \longrightarrow 0, \ \beta, \gamma > 0.$$

Theorem 2.2. *Let $f(x)$ belong to $DLip(p, \beta, \gamma)$. Then*

$$\int_{|\lambda| \geq r} |\mathcal{F}_\alpha(f)(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda = O(r^{-q\beta}(\log r)^{-q\gamma}) \text{ as } r \longrightarrow +\infty,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let $f \in DLip(p, \beta, \gamma)$. Then

$$\|\mathrm{T}_h f(.) - f(.)\|_{p,\alpha} = O\left(\frac{h^\beta}{(\log \frac{1}{h})^\gamma}\right) \text{ as } h \longrightarrow 0.$$

If $|\lambda| \in [\frac{1}{h}, \frac{2}{h}]$ then $|\lambda h| \geq 1$ and (3) of Lemma 1.2 implies that

$$1 \leq \frac{1}{c^q}|1 - e_\alpha(\lambda h)|^q.$$

Therefore

$$\begin{aligned} &\int_{1/h \leq |\lambda| \leq 2/h} |\mathcal{F}_\alpha(f)(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda \\ &\leq \frac{1}{c^q} \int_{1/h \leq |\lambda| \leq 2/h} |1 - e_\alpha(\lambda h)|^q |\mathcal{F}_\alpha(f)(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda \\ &\leq \frac{1}{c^q} \int_{\mathbf{R}} |1 - e_\alpha(\lambda h)|^q |\mathcal{F}_\alpha(f)(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda. \end{aligned}$$

From formulas (1) and (2) we have

$$\int_{\mathbf{R}} |1 - e_\alpha(\lambda h)|^q |\mathcal{F}_\alpha(f)(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda \leq C_1^q \|\mathrm{T}_h f(.) - f(.)\|_{p,\alpha}^q.$$

Then there exists a positive constant K such that

$$\int_{1/h \leq |\lambda| \leq 2/h} |\mathcal{F}_\alpha(f)(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda \leq K \frac{h^{q\beta}}{(\log \frac{1}{h})^{q\gamma}}.$$

We obtain

$$\int_{r \leq |\lambda| \leq 2r} |\mathcal{F}_\alpha(f)(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda \leq K \frac{r^{-q\beta}}{(\log r)^{q\gamma}}.$$

So that

$$\begin{aligned} &\int_{|\lambda| \geq r} |\mathcal{F}_\alpha(f)(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda \\ &= \left[\int_{r \leq |\lambda| \leq 2r} + \int_{2r \leq |\lambda| \leq 4r} + \int_{4r \leq |\lambda| \leq 8r} + \cdots\right] \\ &\quad \times |\mathcal{F}_\alpha(f)(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda \\ &\leq K \frac{r^{-q\beta}}{(\log r)^{q\gamma}} + K \frac{(2r)^{-q\beta}}{(\log 2r)^{q\gamma}} + K \frac{(4r)^{-q\beta}}{(\log 4r)^{q\gamma}} + \cdots \\ &\leq K \frac{r^{-q\beta}}{(\log r)^{q\gamma}} + K \frac{(2r)^{-q\beta}}{(\log r)^{q\gamma}} + K \frac{(4r)^{-q\beta}}{(\log r)^{q\gamma}} + \cdots \\ &\leq K \frac{r^{-q\beta}}{(\log r)^{q\gamma}} [1 + 2^{-q\beta} + (2^{-q\beta})^2 + (2^{-q\beta})^3 + \cdots] \\ &\leq CK \frac{r^{-q\beta}}{(\log r)^{q\gamma}}, \end{aligned}$$

where $C = (1 - 2^{-q\beta})^{-1}$.

Finally, we get

$$\int_{|\lambda| \geq r} |\mathcal{F}_\alpha(f)(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda = O\left(\frac{r^{-q\beta}}{(\log r)^{q\gamma}}\right) \text{ as } r \longrightarrow +\infty.$$

Thus, the proof is finished. □

Definition 2.3. A function $f \in \mathrm{L}_{p,\alpha}$ is said to be in the (p, ψ, γ)-Dini Lipschitz Dunkl, denoted by $DLip(p, \psi, \gamma)$; if

$$\|\mathrm{T}_h f(.) - f(.)\|_{p,\alpha} = O\left(\frac{\psi(h)}{(\log \frac{1}{h})^\gamma}\right) \text{ as } h \longrightarrow 0, \ \gamma > 0,$$

where

1. $\psi(t)$ a continuous increasing function on $[0, \infty)$,
2. $\psi(0) = 0$,
3. $\psi(ts) = \psi(t)\psi(s)$ for all $t, s \in [0, \infty)$.

Theorem 2.4. *Let $f(x)$ belong to $DLip(p,\psi,\gamma)$. Then*

$$\int_{|\lambda|\geq r} |\mathcal{F}_\alpha(f)(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda = O(\psi(r^{-q})(\log r)^{-q\gamma}) \text{ as } r \longrightarrow +\infty,$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Let $f \in DLip(p,\psi,\gamma)$. Then we have

$$\|\mathrm{T}_h f(.) - f(.)\|_{p,\alpha} = O\left(\frac{\psi(h)}{(\log \frac{1}{h})^\gamma}\right) \text{ as } h \longrightarrow 0.$$

If $|\lambda| \in [\frac{1}{h}, \frac{2}{h}]$ then $|\lambda h| \geq 1$ and from (3) of Lemma 1.2, we obtain

$$1 \leq \frac{1}{c^q} |1 - e_\alpha(\lambda h)|^q.$$

Then

$$\begin{aligned}
&\int_{1/h\leq|\lambda|\leq 2/h} |\mathcal{F}_\alpha(f)(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda \\
&\leq \frac{1}{c^q} \int_{1/h\leq|\lambda|\leq 2/h} |1 - e_\alpha(\lambda h)|^q |\mathcal{F}_\alpha(f)(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda \\
&\leq \frac{1}{c^q} \int_{\mathbf{R}} |1 - e_\alpha(\lambda h)|^q |\mathcal{F}_\alpha(f)(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda \\
&\leq K \|\mathrm{T}_h f(.) - f(.)\|_{p,\alpha}^q \\
&= O\left(\frac{(\psi(h))^q}{(\log \frac{1}{h})^{q\gamma}}\right) \\
&= O\left(\frac{\psi(h^q)}{(\log \frac{1}{h})^{q\gamma}}\right),
\end{aligned}$$

where K is a positive constant.

We obtain

$$\int_{r\leq|\lambda|\leq 2r} |\mathcal{F}_\alpha(f)(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda = O\left(\frac{\psi(r^{-q})}{(\log r)^{q\gamma}}\right).$$

Then there exists a positive constant C such that

$$\int_{r\leq|\lambda|\leq 2r} |\mathcal{F}_\alpha(f)(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda \leq C \frac{\psi(r^{-q})}{(\log r)^{q\gamma}}.$$

So that

$$\begin{aligned}
&\int_{|\lambda|\geq r} |\mathcal{F}_\alpha(f)(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda \\
&= \left[\int_{r\leq|\lambda|\leq 2r} + \int_{2r\leq|\lambda|\leq 4r} + \int_{4r\leq|\lambda|\leq 8r} + \cdots\right] \\
&\quad \times |\mathcal{F}_\alpha(f)(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda \\
&\leq C \frac{\psi(r^{-q})}{(\log r)^{q\gamma}} + C \frac{\psi((2r)^{-q})}{(\log 2r)^{q\gamma}} + C \frac{\psi((4r)^{-q})}{(\log 4r)^{q\gamma}} + \cdots \\
&\leq C \frac{\psi(r^{-q})}{(\log r)^{q\gamma}} + C \frac{\psi((2r)^{-q})}{(\log r)^{q\gamma}} + C \frac{\psi((4r)^{-q})}{(\log r)^{q\gamma}} + \cdots \\
&\leq C \frac{\psi(r^{-q})}{(\log r)^{q\gamma}} [1 + \psi(2^{-q}) + \psi^2(2^{-q}) + \psi^3(2^{-q}) + \cdots].
\end{aligned}$$

Therefore

$$\int_{|\lambda|\geq r} |\mathcal{F}_\alpha(f)(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda \leq C_\psi \frac{\psi(r^{-q})}{(\log r)^{q\gamma}},$$

where $C_\psi = C(1-\psi(2^{-q}))^{-1}$ since $\psi(2^{-q}) < 1$.

Finally, we get

$$\int_{|\lambda|\geq r} |\mathcal{F}_\alpha(f)(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda = O\left(\frac{\psi(r^{-q})}{(\log r)^{q\gamma}}\right) \text{ as } r \longrightarrow +\infty.$$

and this ends the proof. □

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