Some notes on the Borel directions of meromorphic functions

By Nan Wu*) and Zu-xing Xuan**)

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Abstract: We prove that the distribution of the Borel radii (indirect Borel points) and that of Borel radii (indirect Borel points) concerning the small functions of a meromorphic function are the same. Furthermore, some equivalent conclusions on the Borel radii (indirect Borel points) of meromorphic functions of order $0 < \rho < \infty$ are established. This is a continuous work of Tsuji [4,5].

Key words: Meromorphic functions; small functions; Borel points.

1. Introduction and results. We suppose that the readers are familiar with the basic notions of value distribution of meromorphic functions such as $n(r, f = a), N(r, f = a), S(r, f)$ and $T(r, f)$ (see [2,4,8]). The singular points of a meromorphic function on the unit circle has been a popular topic in the study of value distribution. Tsuji [4,5] studied the Borel rays of a meromorphic function from two aspects, the first being the rays through the origin directed outward of the unit disk, and the second being the rays through a point on the unit circle $\{z : |z| = 1\}$ directed inward of the unit disk.

Given a sector $\Omega = \{z : \alpha < \arg z < \beta, |z| < R\}$, for a function $f(z)$ meromorphic in Ω, define

$$N(r, \Omega, f = a) = \int_{r_0}^{r} \frac{n(t, \Omega, f = a)}{t}\, dt,$$

where $n(t, \Omega, f = a)$ is the number of the roots of $f(z) = a$ in $\Omega \cap \{r_0 < |z| < t\}$ counted according to the multiplicities, $0 < r_0 < r < R$.

Definition 1.1. Let $f(z)$ be a meromorphic function in the disk $C_R = \{z : |z| < R\} (0 < R \le \infty)$. The order of $f(z)$ is defined as follows:

$$\limsup_{r\to R-} \frac{\log T(r, f)}{\log \frac{R}{R-r}} = \rho.$$

A radius $L(\theta) : \arg z = \theta$ is called a Borel radius of order ρ for $f(z)$, if for any $\varepsilon > 0$,

$$\limsup_{r\to R-} \frac{\log N(r, Z_\varepsilon(\theta), f = a)}{\log \frac{R}{R-r}} = \rho$$

or

$$\limsup_{r\to R-} \frac{\log n(r, Z_\varepsilon(\theta), f = a)}{\log \frac{R}{R-r}} = \rho + 1$$

holds for any complex value a except at most two complex values, where $Z_\varepsilon(\theta) = \{z : \theta - \varepsilon < \arg z < \theta + \varepsilon\}$.

We collect together three characteristic functions:

(I) Ahlfors-Shimuzi characteristic function (see [4]):

$$\mathcal{S}(r, \Omega, f) = \frac{1}{\pi} \iint_{\Omega\cap\{0<|z|<r\}} \frac{|f'(te^{i\theta})|^2}{(1 + |f(te^{i\theta})|)^2}\, t dt d\theta,$$

$$\mathcal{T}(r, \Omega, f) = \int_0^r \frac{\mathcal{S}(t, \Omega, f)}{t}\, dt,$$

where Ω is an angular domain whose vertex is the origin or a point on the unit circle $\{z : |z| = 1\}$.

(II) The Nevanlinna angular characteristic function (see [2,3]):

$$S_{\alpha,\beta}(r, f) = A_{\alpha,\beta}(r, f) + B_{\alpha,\beta}(r, f) + C_{\alpha,\beta}(r, f),$$

where

$$A_{\alpha,\beta}(r, f) = \frac{\lambda}{\pi} \int_{r_0}^{r} \left(\frac{1}{t^\lambda} - \frac{t^\lambda}{r^{2\lambda}}\right) \{\log^+ |f(te^{i\alpha})| + \log^+ |f(te^{i\beta})|\} \frac{dt}{t},$$

$$B_{\alpha,\beta}(r, f) = \frac{2\lambda}{\pi r^\lambda} \int_\alpha^\beta \log^+ |f(re^{i\theta})| \sin \lambda(\theta - \alpha) d\theta,$$

$$C_{\alpha,\beta}(r, f) = 2 \sum_{r_0<|b_n|<r} \left(\frac{1}{|b_n|^\lambda} - \frac{|b_n|^\lambda}{r^{2\lambda}}\right) \sin \lambda(\theta_n - \alpha),$$

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*) Department of Mathematics, School of Science, China University of Mining and Technology (Beijing), Beijing, 100083, People's Republic of China.

**) Beijing Key Laboratory of Information Service Engineering, Department of General Education, Beijing Union University, No. 97 Bei Si Huan Dong Road, Chaoyang District, Beijing, 100101, People's Republic of China.

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$\lambda = \pi/(\beta-\alpha)$.

(III) (see [2,3])

$$\dot{S}_{\alpha,\beta}(r,f) = \frac{\lambda}{\pi}\int_{r_0}^{r}\int_{\alpha}^{\beta}\left(\frac{1}{t^{\lambda}} - \frac{t^{\lambda}}{r^{2\lambda}}\right)\frac{|f'(te^{i\theta})|^2}{(1+|f(te^{i\theta})|^2)^2} \cdot \sin\lambda(\theta-\alpha)td\theta dt,$$

where $0 < r_0 < r < R, R = \{1, \infty\}$.

In 1989, Pang [6] discussed some equivalent relations and obtained a theorem as follows:

Theorem A. *Let $f(z)$ be a meromorphic function of order ρ in the whole complex plane, $\rho(r)$ be its precise order, $U(r) = r^{\rho(r)}$. Then the following properties are equivalent.*

1) *For any $\varepsilon > 0$,*

$$\limsup_{r\to\infty}\frac{n(r, Z_\varepsilon(\theta), f = a)}{U(r)} > 0$$

holds for any $a \in \mathbf{C}$, with two exceptions at most. The half line $L(\theta) : \arg z = \theta$ is called the Borel direction of maximal kind.

2) *For any $\varepsilon > 0$,*

$$\limsup_{r\to\infty}\frac{\mathcal{T}(r, Z_\varepsilon(\theta), f)}{U(r)} > 0.$$

3) *For any $\varepsilon > 0$ and any meromorphic function $a(z)$ with $T(r,a) = o(U(r))$*

$$\limsup_{r\to\infty}\frac{n(r, Z_\varepsilon(\theta), f = a(z))}{U(r)} > 0$$

holds for any meromorphic function $a(z)$, with two exceptions at most. The half line $L(\theta) : \arg z = \theta$ is called the Borel direction of maximal kind respect to small functions.

For meromorphic functions of infinite order, Chuang [1] proved the following result.

Theorem B. *Let $f(z)$ be a meromorphic function of infinite order in the complex plane, $\rho(r)$ be a precise order. A half line $L(\theta) : \arg z = \theta$ be a $\rho(r)$ order of Borel direction if and only if for any $0 < \varepsilon < \pi$,*

$$\limsup_{r\to\infty}\frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)}{\rho(r)\log r} = 1.$$

Zhang [9] established the following theorem:

Theorem C. *Let $f(z)$ be a meromorphic function of order $\rho \in (0, +\infty)$ in the complex plane. Then $L(\theta) : \arg z = \theta$ is a ρ order Borel direction, if and only if for any $0 < \varepsilon < \pi$,*

$$\limsup_{r\to\infty}\frac{\log \mathcal{T}(r, Z_\varepsilon(\theta), f)}{\log r} = \rho.$$

Motivated by Theorems B and C, we establish the following:

Theorem 1.1. *Let $f(z)$ be a meromorphic function of order $\rho \in (\frac{1}{2}, +\infty)$ in the complex plane. Assume that $\Omega = \{z : \alpha < \arg z < \beta\}(-\frac{\pi}{2} \le \alpha < \beta < \frac{3\pi}{2})$ is an angular domain such that*

$$\beta - \alpha > \frac{\pi}{\rho}.$$

Then the angular domain Ω possesses a Borel direction of $f(z)$ of order ρ if and only if

$$\limsup_{r\to\infty}\frac{\log S_{\alpha,\beta}(r,f)}{\log r} = \rho - \frac{\pi}{\beta-\alpha}, \tag{1.1}$$

or

$$\limsup_{r\to\infty}\frac{\log S_{\alpha,\beta}(r,f)}{\left(\rho - \frac{\pi}{\beta-\alpha}\right)\log r} = 1. \tag{1.2}$$

Example 1. The function $f(z) = e^z$ has two Borel directions $L(\frac{\pi}{2}) = \{z : \arg z = \frac{\pi}{2}\}$, $L(-\frac{\pi}{2}) = \{z : \arg z = -\frac{\pi}{2}\}$ of order $\rho = 1$.

Now we will show that $f(z) = e^z$ supports Theorem 1.1.

Consider the angular domain $\Omega = \{z : \alpha < \arg z < \beta\}(-\frac{\pi}{2} < \alpha < 0, \pi < \beta < \frac{3\pi}{2})$. Then, $\beta - \alpha > \pi/\rho, \rho = 1$. Ω satisfies the condition of Theorem 1.1 and contains a Borel direction $L(\frac{\pi}{2})$. Noticing that

$$\log^+|f(re^{i\theta})| = \log^+|e^{re^{i\theta}}| = \log^+(e^{r\cos\theta}) = \begin{cases} r\cos\theta, & \cos\theta > 0, \\ 0, & \cos\theta \le 0 \end{cases}$$

and $\cos\alpha > 0, \cos\beta < 0$, it follows that

$$A_{\alpha,\beta}(r, e^z) = \frac{1}{\beta-\alpha}\int_1^r\left(\frac{1}{t^{\lambda+1}} - \frac{t^{\lambda-1}}{r^{2\lambda}}\right)t\cos\alpha dt = \frac{\cos\alpha}{\beta-\alpha}\left[\frac{2\lambda}{1-\lambda^2}r^{1-\lambda} - \frac{1}{1-\lambda} + \frac{r^{-2\lambda}}{\lambda+1}\right]$$

and

$$B_{\alpha,\beta}(r, e^z) = \frac{2}{(\beta-\alpha)r^{\lambda}}\int_{\alpha}^{\pi/2} r\cos\theta\sin\lambda(\theta-\alpha)d\theta = \frac{2r^{1-\lambda}}{\beta-\alpha}J,$$

where $J = \int_{\alpha}^{\pi/2}\cos\theta\sin\lambda(\theta-\alpha)d\theta$. Therefore,

$$S_{\alpha,\beta}(r, e^z) = O(r^{1-\lambda}), r \to \infty.$$

This coincides with (1.1) or (1.2).

Regarding the Borel direction of maximal kind, we have the following theorem.

Theorem 1.2. *Let $f(z)$ be a meromorphic function of order $\rho \in (\frac{1}{2}, +\infty)$ in the complex plane. Assume that $\Omega = \{z : \alpha < \arg z < \beta\}(-\frac{\pi}{2} \le \alpha < \beta < \frac{3\pi}{2})$ is an angular domain such that*

$$\beta - \alpha > \frac{\pi}{\rho}.$$

Then the angular domain Ω possesses a Borel direction of maximal kind for $f(z)$ if and only if

$$\limsup_{r\to\infty} \frac{r^{\frac{\pi}{\beta-\alpha}} S_{\alpha,\beta}(r,f)}{U(r)} > 0. \tag{1.3}$$

Next, by using the above characteristic functions, we establish an equivalent conclusion as follows:

Theorem 1.3. *Let $f(z)$ be a meromorphic function in the unit disk of order $0 < \rho < \infty$ and z_0 be a point on the unit circle and J be a line through z_0, directed inward of the unit disk, which may coincide with the tangent of $|z| = 1$. Let ω be an any small angular domain, which contains J and is bounded by two lines through z_0. Then the following statements are equivalent.*

(1)

$$\limsup_{r\to 1-} \frac{\log n(r,\omega, f = a)}{\log \frac{1}{1-r}} = \rho + 1$$

with two possible exceptional numbers $a \in \widehat{\mathbf{C}}$.

(2)

$$\limsup_{r\to 1-} \frac{\log \mathcal{S}(r,\omega, f)}{\log \frac{1}{1-r}} = \rho + 1.$$

(3)

$$\limsup_{r\to 1-} \frac{\log n(r,\omega, f = a(z))}{\log \frac{1}{1-r}} = \rho + 1,$$

with two possible exceptional functions $a(z)$ satisfying $T(r,a) = o(T(r,f))$. Here z_0 is called an indirect Borel point of $f(z)$.

If ω is a sector whose vertex is the origin, Theorem 1.3 is also true. Moreover, we have the following two theorems.

Theorem 1.4. *Let $f(z)$ be meromorphic in the unit disk of order $0 < \rho < \infty$. Then $L(\theta)$: $\arg z = \theta$ is a Borel radius of order ρ for $f(z)$ if and only if for arbitrary $0 < \varepsilon < \pi$,*

$$\limsup_{r\to 1-} \frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)}{\log \frac{1}{1-r}} = \rho. \tag{1.4}$$

Theorem 1.5. *Let $f(z)$ be meromorphic in the unit disk of order $0 < \rho < \infty$ and let $\{\varphi(z)\}$ be a family of small functions in the unit disk such that*

$$T(r,\varphi) = o(T(r,f)).$$

Then the Borel radii of $f(z)$ respect to complex numbers and the Borel radii of $f(z)$ respect to small functions $\varphi(z)$ are common.

2. Some lemmas. We need some lemmas for the proofs of the theorems.

Lemma 2.1. [2,3] *Let f be a nonconstant meromorphic function in the unit disk. Then for any complex number $a \in \widehat{\mathbf{C}}$,*

$$S_{\alpha,\beta}(r,f) = S_{\alpha,\beta}\left(r, \frac{1}{f-a}\right) + O(1), \quad r \to 1- .$$

For any $q(\ge 3)$ complex numbers $a_j \in \widehat{\mathbf{C}}(j = 1, 2, \ldots, q)$,

$$\begin{aligned}(q-2)&S_{\alpha,\beta}(r,f)\\ &\le \sum_{j=1}^{q} \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-a_j}\right) + Q_{\alpha,\beta}(r,f),\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}Q_{\alpha,\beta}(r,f) = (A+B)_{\alpha,\beta}&\left(r, \frac{f'}{f}\right)\\ &+ \sum_{j=1}^{q} (A+B)_{\alpha,\beta}\left(r, \frac{f'}{f-a_j}\right) + O(1).\end{aligned}$$

Let

$$Q(r,f) = A_{\alpha,\beta}\left(r, \frac{f'}{f}\right) + B_{\alpha,\beta}\left(r, \frac{f'}{f}\right).$$

Then

(1) $Q(r,f) = O(\log \frac{1}{1-r})$ as $r \to 1-$, when $\lambda(f) < \infty$.

(2) $Q(r,f) = O(\log \frac{1}{1-r} + \log T(r,f))$ as $r \to 1-$ and $r \notin E$ when $\lambda(f) = \infty$, where E is a set such that $\int_E \frac{dr}{1-r} < \infty$.

The following lemma is well known to us and we omit the proof of it.

Lemma 2.2. *Let $h(t)$ be a positive increasing and continuous function defined on $0 < t < 1$, $H(r) = \int_{r_0}^{r} \frac{h(t)}{t} dt, (0 < r_0 < r < 1)$. Then*

$$\limsup_{r\to 1-} \frac{\log h(r)}{\log \frac{1}{1-r}} = \rho + 1 \iff \limsup_{r\to 1-} \frac{\log H(r)}{\log \frac{1}{1-r}} = \rho.$$

Lemma 2.3. [7] *Let $f(z)$ be meromorphic in* *the sector* $\Omega = \{z : \alpha < \arg z < \beta, |z| < 1\}$. *Then the following hold*

$$S_{\alpha,\beta}(r,f) \le \frac{3\lambda^2}{r_0^\lambda} \mathcal{T}(r,\Omega,f) + O(1),$$
$$S_{\alpha,\beta}(r,f) \ge \lambda^2 \sin(\lambda\delta) \mathcal{T}(r,\Omega_\delta,f) + O(1),$$

where $\Omega_\delta = \{z : \alpha + \delta < \arg z < \beta - \delta, |z| < 1\}, \lambda = \pi/(\beta - \alpha)$.

The following two lemmas are from [10].

Lemma 2.4. [10] *Let* $f(z)$ *be meromorphic in the angular domain* $\Omega = \{\alpha < \arg z < \beta\}$. *Then*

$$S_{\alpha,\beta}(r,f) = \dot{S}_{\alpha,\beta}(r,f) + O(1),$$
$$\dot{S}_{\alpha,\beta}(r,f) \le 2\lambda^2 \frac{\mathcal{T}(r,\Omega,f)}{r^\lambda} + \lambda^3 \int_{r_0}^r \frac{\mathcal{T}(t,\Omega,f)}{t^{\lambda+1}} dt + O(1),$$
$$\dot{S}_{\alpha,\beta}(r,f) \ge \lambda^2 \sin(\lambda\varepsilon) \frac{\mathcal{T}(r,\Omega_\varepsilon,f)}{r^\lambda} + \lambda^3 \sin(\lambda\varepsilon) \int_{r_0}^r \frac{\mathcal{T}(t,\Omega_\varepsilon,f)}{t^{\lambda+1}} dt + O(1),$$

as $r \to \infty$, *where* $\Omega_\varepsilon = \{\alpha + \varepsilon < \arg z < \beta - \varepsilon\}$ *and* $\lambda = \frac{\pi}{\beta-\alpha}$.

Lemma 2.5. [10] *Let* $T(r)$ *be a non-negative and non-decreasing function in* $0 < r < \infty$. *If*

$$\liminf_{r\to\infty} \frac{T(dr)}{T(r)} > d^\omega$$

for some $d > 1$ *and* $\omega > 0$, *then*

$$\int_1^r \frac{T(t)}{t^{\omega+1}} dt \le K \frac{T(r)}{r^\omega} + O(1), \tag{2.2}$$

as $r \to \infty$, *where* K *is a positive constant.*

3. Proof of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Sufficiency. Lemma 2.4 yields that

$$S_{\alpha,\beta}(r,f) \le 2\lambda^2 \frac{\mathcal{T}(r,\Omega,f)}{r^\lambda} + \lambda^3 \int_{r_0}^r \frac{\mathcal{T}(t,\Omega,f)}{t^{\lambda+1}} dt + O(1), \quad r \to \infty. \tag{3.1}$$

Suppose that the angular domain Ω contains no Borel direction of order ρ for $f(z)$. Using Theorem C gives

$$\limsup_{r\to\infty} \frac{\log \mathcal{T}(r,\Omega,f)}{\log r} < \rho.$$

Therefore, there exists a number ρ_1 such that $\lambda < \rho_1 < \rho$ and

$$\mathcal{T}(t,\Omega,f) \le t^{\rho_1}, \quad t \in (r_0, r).$$

By a simple calculation, we have

$$\int_{r_0}^r \frac{\mathcal{T}(t,\Omega,f)}{t^{\lambda+1}} dt \le \frac{r^{\rho_1-\lambda}}{\rho_1 - \lambda}.$$

Substituting the above to (3.1) yields

$$S_{\alpha,\beta}(r,f) \le \left(2\lambda^2 + \frac{\lambda^3}{\rho_1 - \lambda}\right) r^{\rho_1-\lambda} + O(1), \quad \lambda = \pi/(\beta - \alpha). \tag{3.2}$$

This gives

$$\limsup_{r\to\infty} \frac{\log S_{\alpha,\beta}(r,f)}{\log r} \le \rho_1 - \lambda < \rho - \lambda.$$

This contradicts to (1.1).

Necessary. Lemma 2.4 implies that

$$S_{\alpha,\beta}(r,f) \ge \lambda^2 \sin(\lambda\varepsilon) \frac{\mathcal{T}(r,\Omega_\varepsilon,f)}{r^\lambda} + O(1), \quad \forall \varepsilon \in \left(0, \frac{\beta-\alpha}{2}\right). \tag{3.3}$$

Suppose that $\arg z = \theta \in (\alpha, \beta)$ is a Borel direction of order ρ for $f(z)$. We choose $\varepsilon > 0$ such that $\theta \in (\alpha + \varepsilon, \beta - \varepsilon)$. The fact that Ω_ε contains a Borel direction of order ρ for $f(z)$ gives

$$\limsup_{r\to\infty} \frac{\log \mathcal{T}(r,\Omega_\varepsilon,f)}{\log r} \ge \rho. \tag{3.4}$$

Combining (3.4) with (3.3), we have

$$\limsup_{r\to\infty} \frac{\log S_{\alpha,\beta}(r,f)}{\log r} \ge \rho - \lambda.$$

By the inequality (3.1), we can obviously find that

$$\limsup_{r\to\infty} \frac{\log S_{\alpha,\beta}(r,f)}{\log r} \le \rho - \lambda.$$

Theorem 1.1 follows. □

Proof of Theorem 1.2. We use the same notations in the proof of Theorem 1.1.

Sufficiency. Suppose that Ω contains no Borel direction of maximal kind for $f(z)$. Then, Theorem A yields

$$\mathcal{T}(r,\Omega,f) = o(U(r)). \tag{3.5}$$

In view of Lemma 2.5 and noticing $U(r)$ satisfying $\lim_{r\to\infty} \frac{U(dr)}{U(r)} = d^\rho (d > 1)$, $\rho > \lambda = \frac{\pi}{\beta-\alpha}$, we get

$$\int_1^r \frac{U(t)}{t^{\lambda+1}} dt \le K \frac{U(r)}{r^\lambda} + O(1), \tag{3.6}$$

where K is a positive constant. Combining (3.1), (3.5) and (3.6), it follows that

$$S_{\alpha,\beta}(r,f)=o\left(\frac{U(r)}{r^{\lambda}}\right).$$

This contradicts to (1.3).

Necessary. Suppose that $\arg z=\theta\in(\alpha,\beta)$ is a Borel direction of order ρ for $f(z)$. We choose $\varepsilon>0$ such that $\theta\in(\alpha+\varepsilon,\beta-\varepsilon)$. The fact that Ω_ε contains a Borel direction of maximal kind for $f(z)$ gives

$$\limsup_{r\to\infty}\frac{\mathcal{T}(r,\Omega_\varepsilon,f)}{U(r)}>0. \tag{3.7}$$

Combining (3.3) with (3.7), we have (1.3).

Theorem 1.2 follows. □

4. Proof of Theorems 1.3–1.5.

Proof of Theorem 1.3. (1)⇒(2) Suppose that (2) is false. Then there exist an angular domain ω_1 and $0<\rho_1<\rho+1$, such that

$$\mathcal{S}(r,\omega_1,f)<\left(\frac{1}{1-r}\right)^{\rho_1}, \tag{4.1}$$

as $r\to 1-$. Take $r=1-2^{-k}$, and denote by E_k the set of a which satisfies

$$n\left(1-\frac{1}{2^k},\omega,a\right)\geq k^2 2^{k\rho_1},$$

for sufficiently large k. The spherical measure of E_k is not larger than πk^{-2}, otherwise we have

$$\mathcal{S}\left(1-\frac{1}{2^k},\omega,f\right)>\frac{1}{\pi}k^2 2^{k\rho_1}\pi k^{-2},$$

which contradicts to (4.1). Put

$$E=\bigcap_{\mu=1}^{\infty}\bigcup_{k=\mu}^{\infty}E_k.$$

The spherical measure of E is zero. For any $a\notin E$ and $1-2^{-k}\leq r<1-2^{-(k+1)}$, we have

$$\begin{aligned}n(r,\omega_1,a)&\leq n\left(1-\frac{1}{2^{k+1}},\omega_1,a\right)\\&<(k+1)^2 2^{(k+1)\rho_1}\leq (k+1)^2 2^{\rho_1}\left(\frac{1}{1-r}\right)^{\rho_1}.\end{aligned}$$

Notice that $k\leq\frac{\log\frac{1}{1-r}}{\log 2}$, which implies that

$$n(r,\omega_1,a)<\left(\frac{1}{1-r}\right)^{\rho_2},\quad(\rho_1<\rho_2<\rho+1).$$

Since E has a zero measure, it follows that for almost every a,

$$n(r,\omega_1,a)<\left(\frac{1}{1-r}\right)^{\rho_2},$$

as $r\to 1-$. This contradicts to (1).

(2) ⇒ (3)

From [4] we gain an inequality as follows:

$$\begin{aligned}\mathcal{S}(r,\omega,f)\leq K\sum_{j=1}^{3}n\left(\frac{r+255}{256},\omega_0,f=a_j(z)\right)\\+O\left(\int_0^{\frac{r+127}{128}}\frac{T(r,a)}{(1-r)^2}\,dr\right),\end{aligned}$$

where $\omega\subseteq\omega_0$ are two sectors, and $a_j(z)(j=1,2,3)$ are small functions of $f(z)$.

Since $T(r,a)=o(T(r,f))$ and $T(r,f)=O((\frac{1}{1-r})^{\rho+\varepsilon})$, we get $T(r,a)=o((\frac{1}{1-r})^{\rho+\varepsilon})$. Then we have

$$\begin{aligned}\int_0^{\frac{r+127}{128}}\frac{T(r,a)}{(1-r)^2}\,dr&=o\left(\int_0^{\frac{r+127}{128}}\frac{1}{(1-r)^{\rho+\varepsilon+2}}\,dr\right)\\&=o\left(\frac{1}{(1-r)^{\rho+1+\varepsilon}}\right)\end{aligned}$$

and

$$\begin{aligned}O\left(\log\left(\int_0^{\frac{r+127}{128}}\frac{T(r,a)}{(1-r)^2}\,dr\right)\right)\\=o\left((\rho+1+\varepsilon)\log\frac{1}{1-r}\right).\end{aligned}$$

Thus we have

$$\limsup_{r\to 1-}\frac{\log n(r,\omega_0,f=a(z))}{\log\frac{1}{1-r}}\geq\rho+1,$$

with two possible exceptional for $a(z)$.

On the other hand, the inequality

$$\limsup_{r\to 1-}\log n(r,\omega_0,f=a(z))\Big/\log\frac{1}{1-r}\leq\rho+1$$

is obvious. Hence,

$$\limsup_{r\to 1-}\frac{\log n(r,\omega_0,f=a(z))}{\log\frac{1}{1-r}}=\rho+1.$$

(3) ⇒ (1) is obvious.

Theorem 1.3 follows. □

Proof of Theorem 1.4. In view of Lemma 2.2, we obtain

$$\limsup_{r\to 1-}\frac{\log\mathcal{S}(r,\Omega,f)}{\log\frac{1}{1-r}}=\rho+1$$

$$\Longleftrightarrow \limsup_{r\to 1-} \frac{\log \mathcal{T}(r,\Omega,f)}{\log \frac{1}{1-r}} = \rho.$$

From Lemma 2.3, there exit two constants $K_1, K_2 > 0$ such that

$$S_{\alpha,\beta}(r,f) \le K_1 \mathcal{T}(r,\Omega,f) + O(1),$$
$$S_{\alpha,\beta}(r,f) \ge K_2 \mathcal{T}(r,\Omega_\varepsilon,f) + O(1), \quad r\to 1-.$$

By using Theorem 1.3, we can get the result. □

Proof of Theorem 1.5. Suppose that $\arg z = \theta$ is a Borel radius of $f(z)$ respect to constant a of order ρ, then we have (1.4).

Let

$$g(z) = \frac{f(z)-a_1(z)}{f(z)-a_2(z)} \frac{a_3(z)-a_2(z)}{a_3(z)-a_1(z)},$$

where $a_j(z) (j = 1,2,3)$ are three small functions. Then

$$S_{\alpha,\beta}(r,f) - o(T(r,f)) \le S_{\alpha,\beta}(r,g) \le S_{\alpha,\beta}(r,f) + o(T(r,f)).$$

Therefore, we have

$$\limsup_{r\to 1-} \frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,g)}{\log \frac{1}{1-r}} = \rho.$$

Let b_i be $0, 1, \infty$, respectively. Then, we have

$$N(r, Z_\varepsilon(\theta), g = b_i) = N(r, Z_\varepsilon(\theta), f = a_i(z)) + o(T(r,f)).$$

Suppose that there exist three small functions $a_j(z), j = 1,2,3$, such that

$$N(r, Z_\varepsilon(\theta), f = a_i(z)) < \left(\frac{1}{1-r}\right)^{\rho_1} \quad (0 < \rho_1 < \rho).$$

Note that

$$\begin{aligned} &C_{\theta-\varepsilon,\theta+\varepsilon}(r, g = b_i) \\ &\le 4\lambda \frac{N(r, Z_\varepsilon(\theta), g = b_i)}{r^\lambda} \\ &\quad + 2\lambda^2 \int_{r_0}^{r} \frac{N(t, Z_\varepsilon(\theta), g = b_i)}{t^{\lambda+1}}\, dt \\ &= O(N(r, Z_\varepsilon(\theta), g = b_i)), \quad r \to 1-, \end{aligned}$$

where $\lambda = \frac{\pi}{2\varepsilon}$. In view of Lemma 2.1, we have

$$S_{\alpha,\beta}(r,g) < \sum_{j=1}^{3} C_{\alpha,\beta}(r, g = b_j) + O\left(\log T(r,f) + \log\frac{1}{1-r}\right).$$

Then we have

$$\limsup_{r\to 1-} \frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,g)}{\log \frac{1}{1-r}} < \rho.$$

This leads to a contradiction. Theorem 1.5 follows. □

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