Properties of some difference polynomials

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Abstract: In this article, we investigate some properties of some difference polynomials. The results in this article improve some theorems of Liu and Laine. Several examples are provided to show that our results are best possible.

Key words: Meromorphic functions; difference equations; value distribution; finite order.

1. Introduction and results. In this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory (see, e.g., [12,16]). In addition, we will use the notation $\sigma(f)$ to denote the order of the meromorphic function $f(z)$, and $\lambda(f)$ and $\lambda(\frac{1}{f})$ to denote the exponent of convergence of zeros and poles of $f(z)$, respectively.

Hayman [11] proved two classical theorems which can be combined as follows:

Theorem A [11]. *Let $f(z)$ be a transcendental meromorphic function and $a \neq 0, b$ be finite complex constants. Then $f^n(z) + af'(z) - b$ has infinitely many zeros for $n \geq 5$. If $f(z)$ is transcendental entire, this holds for $n \geq 3$, resp. $n \geq 2$, if $b = 0$.*

Recently, a number of papers (see, e.g., [1–3,5–10,13–15]) have focused on complex difference equations and difference analogues of Nevanlinna's theory.

Liu and Laine [15] established partial difference counterparts of Theorem A, and obtained the following

Theorem B [15]. *Let $f(z)$ be a transcendental entire function of finite order, not of period c, and let $s(z)$ be a nonzero function, small compared to f. Then $f^n(z) + f(z+c) - f(z) - s(z)$ has infinitely many zeros, provided $n \geq 3$, resp. $n \geq 2$, if $s = 0$.*

In this paper, we consider the zeros of the difference polynomial

$$F_n(z) = \sum_{j=1}^{k} a_j(z) f(z+c_j) - a(z) f^n(z),$$

and obtain the following results which generalize Theorem B. In Theorems 1.1 and 1.5, we consider the case when the coefficients of $F_n(z)$ are constants.

Theorem 1.1. *Let $f(z)$ be a transcendental entire function of finite order $\rho(f)$, let $b, a, c_j, a_j (j = 1, 2, \cdots, k)$ be complex constants. Set $F_n(z) = \sum_{j=1}^{k} a_j f(z+c_j) - af^n(z)$, where $n \geq 3$ is an integer. Then $F_n(z)$ have infinitely many zeros and $\lambda(F_n(z) - b) = \rho(f)$ provided that $\sum_{j=1}^{k} a_j(z) f(z+c_j) \not\equiv b$.*

In the previous theorem, we consider difference polynomial $F_n(z)$ with $n \geq 3$. The following theorem is about the case $n = 2$:

Theorem 1.2. *Suppose that $f(z)$ be a finite order transcendental entire function with a Borel exceptional value d. Let $b(z), a(z) (\not\equiv 0), a_j(z) (j = 1, 2, \cdots, k)$ be polynomials, and let $c_j (j = 1, 2, \cdots, k)$ be complex constants. If either $d = 0$ and $\sum_{j=1}^{k} a_j(z) f(z+c_j) \not\equiv 0$, or, $d \neq 0$ and $\sum_{j=1}^{k} da_j(z) - d^2 a(z) - b(z) \not\equiv 0$, then $F_2(z) - b(z) = \sum_{j=1}^{k} a_j(z) f(z+c_j) - a(z) f^2(z) - b(z)$ has infinitely many zeros and $\lambda(F_2(z) - b(z)) = \rho(f)$.*

Example 1.3. For $f(z) = \exp\{z\} + z, a(z) = 4, c_1 = 3\pi i, c_2 = \pi i, c_3 = 0, c_4 = 5\pi i, c_5 = 7\pi i, a_1(z) = z, a_2(z) = -3z, a_3(z) = 6z, a_4(z) = -1, a_5(z) = 1, a_6(z) = \cdots = a_k(z) = 0, b(z) = 2\pi i,$ we have $F_2(z) - b(z) = \sum_{j=1}^{k} a_j(z) f(z+c_j) - a(z) f^2(z) - b(z) = -4\exp\{2z\}$. Here $f(z)$ has no Borel exceptional values, but $F_2(z) - b(z)$ has no zeros. Hence the condition that $f(z)$ has a Borel exceptional value cannot be omitted in Theorem 1.2.

Example 1.4. For $f(z) = \exp\{z\} + 1, a = 2, c_1 = \ln 2, c_2 = \ln 4, c_3 = \ln 3, a_1 = 3, a_2 = 1, a_3 = -2,$

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$a_4 = \cdots = a_k = 0$, we have $F_2(z) = \sum_{j=1}^{k} a_j f(z + c_j) - af^2(z) = -2\exp\{2z\}$. Here $F_2(z)$ has no zero, but $f(z)F_2(z) = -2\exp\{2z\}(\exp\{z\} + 1)$ has infinitely many zeros. This shows that $f(z)F_2(z)$ is different from $F_2(z)$.

What can we say about $f(z)F_2(z)$ when $f(z)$ has infinitely many multi-order zeros. For this question, we obtain the following Theorem 1.5:

Theorem 1.5. *Let $f(z)$ be a finite order transcendental entire function, and let $b, a, a_j, c_j (j = 1, 2, \cdots, k)$ be complex constants. If $f(z)$ has infinitely many multi-order zeros, then $H(z) = f(z)(\sum_{i=1}^{k} a_j f(z + c_j) - af^2(z)) - b$ has infinitely many zeros.*

Example 1.6. For $f(z) = \exp\{\exp\{z\}\}, c_1 = \ln 3$, $c_2 = \ln 3$, $c_3 = 0$, $a = 2$, $a_1 = 1$, $a_2 = 1$, $a_3 = 2$, $a_4 = \cdots = a_k = 0$, we obtain

$$F_3(z) = \sum_{j=1}^{3} a_j f(z + c_j) - af^3(z) = 2\exp\{\exp\{z\}\}.$$

Here $F_3(z) \not\equiv 0$. This shows that Theorem 1.1 may fail for entire functions of infinite order.

Example 1.7. For $f(z) = \exp\{z\} + 2, a = 2$, $c_1 = \ln 3, c_3 = \ln 4, a_1 = 1, a_2 = 1, a_3 = -1, a_4 = \cdots = a_k = 0$, we have $F_2(z) = \sum_{j=1}^{k} a_j f(z + c_j) - af^2(z) = -2\exp\{2z\} - 6$. Here $F_2(z) \neq -6$. This shows that Theorem 1.1 may be fail for $n = 2$ and that the condition $n \geq 3$ in Theorem 1.1 is the best possible.

Example 1.8. For $f(z) = \exp\{z\} + 1, a = 1$, $a_1 = 2, a_2 = 3, a_3 = -4, a_4 = \cdots = a_k = 0, b = 0$, $c_1 = \ln 4, c_2 = \ln 3, c_3 = \ln 2$, we have $F_2(z) = \exp\{z\}(7 - \exp\{z\})$, which assume all finite values infinitely often.

2. Preliminary lemmas. In order to prove our theorems, we need the following lemmas.

The following lemma is a generalisation of Borel's Theorem on linear combinations of entire functions.

Lemma 2.1 [16, pp. 79–80]. *Let $f_j(z)(j = 1, 2, \cdots, n)(n \geq 2)$ be meromorphic function, $g_j(z)(j = 1, 2, \cdots, n)$ be entire functions, and let them satisfy*
(i) $f_1(z)e^{g_1(z)} + \cdots + f_k(z)e^{g_k(z)} \equiv 0$;
(ii) when $1 \leq j < k \leq n$, then $g_j(z) - g_k(z)$ is not a constant.
(iii) when $1 \leq j \leq n$, $1 \leq h < k \leq n$, then

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\} \quad (r \to \infty,\ r \notin E),$$

where $E \subset (1, \infty)$ is of finite linear measure or finite logarithmic measure. Then $f_j \equiv 0 (j = 1, \cdots, n)$.

Let $c_j, j = 1, \cdots, n$, be a finite collection of complex numbers. Then a difference polynomial in $f(z)$ is a function which is polynomial in $f(z + c_j), j = 1, \cdots, n$, with meromorphic coefficients $a_\lambda(z)$ such that $T(r, a_\lambda) = S(r, f)$ for all λ. As for difference counterparts of the Clunie lemma [4], see [8, Corollary 3.3]. The following lemma due to Laine and Yang [14] is a more general version.

Lemma 2.2 [14]. *Let f be a transcendental meromorphic solution of finite order of a difference equation of the form*

$$U(z, f)P(z, f) = Q(z, f), \tag{2.1}$$

where $U(z, f), P(z, f)$, and $Q(z, f)$ are difference polynomials such that the total degree $\deg U(z, f) = n$ in $f(z)$ and its shifts, and $\deg Q(z, f) \leq n$. Moreover, we assume that $U(z, f)$ contains just one term of maximal total degree in $f(z)$ and its shifts. Then

$$m(r, P(z, f)) = S(r, f).$$

The following lemma is a difference analogue of the logarithmic derivative lemma.

Lemma 2.3 [8,10]. *Let $f(z)$ be a meromorphic function of finite order and let c be a non-zero complex number. Then we have*

$$m\left(r, \frac{f(z + c)}{f(z)}\right) = S(r, f).$$

3. Proof of Theorem 1.1. The main idea of this proof is from [15, case $n \geq 3$], while the details are somewhat different. For the convenience of the reader, we give a complete proof. Firstly, we prove that $\rho(F_n - b) = \rho(f)$. Lemma 2.3 and $F_n(z) = \sum_{j=1}^{k} a_j f(z + c_j) - af^n(z)$ yield that

$$\begin{aligned} nT(r, f(z)) &= T(r, af^n(z)) + O(1) \\ &= T\left(r, \sum_{j=1}^{n} a_j f(z + c_j) - F_n(z)\right) + O(1) \\ &\leq m\left(r, f(z)\frac{\sum_{i=1}^{k} a_j f(z + c_j)}{f(z)}\right) \\ &\quad + m(r, F_n(z)) + O(1) \\ &\leq m(r, f(z)) + \sum_{j=1}^{k} a_j m\left(r, \frac{f(z + c_j)}{f(z)}\right) \\ &\quad + m(r, F_n(z)) + O(1) \\ &= m(r, f(z)) + m(r, F_n(z)) + S(r, f) \\ &= T(r, f(z)) + T(r, F_n(z)) + S(r, f). \end{aligned} \tag{3.1}$$

On the other hand, from Lemma 2.3 and $F_n(z) = \sum_{j=1}^{k} a_j f(z+c_j) - af^n(z)$, we have

$$(3.2)\qquad \begin{aligned} T(r, F_n(z)) &= m(r, F_n(z)) \\ &= m\left(r, \sum_{j=1}^{k} a_j f(z+c_j) - af^n(z)\right) \\ &\leq m(r, f(z)) + \sum_{j=1}^{k} a_j m\left(r, \frac{f(z+c_j)}{f(z)}\right) \\ &\quad + m(r, af^n(z)) + O(1) \\ &= (n+1)T(r, f(z)) + S(r, f). \end{aligned}$$

(3.1) and (3.2) imply that $\rho(f) = \rho(F_n)$. Hence $\rho(F_n - b) = \rho(f)$. We next discuss the following two cases:

Case 1. $\rho(f) = 0$. By $0 \leq \lambda(F_n - b) \leq \rho(F_n - b) = \rho(f) = 0$, we have $\lambda(F_n - b) = \rho(f)$. Theorem 1.1 thus holds.

Case 2. $\rho(f) > 0$. Suppose, contrary to the assertion, that $\lambda(F_n - b) < \rho(f)$. By this and $\rho(F_n - b) = \rho(f)$, we can rewritten $F_n(z) - b$ as

$$(3.3)\qquad \begin{aligned} F_n(z) - b &= \sum_{j=1}^{k} a_j f(z+c_j) - af^n(z) - b \\ &= p(z)\exp\{q(z)\}, \end{aligned}$$

where $q(z) \not\equiv 0$ is a polynomial, $p(z)$ is an entire function with $\rho(p) < \rho(f)$. Differentiating (3.3) and eliminating $\exp\{q(z)\}$, we obtain

$$(3.4)\qquad \begin{aligned} f^{(n-1)}(z)&(anp(z)f'(z) - a(p'(z) + q'(z)p(z))f(z)) \\ &= p(z)\sum_{j=1}^{k} a_j f'(z+c_j) + b(p'(z) + p(z)q'(z)) \\ &\quad - \{p'(z) + p(z)q'(z)\}\sum_{i=1}^{k} a_j f(z+c_j). \end{aligned}$$

Suppose that

$$(3.5)\qquad anp(z)f'(z) - a(p'(z) + q'(z)p(z))f(z) \equiv 0.$$

Integrating (3.5), we have

$$(3.6)\qquad f^n(z) = dp(z)\exp\{q(z)\},$$

where d is a nonzero constant. Hence (3.3) and (3.6) yield that

$$(3.7)\qquad \begin{aligned} F_n(z) - b &= \sum_{j=1}^{k} a_j f(z+c_j) - af^n(z) - b \\ &= \frac{1}{d} f^n(z), \end{aligned}$$

that is

$$(3.8)\qquad d\left(\sum_{j=1}^{k} a_j f(z+c_j) - b\right) = (ad+1)f^n(z).$$

Since $\sum_{j=1}^{k} a_j f(z+c_j) \not\equiv b$, we have $ad + 1 \neq 0$. Differentiating (3.8), we have

$$(3.9)\qquad d\left(\sum_{j=1}^{k} \frac{a_j f'(z+c_j)}{f'(z)}\right) = n(ad+1)f^{n-1}(z).$$

By using Lemma 2.3 we obtain from (3.9) that

$$(n-1)T(r, f) = (n-1)m(r, f) = S(r, f') = S(r, f),$$

which is a contradiction, since $n \geq 3$. Therefore $P(z, f) \not\equiv 0$. Since $n \geq 3$, by Lemma 2.2 and (3.4), we obtain that

$$(3.10)\qquad \begin{aligned} T(r, &anp(z)f'(z) - a(p'(z) + q'(z)p(z))f(z)) \\ &= m(r, anp(z)f'(z) \\ &\quad - a(p'(z) + q'(z)p(z))f(z)) \\ &= S(r, f), \end{aligned}$$

and

$$(3.11)\qquad \begin{aligned} T(r, &f(z)(anp(z)f'(z) \\ &- a(p'(z) + q'(z)p(z))f(z))) \\ &= m(r, f(z)(anp(z)f'(z) \\ &\quad - a(p'(z) + q'(z)p(z))f(z))) \\ &= S(r, f), \end{aligned}$$

for all r outside of an exceptional set of finite logarithmic measure. Thus, by (3.10) and (3.11), we have

$$T(r, f) = S(r, f),$$

for all r outside of an exceptional set of finite logarithmic measure. This is a contradiction. Hence $\lambda(F_n(z) - b) = \rho(f)$. Theorem 1.1 is thus proved.

4. Proof of Theorem 1.2. Suppose that d is the Borel exceptional value of $f(z)$. Then we can write $f(z)$ as

$$(4.1)\qquad f(z) = d + g(z)\exp\{\alpha z^k\},$$

where α is a nonzero constant, $k \geq 1$ is an integer, and $g(z)$ is an entire function such that $g(z)(\not\equiv 0), \rho(g) < k$. By (4.1), we have

$$(4.2)\qquad f(z+c_j) = d + g(z+c_j)g_j(z)\exp\{\alpha z^k\}, \quad (j = 1, 2, \cdots, k)$$

where $g_j(z) = \exp\{\alpha\binom{k}{1}z^{k-1}c_j + \alpha\binom{k}{2}z^{k-2}c_j^2 + \cdots + \alpha c_j^k\}, \rho(g_j) = k-1$. Suppose that $F_2(z) - b(z)$ is a polynomial. Then

$$(4.3)\qquad \sum_{j=1}^{k} a_j(z)f(z+c_j) - a(z)f^2(z) - b(z) = p(z),$$

where $p(z)$ is a polynomial. By Lemma 2.3 and (4.3), we have

$$\begin{aligned}2T(r,f(z)) &= 2m(r,f(z))\\ &= m(r,a(z)f^2(z)) + S(r,f)\\ &= m\left(r, \sum_{j=1}^{k} a_j(z)f(z+c_j) - b(z) - p(z)\right) + S(r,f)\\ &\le m(r,f(z)) + \sum_{j=1}^{k} m\left(r, \frac{f(z+c_j)}{f(z)}\right) + m(r,b(z))\\ &\quad + m(r,p(z)) + \sum_{j=1}^{k} m(r,a_j(z)) + S(r,f)\\ &= T(r,f(z)) + S(r,f) + O(\log r),\end{aligned}$$

which is a contradiction. Therefore, $F_2(z) - b(z)$ is transcendental. By (4.1) and (4.2), we have

$$\begin{aligned}(4.4)\quad F_2(z) - b(z) &= \left(\sum_{j=1}^{k} a_j(z)g(z+c_j)g_j(z) - 2da(z)g(z)\right)\\ &\quad \times \exp\{\alpha z^k\} - a(z)g^2(z)\exp\{2\alpha z^k\}\\ &\quad + \sum_{j=1}^{k} da_j(z) - d^2a(z) - b(z).\end{aligned}$$

Since $g(z) \not\equiv 0$, we have $\rho(F_2(z) - b(z)) = \rho(f) = k$. Next, we prove $\lambda(F_2(z) - b(z)) = k$. Suppose, contrary to the assertion, that $\lambda(F_2(z) - b(z)) < \rho(f)$. Then

$$(4.5)\qquad F_2(z) - b(z) = l(z)\exp\{\beta z^k\},$$

where β is a nonzero constant, $l(z)$ is an entire function such that $\rho(l) < k$.

(i) Suppose that $d = 0$. (4.1), (4.2) and (4.5) imply that

$$\begin{aligned}(4.6)\quad &\sum_{j=1}^{k} a_j(z)g(z+c_j)g_j(z)\exp\{\alpha z^k\}\\ &\quad - a(z)g^2(z)\exp\{2\alpha z^k\} - b(z) = l(z)\exp\{\beta z^k\}.\end{aligned}$$

Since $g(z) \not\equiv 0$, by comparing the growth of both sides of (4.6), we have $\beta = 2\alpha$. Hence (4.6) can be rewritten as

$$\begin{aligned}(4.7)\quad &\sum_{j=1}^{k} a_j(z)g(z+c_j)g_j(z)\exp\{\alpha z^k\}\\ &\quad - (a(z)g^2(z) + l(z))\exp\{2\alpha z^k\} - b(z) = 0.\end{aligned}$$

By Lemma 2.1 and (4.7), we obtain that $\sum_{j=1}^{k} a_j(z)g(z+c_j)g_j(z) \equiv 0$, a contradiction, since $\sum_{j=1}^{k} a_j(z)f(z+c_j) \not\equiv 0$. Hence, we obtain that $\lambda(F_2(z) - b(z)) = k$.

(ii) Suppose that $d \neq 0$. (4.1), (4.2) and (4.5) yield that

$$\begin{aligned}(4.8)\quad &\left(\sum_{j=1}^{k} a_j(z)g(z+c_j)g_j(z) - 2da(z)g(z)\right)\exp\{\alpha z^k\}\\ &\quad - a(z)g^2(z)\exp\{2\alpha z^k\} - l(z)\exp\{\beta z^k\}\\ &\quad + \sum_{j=1}^{k} da_j(z) - d^2a(z) - b(z) = 0.\end{aligned}$$

If $\beta \neq \alpha$ and $\beta \neq 2\alpha$, By Lemma 2.1 and (4.8), we get $\sum_{i=1}^{k} da_j(z) - d^2a(z) - b(z) \equiv 0$, which contradicts our assumption that $\sum_{i=1}^{k} da_j(z) - d^2a(z) - b(z) \not\equiv 0$.

If $\beta = 2\alpha$ or $\beta = \alpha$, using the same method as above, we also get $\sum_{i=1}^{k} da_j(z) - d^2a(z) - b(z) \equiv 0$, a contradiction. Hence $\lambda(F_2(z) - b(z)) = k$.

5. Proof of Theorem 1.5. Suppose that $f(z)$ has infinitely many multi-order zeros. If $b = 0$, then $H(z)$ has infinitely many zeros. Next we suppose that $b \neq 0$. If $H(z) - b$ has only finitely many zeros, then we can rewrite $H(z)$ as

$$\begin{aligned}(5.1)\quad H(z) &= f(z)\left(\left(\sum_{i=1}^{k} a_i f(z+c_i)\right) - af^2(z)\right) - b\\ &= p(z)e^{q(z)},\end{aligned}$$

where $p(z), q(z)$ are polynomials. Suppose that $H(z)$ is a polynomial. Then

$$\begin{aligned}(5.2)\quad H(z) &= f(z)\left(\left(\sum_{i=1}^{k} a_i f(z+c_i)\right) - af^2(z)\right) - b\\ &= P(z),\end{aligned}$$

where $P(z)$ is a polynomial. By (5.2), we have

$$\begin{aligned}3T(r,f(z)) &= 3m(r,f(z))\\ &= m(r,af^3(z)) + O(1)\\ &\le m(r,P(z)) + m(r,f^2(z))\\ &\quad + \sum_{j=1}^{k} a_j m\left(r, \frac{f(z+c_j)}{f(z)}\right) + O(1)\\ &= 2T(r,f) + S(r,f),\end{aligned}$$

a contradiction. Hence $H(z)$ is transcendental, and so $p(z) \not\equiv 0, \deg q(z) \geq 1$. By this, we have $p'(z) + p(z)q(z) \not\equiv 0$. Differentiating (5.1) and eliminating $e^{q(z)}$, we have

$$(5.3)\quad \frac{3af(z)f'(z)}{\sum_{i=1}^{k}a_if(z+c_i)} - \frac{(f(z)\sum_{i=1}^{k}a_if(z+c_i))'}{f(z)\sum_{i=1}^{k}a_if(z+c_i)} = \left(\frac{p'(z)+p(z)q(z)}{p(z)}\right)\left(\frac{af^2(z)}{\sum_{i=1}^{k}a_if(z+c_i)} - 1 + \frac{b}{f(z)\sum_{i=1}^{k}a_if(z+c_i)}\right).$$

Since $f(z)$ has infinitely many multi-order zero, there exists a sufficiently large point z_0 such that the multiplicity of the zero of $f(z)$ at z_0 is $k(\geq 2)$, and $p'(z_0)+p(z_0)q(z_0)\neq 0, p(z_0)\neq 0$.

If $\sum_{i=1}^{k}a_if(z_0+c_i)=0$ with the multiplicity $k_1(\geq 1)$, then the multiplicity of $\frac{(f(z_0)\sum_{i=1}^{k}a_if(z_0+c_i))'}{f(z_0)\sum_{i=1}^{k}a_if(z_0+c_i)}=\infty$ is 1, the multiplicity of $\frac{3af(z_0)f'(z_0)}{\sum_{i=1}^{k}a_if(z_0+c_i)}=\infty$ is k_1-2k+1, the multiplicity of $\frac{af^2(z_0)}{\sum_{i=1}^{k}a_if(z_0+c_i)}=\infty$ is k_1-2k, but the multiplicity of $\frac{b}{f(z_0)\sum_{i=1}^{k}a_if(z_0+c_i)})=\infty$ is k_1+k. By (5.3), we get a contradiction.

If $\sum_{i=1}^{k}a_if(z_0+c_i)\neq 0$, then the multiplicity of $\frac{(f(z_0)\sum_{i=1}^{k}a_if(z_0+c_i))'}{f(z_0)\sum_{i=1}^{k}a_if(z_0+c_i)}=\infty$ is 1, $\frac{3af(z_0)f'(z_0)}{\sum_{i=1}^{k}a_if(z_0+c_i)}=0$, $\frac{af^2(z_0)}{\sum_{i=1}^{k}a_if(z_0+c_i)}=0$, but the multiplicity of $\frac{b}{f(z_0)\sum_{i=1}^{k}a_if(z_0+c_i)})=\infty$ is $k(\geq 2)$. By (5.3), we also get a contradiction. Hence $H(z)$ takes every value b infinitely often.

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