Absorbent property, Krasner type lemmas and spectral norms for a class of valued fields

Dedicated to the memory of our Professor Nicolae Popescu

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Abstract: Let (K,φ) be a perfect valued field of rank 1, let $\overline{\varphi}$ be an extension of the absolute (multiplicative) value φ to a fixed algebraic closure $\overline{K}$ and let $\|.\|_{\varphi}$ be the corresponding spectral norm on K. Let $(\widetilde{\overline{K}},\|.\|_{\tilde{\varphi}})$ be a fixed completion of $(\overline{K},\|.\|_{\varphi})$. In this paper we generalize a result of A. Ostrowski [8] relative to the absorbent property of a subfield, from the case of a complete non-Archimedean valued field of characteristic 0 to our ring $(\widetilde{\overline{K}},\|.\|_{\tilde{\varphi}})$ (see Theorem 1, Theorem 4). We also apply these results to discuss in a more general context the following conjecture due to A. Zaharescu (2009): $\langle$For any $x,y\in\mathbf{C}_p$-the complex p-adic field, there exists $t\in\mathbf{Q}_p$-the p-adic number field, such that $\widetilde{\mathbf{Q}_p(x,y)}=\widetilde{\mathbf{Q}_p(x+ty)}$, where $\widetilde{L}$ means the p-adic topological closure of a subfield L of $\mathbf{C}_p$ in $\mathbf{C}_p\rangle$.

Key words: Valued fields; Krasner Lemma; spectral norms.

Introduction. In [8] (see also [11] or [5]) A. Ostrowski proved the following "mysterious" result: $\langle$Let (K,φ) be a perfect complete non-Archimedean valued field relative to a nontrivial multiplicative valuation φ and let $\overline{\varphi}$ be the unique extension of φ to a fixed algebraic closure $\overline{K}$ of K. Let $\alpha\in\overline{K}\setminus K$ and let L be a subfield of $\overline{K}$ which contains K, such that the distance from α to L is strictly less than the distance of α to the nearest conjugate of α. Then L "absorbs" α, i.e., $\alpha\in L\rangle$.

It appears that this result is stronger than the classical Krasner Lemma. We shall prove later (see Section 2) that in fact they are equivalent in a more general context. The main point in proving the above result of Ostrowski or that one of Krasner is the equivariance property of the valuation $\overline{\varphi}$ with respect to the absolute Galois group $G=Gal(\overline{K}/K)$. This means that $\overline{\varphi}(\sigma(x))=\overline{\varphi}(x)$ for any $x\in\overline{K}$ and $\sigma\in G$ (see [7], [5], or [4]). If (K,φ) is not a henselian field, this $\overline{\varphi}$ can be substituted with a special equivariant norm $\|.\|_{\varphi}$ which extends φ from K to $\overline{K}$. Now $\overline{\varphi}$ is not unique and a candidate for such a norm is the so called *φ-spectral norm* (Archimedean or non-Archimedean) defined on $\overline{K}$ as follows:

$$\text{(0.1)}\qquad \|x\|_{\varphi}=\max\{\overline{\varphi}(\sigma(x)):\sigma\in G\}, x\in\overline{K}.$$

(See also [1], [2], [9], [10]). In the case of a henselian field (K,φ), since for any $\sigma\in G$, $\overline{\varphi}\circ\sigma$ is a new multiplicative absolute value on $\overline{K}$, one has that $\overline{\varphi}\circ\sigma=\overline{\varphi}$ and then $\|x\|_{\varphi}=\overline{\varphi}(x)$ for any $x\in\overline{K}$. It is very easy to see that the φ-spectral norm depends only on φ and not on the fixed extension $\overline{\varphi}$ of it (see also [1]). This is true because any other valuation on $\overline{K}$ which extends φ is of the form $\overline{\varphi}\circ\sigma$ for a K-automorphism σ of $\overline{K}$ (see for instance [7], or [5]). The philosophy of this paper is to substitute the unique extension $\overline{\varphi}$ of φ in the complete or henselian cases with the above defined φ-spectral norm in the case of a general separable valued field of rank 1 (non-Archimedean or Archimedean).

Some other interesting results connected with this paper one can find, for the particular case $K=\mathbf{Q}_p$-the p-adic number field, in [6] and in [3].

By using the above defined φ-spectral norm $\|.\|_{\varphi}$ on a fixed algebraic closure $\overline{K}$ of K, in both cases, non-Archimedean or Archimedean, we generalize Ostrowski's and Krasner's results (Theorem 1 and Theorem 2) for the valued field $(\overline{K},\|.\|_{\varphi})$. If instead of $(\overline{K},\|.\|_{\varphi})$ one takes its completion $(\widetilde{\overline{K}},\|.\|_{\tilde{\varphi}})$ relative to the φ-spectral norm $\|.\|_{\varphi}$, one

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obtains another two variants for Ostrowki's and Krasner's results, this time for a class of closed subrings of the ring $\widetilde{\overline{K}}$ (Theorem 4 and Corollary 1).

In Section 2 we prove that the class of triplets $(K, \varphi, \|.\|)$, where $\|.\|$ is an arbitrary equivariant (relative to $G = Gal(\overline{K}/K)$ norm, for which the Ostrowski's absorbent property for closed subrings of $\widetilde{\overline{K}}_{\|.\|}$ (the completion of $(\overline{K}, \|.\|)$) works, is the same with the class of triplets $(K, \varphi, \|.\|)$ for which Krasner's Lemma works (Theorem 5). In Definition 4 we introduce a new class of triplets $(K, \varphi, \|.\|)$, called *appropriate triples*. Shortly speaking, for such a triplet, any closed subring L of $\widetilde{\overline{K}}_{\|.\|}$ is completely defined by its algebraic part, i.e., $L = \widetilde{L \cap \overline{K}}$. They are important because for such triples one could have a Galois type theory which connects the set of closed subfields of $\widetilde{\overline{K}}_{\|.\|}$ and the set of closed subgroups of G. Moreover, this last group can be identified with the group of all continuous K-automorphisms of $\widetilde{\overline{K}}$. In 1 we discuss such a situation.

We also discuss the state of art of a Zaharescu's conjecture (Conjecture 1) for a more general case (see Corollary 2).

1. The spectral norm case. Let (K, φ) be a perfect valued field of rank 1, where φ is a nontrivial multiplicative Archimedean or non-Archimedean absolute value on K. Let $\overline{K}$ be a fixed algebraic closure of K and let $\overline{\varphi}$ be a fixed extension of φ to $\overline{K}$. We define on $\overline{K}$ the following norm, which will be called the φ-spectral norm of $\overline{K}$ (it does not depend on $\overline{\varphi}$!):

$$\|x\|_{\varphi} = \max\{\overline{\varphi}(\sigma(x)) : \sigma \in G\}, \tag{1.1}$$

where $x \in \overline{K}$ and $G = Gal(\overline{K}/K)$ is the absolute Galois group of K.

Remark 1. Since any other multiplicative valuation on $\overline{K}$ is of the form $\overline{\varphi} \circ \mu$, where $\mu \in G$ (see [7] or [5]) the φ-spectral norm does not depend on the choice of extension $\overline{\varphi}$ of φ to K. It is not complicated to prove (see also [1]) that this φ-spectral norm is indeed a K-norm on $\overline{K}$:

i) $\|x\|_{\varphi} = 0$ if and only if $x = 0$ for any x in $\overline{K}$.

ii) $\|\alpha x\|_{\varphi} = \varphi(\alpha)\|x\|_{\varphi}$ for any x in $\overline{K}$ and for any $\alpha \in K$.

iii) $\|xy\|_{\varphi} \leq \|x\|_{\varphi}\|y\|_{\varphi}$ for any x and y in $\overline{K}$.

iv) $\|x+y\|_{\varphi} \leq \max\{\|x\|_{\varphi}, \|y\|_{\varphi}\}$, if φ is non-Archimedean and $\|x+y\|_{\varphi} \leq \|x\|_{\varphi} + \|y\|_{\varphi}$, if φ is Archimedean.

v) $\|\sigma(x)\|_{\varphi} = \|x\|_{\varphi}$ for any x in $\overline{K}$ and for any $\sigma \in G$, i.e., the φ-spectral norm is G-equivariant.

Let c_* be equal to 1/2 if φ is Archimedean and $c_* = 1$ if φ is non-Archimedean. Let $L \subset \overline{K}$ be a subfield of the algebraic closure $\overline{K}$ of K such that $K \subset L$. For any $\alpha \in \overline{K}$ we define the φ-spectral distance of α to L as follows:

$$distspec_{\varphi}(L, \alpha) = \inf_{\beta \in L} \|\alpha - \beta\|_{\varphi}. \tag{1.2}$$

We shall prove later that $\alpha \in L$ if and only if $distspec_{\varphi}(L, \alpha) = 0$. Using a deep idea of Ostrowski ([8], or [5]) and looking at it at a more general level, we find the following result.

Theorem 1 (The absorbent theorem)**.** *Let $K \subset L \subset \overline{K}$ as above and let*

$$\omega(\alpha) = \min_{\sigma \in G}\Big\{\|\alpha - \sigma(\alpha)\|_{\varphi} : \alpha \neq \sigma(\alpha)\Big\},$$

if $\alpha \notin K$ and $\omega(\alpha) = 0$ if $\alpha \in K$. Let now $\alpha \in \overline{K} \setminus K$ such that $distspec_{\varphi}(L, \alpha) < c_*\omega(\alpha)$. Then $\alpha \in L$, i.e., L absorbs α. The same statement is true if instead of the φ-spectral norm $\|.\|_{\varphi}$ we take any φ-norm $\|.\|$ on $\overline{K}$, which is G-equivariant.

Proof. We assume on contrary that $\alpha \notin L$. Then, by using the classical Galois theory, there exists at least one $\sigma_0 \in G$ such that $\sigma_0(x) = x$ for all $x \in L$ and $\sigma_0(\alpha) \neq \alpha$. a) If φ is a non-Archimedean valuation ($c_* = 1$) then,

$$distspec_{\varphi}(L, \alpha) < \omega(\alpha) \leq \|\alpha - \sigma_0(\alpha)\|_{\varphi}$$
$$\leq \max\Big\{\|\alpha - x\|_{\varphi}, \|x - \sigma_0(\alpha)\|_{\varphi}\Big\},$$

for any $x \in L$. Since $\sigma_0(x) = x$ for any $x \in L$ and since

$$\|\alpha - x\|_{\varphi} = \|\sigma_0(\alpha) - \sigma_0(x)\|_{\varphi} = \|\sigma_0(\alpha) - x\|_{\varphi}, \tag{1.3}$$

we finally get:

$$distspec_{\varphi}(L, \alpha) < \omega(\alpha) \leq \|\alpha - x\|_{\varphi}$$

for any $x \in L$. Taking infimum on the right, we obtain:

$$distspec_{\varphi}(L, \alpha) < \omega(\alpha) \leq distspec_{\varphi}(L, \alpha),$$

a contradiction. b) If φ is an Archimedean valuation ($c_* = 1/2$) then:

$$distspec_{\varphi}(L, \alpha) < \frac{1}{2}\omega(\alpha) \leq \frac{1}{2}\|\alpha - \sigma_0(\alpha)\|_{\varphi}$$
$$\leq \frac{1}{2}\|\alpha - x\|_{\varphi} + \frac{1}{2}\|x - \sigma_0(\alpha)\|_{\varphi}.$$

But, as in (1.3), one has that

$$\|\alpha - x\|_{\varphi} = \|x - \sigma_0(\alpha)\|_{\varphi}$$

for any $x \in L$. So

$$distspec_{\varphi}(L, \alpha) < \frac{1}{2}\omega(\alpha) \leq \|\alpha - x\|_{\varphi}$$

for any $x \in L$. Taking infimum on the right, we get:

$$distspec_{\varphi}(L, \alpha) < \frac{1}{2}\omega(\alpha) \leq distspec_{\varphi}(L, \alpha),$$

a contradiction. Thus, in any of the two cases we obtain a contradiction. So $\alpha \in L$. □

Remark 2. Let $K, \overline{K}, L, \alpha$ be as above and assume that $\alpha \in \widetilde{L} \cap \overline{K}$, where $\widetilde{L}$ is the topological completion of L with respect to the φ-spectral norm $\|.\|_{\varphi}$. Then $distspec_{\varphi}(L, \alpha) = 0$ and, from the last theorem, one has that $\alpha \in L$. This means that L is topologically closed in $\overline{K}$. But this does not mean that L is complete relative to the φ-spectral norm, i.e., it is not closed in $\widetilde{\overline{K}}$, the completion of $\overline{K}$ relative to the same φ-spectral norm. In other words, its closure in $\widetilde{\overline{K}}$ does not contain algebraic elements besides those of L itself. To see that L is not complete in general, let us take $K = \mathbf{Q}_p$ and $L = \overline{K} = \overline{\mathbf{Q}}_p$. Then $\widetilde{L} = \mathbf{C}_p$ and we know (see [4] for instance) that $L \neq \widetilde{L}$ in this case. Moreover, it is not difficult to prove that for any infinite extension L of $\mathbf{Q}_p$, $L \neq \widetilde{L}$, where $\widetilde{L}$ is the topological closure of L in $\mathbf{C}_p$, the complex p-adic number field.

In particular we also get a generalization of the classical Krasner's lemma ([7], [4] or [5]).

Theorem 2 (Krasner's Lemma for $\overline{K}$)**.** *Let K, $\overline{K}$, φ, $\overline{\varphi}$ be as above and let α be an element of $\overline{K} \setminus K$. Let $y \in \overline{K}$ be such that $\|\alpha - y\|_{\varphi} < c_*\omega(\alpha)$, where $\omega(\alpha) = \min_{\sigma \in G}\{\|\alpha - \sigma(\alpha)\|_{\varphi} : \alpha \neq \sigma(\alpha)\}$. Then $K(\alpha) \subset K(y)$.*

Proof. It is sufficient to prove that $\alpha \in K(y)$. In view of Theorem 1, it is also sufficient to prove that $distspec_{\varphi}(K(y), \alpha) < c_*\omega(\alpha)$. Since

$$distspec_{\varphi}(K(y), \alpha) \leq \|\alpha - y\|_{\varphi} < c_*\omega(\alpha),$$

the desired condition is satisfied and the proof of the theorem is completed. □

Let $\widetilde{\overline{K}}$ be the completion of $\overline{K}$ with respect to the φ-spectral norm $\|.\|_{\varphi}$. It is easy to see that $\widetilde{\overline{K}}$ is in general a ring and that it is a field if and only if $\|.\|_{\varphi}$ is a multiplicative absolute value, i.e., if and only if $\overline{\varphi}$ is the unique extension of φ to $\overline{K}$, i.e., if and only if (K, φ) is henselian (see also [1]). Since $\widetilde{K}$, the topological closure of K in $\widetilde{\overline{K}}$, is a completion of (K, φ), we have enough (infinite) transcendental elements in $\widetilde{\overline{K}}$ over K. $\widetilde{\overline{K}}$ becomes a normed ring as follows. Let $x = \{\widehat{x_n}\}$ be the class of a Cauchy sequence $\{x_n\}$ with respect to the φ-spectral norm on $\overline{K}$, $x_n \in \overline{K}$ for any $n \in \mathbf{N}$. Since

$$\Big| \|x_{n+p}\|_{\varphi} - \|x_n\|_{\varphi} \Big| \leq \|x_{n+p} - x_n\|_{\varphi},$$

the sequence $\{\|x_n\|_{\varphi}\}$ is a Cauchy sequence and one can easily define

$$\|x\|_{\varphi}^{\sim} \stackrel{def}{=} \lim_{n\to\infty} \|x_n\|_{\varphi}.$$

This definition does not depend on the choice of the Cauchy sequence $\{x_n\}$ in the class of x. Now, if $x \in \overline{K}$, we can embed x in $\widetilde{\overline{K}}$ by the following ring morphism $x \rightsquigarrow (x, x, \ldots, x, \ldots)$. It is easy to see that $\|x\|_{\varphi}^{\sim} = \|x\|_{\varphi}$ for any $x \in \overline{K}$.

Assume in the following that $\widetilde{\overline{K}} \neq \overline{K}$, i.e., that there exists at least one element y in $\widetilde{\overline{K}}$ which is transcendental over K. Moreover, if $\alpha \in \overline{K}$, $y \in \widetilde{\overline{K}}$, transcendental over K, and if $\varepsilon > 0$, the "spectral open ball"

$$B(\alpha, \varepsilon) = \{z \in \widetilde{\overline{K}} : \|z - \alpha\|_{\varphi}^{\sim} < \varepsilon\}$$

contains an infinite number of transcendental elements of the form: $\alpha + ty$, $t \in K$, with $\varphi(t) < \frac{\varepsilon}{\|y\|_{\varphi}}$. Since φ is not the trivial absolute value, the set

$$\left\{ t \in K : \varphi(t) < \frac{\varepsilon}{\|y\|_{\varphi}^{\sim}} \right\}$$

is infinite.

Therefore, one can find in $\widetilde{\overline{K}}$ subfields $L \subsetneq \widetilde{\overline{K}}$ such that

$$distspec^{\sim}(L, \alpha) = \inf_{z \in L}\{\|z - \alpha\|_{\varphi}^{\sim}\}$$

is as small as we want. Take for instance $t \in K$ with $\varphi(t) < \frac{\varepsilon}{\|y\|_{\varphi}}$ and put $L = K(\alpha + ty)$. Then

$$distspec^{\sim}(L, \alpha) \leq \|\alpha + ty - \alpha\|_{\varphi}^{\sim} = \varphi(t)\|y\|_{\varphi}^{\sim} < \varepsilon.$$

Let us denote by the same letter G the group of all continuous (with respect to $\|.\|_{\varphi}^{\sim}$) automorphisms of $\widetilde{\overline{K}}$ over K. Each such automorphism σ is completely determined by its restriction to $\overline{K}$. Since $\|\sigma(x)\|_{\varphi} = \|x\|_{\varphi}$ for any ring automorphism σ of $\widetilde{\overline{K}}$ over K and for any $x \in \overline{K}$, we see that the restriction to $\overline{K}$ of any such ring automorphism of $\widetilde{\overline{K}}$ is continuous on $\overline{K}$, even it is not continuous on $\widetilde{\overline{K}}$. But, given $\mu \in Gal(\overline{K}/K)$, there is a unique extension of μ to a ring continuous automorphism $\widetilde{\mu}$ of $\widetilde{\overline{K}}$ over K. In what follows we consider only such continuous extensions. This is why $G = Gal(\overline{K}/K)$.

Theorem 3. *For any $\sigma \in G$ and $x \in \widetilde{\overline{K}}$ one has that*

$$\|\sigma(x)\|_{\varphi}^{\sim} = \|x\|_{\varphi}^{\sim},$$

i.e., $\|.\|_{\varphi}^{\sim}$ is an equivariant norm with respect to G.

Proof. Let $x_n \to x$, $x_n \in \overline{K}$, relative to $\|.\|_{\varphi}^{\sim}$. Since σ is continuous, one has that

$$\|\sigma(x_n)\|_{\varphi} \to \|\sigma(x)\|_{\varphi}^{\sim}.$$

But $\|.\|_{\varphi}$ is equivariant with respect to $G = Gal(\overline{K}/K)$ (see Remark 1), so $\|\sigma(x_n)\|_{\varphi} = \|x_n\|_{\varphi}$. Since $x_n \to x$ relative to $\|.\|_{\varphi}^{\sim}$, one has that

$$\|x_n\|_{\varphi} = \|x_n\|_{\varphi}^{\sim} \to \|x\|_{\varphi}^{\sim}.$$

The uniqueness of the limit of a sequence in a metric space implies that

$$\|\sigma(x)\|_{\varphi}^{\sim} = \|x\|_{\varphi}^{\sim},$$

i.e., the statement of the theorem. □

Definition 1. A perfect valued field (K, φ) of rank 1 with a nontrivial absolute value φ is said to be an appropriate field if for any closed subring $L \subset \widetilde{\overline{K}}$ one has that

$$\widetilde{L \cap \overline{K}} = L.$$

Here $\widetilde{M}$ means the topological closure of M in $\widetilde{\overline{K}}$ with respect to the norm $\|.\|_{\varphi}^{\sim}$ on $\overline{K}$.

For instance, if (K, φ) is a perfect complete field, then in [6] it is proved that (K, φ) is an appropriate field.

Remark 3. If (K, φ) is a henselian field then it is an appropriate field.

Example 1. Let $K = \mathbf{Q}$ be the rational number field and let $\varphi = |.|_p$ be the p-adic absolute value on $\mathbf{Q}$ for a fixed prime number p. Let $\|.\|_p$ be the $|.|_p$-spectral norm on $\overline{\mathbf{Q}}$, the field of algebraic numbers. Let $(\widetilde{\mathbf{Q}}_p, \|.\|_p^{\sim})$ be the completion of $\overline{\mathbf{Q}}$ with respect to $\|.\|_p$. Then, Theorem 6.3 of [10] says that $(\mathbf{Q}, |.|_p)$ is an appropriate field which is not henselian.

Remark 4. If (K, φ) is an appropriate field and if for any subring L, $K \subset L \subset \widetilde{\overline{K}}$, one defines:

$$distspec_{\varphi}^{\sim}(L, z) = \inf_{y \in L}\left\{\|y - z\|_{\varphi}^{\sim}\right\},$$

for any $z \in \widetilde{\overline{K}}$, the extended spectral distance with respect to φ, then we easily get:

$$(1.4) \quad distspec_{\varphi}^{\sim}(L, z) = distspec_{\varphi}^{\sim}(\widetilde{L}, z) = distspec_{\varphi}^{\sim}(\widetilde{\widetilde{L} \cap \overline{K}}, z) = distspec_{\varphi}^{\sim}(\widetilde{L} \cap \overline{K}, z).$$

We now extend Theorem 1 to closed subrings $L \subset \widetilde{\overline{K}}$ which are not necessarily algebraic over K.

Theorem 4. *Let (K, φ) be an appropriate field and L be a closed subring of $\widetilde{\overline{K}}$, $K \subset L$. Let $\alpha \in \overline{K} \setminus K$ such that $distspec_{\varphi}^{\sim}(L, \alpha) < c_* \omega(\alpha)$. Then $\alpha \in L$.*

Proof. Assume by contradiction that $\alpha \notin L$. Then $\alpha \notin L \cap \overline{K}$ which is an algebraic extension of K. Being a ring and an algebraic extension of K, it is a field. Then, the classical Galois theory says that there exists $\sigma_0 \in G = Gal(\overline{K}/K)$ such that $\sigma_0(\alpha) \neq \alpha$ and $\sigma_0(x) = x$ for all $x \in L \cap \overline{K}$.

Now the proof follows in the same manner like the proof of Theorem 1 by simply substituting L with $L \cap \overline{K}$. Finally we obtain that $\alpha \in L \cap \overline{K}$, i.e., $\alpha \in L$ and the proof of the theorem is completed. □

Corollary 1 (Krasner's Lemma for $\widetilde{\overline{K}}$). *Let (K, φ) be an appropriate field and let y be an element of $\widetilde{\overline{K}}$. Let α be in $\overline{K} \setminus K$ such that $\|\alpha - y\|_{\varphi}^{\sim} < c_* \omega(\alpha)$, where $\omega(\alpha) = \min_{\sigma \in G}\{\|\alpha - \sigma(\alpha)\|_{\varphi} : \alpha \neq \sigma(\alpha)\}$. Then $K(\alpha) \subset \widetilde{K(y)}$.*

The proof of this corollary is similar to the proof of Theorem 2 and we omit it.

Corollary 2 (a primitive element theorem for $\widetilde{\overline{K}}$). *Let (K, φ) be an appropriate field and let $\alpha \in \overline{K}$, $y \in \widetilde{\overline{K}}$. Then there exists an infinite number of elements $t \in K$ such that $\widetilde{K(\alpha, y)} = \widetilde{K(\alpha + ty)}$.*

Proof. Since $\alpha + ty \in K(\alpha, y)$ for any $t \in K$, it remains to prove that for some restrictions on $t \in K$ one has that $\alpha \in \widetilde{K(\alpha + ty)}$. In Theorem 4 we take $L = \widetilde{K(\alpha + ty)}$. If $y = 0$ we have nothing to prove. The same is true if $\alpha \in K$. Assume that $y \neq 0$ and $\alpha \notin K$. There exists an infinite number of elements $t \neq 0$ in K such that $\varphi(t) < \frac{c_* \omega(\alpha)}{\|y\|_{\varphi}^{\sim}}$ (φ is a nontrivial multiplicative absolute value!). For such a t one has:

$$distspec_{\varphi}^{\sim}(L, \alpha) \leq \|\alpha + ty - \alpha\|_{\varphi}^{\sim} = \varphi(t)\|y\|_{\varphi}^{\sim} < c_* \omega(\alpha).$$

Let us apply now Theorem 4 and find that $\alpha \in L$ and the theorem is completely proved. □

Remark 5. Let $K = \mathbf{Q}_p$, the p-adic number field and let $\varphi = |.|_p$ be the usual p-adic absolute value on $\mathbf{Q}_p$. Let $\overline{\mathbf{Q}}_p$ be a fixed algebraic closure of $\mathbf{Q}_p$ and let denote by the same letter φ the unique extension of φ to $\overline{\mathbf{Q}}_p$. Since $\mathbf{Q}_p$ is complete, the corresponding spectral norm on $\overline{\mathbf{Q}}_p$ is exactly φ. Hence, $\widetilde{\overline{\mathbf{Q}}}_p$, the completion of $\overline{\mathbf{Q}}_p$ with respect to this last spectral norm is exactly $\mathbf{C}_p$, the complex p-adic number field. Now, if one takes an arbitrary $y \in \mathbf{C}_p$ and an element $\alpha \in \overline{\mathbf{Q}}_p$, then Corollary 2 says that for any t small enough ($\varphi(t) < \frac{\omega(\alpha)}{\|y\|_\varphi}$, if $y \neq 0$ and $\alpha \notin \mathbf{Q}_p$) one has that $\widetilde{\mathbf{Q}_p(\alpha, y)} = \widetilde{\mathbf{Q}_p(\alpha + ty)}$. This is a proof of a particular case of an intricate conjecture proposed by Prof. Alexandru Zaharescu (Illinois University) in 2009.

Conjecture 1 (Zaharescu's conjecture)**.** Let x, y be two arbitrary elements in $\mathbf{C}_p$, the complex p-adic number field. Then there exists $t \in \mathbf{Q}_p$, the p-adic number field, such that $\widetilde{\mathbf{Q}_p(x, y)} = \widetilde{\mathbf{Q}_p(x + ty)}$. Here, tilde means the topological closure of the corresponding subfield of $\mathbf{C}_p$ with respect to the p-adic topology.

From [6] we know that there exists an element $z \in \mathbf{C}_p$ with $\widetilde{\mathbf{Q}_p(x, y)} = \widetilde{\mathbf{Q}_p(z)}$, but we do not know if there exists such a z (called a topological generator!) of the particular form $z = x + ty$, $t \in \mathbf{Q}_p$ like in the primitive element theorem case. Remark 5 says that Zaharescu's conjecture is true if one of the two elements x or y is algebraic over $\mathbf{Q}_p$. In general we have no answer for this interesting conjecture.

2. The case of a general norm. Let (K, φ) be a perfect valued field with a nontrivial multiplicative valuation φ. Let $\overline{K}$ be a fixed algebraic closure of K and let $\|.\|$ be an equivariant norm on $\overline{K}$ with respect to $G = Gal(\overline{K}/K)$, which extends φ. Let $\widetilde{\overline{K}}_{\|.\|}$ be a completion of $\overline{K}$ relative to $\|.\|$ and let $\|.\|^{\tilde{}}$ be the canonical extension of $\|.\|$ to $\widetilde{\overline{K}}_{\|.\|}$.

Definition 2. We say that the triplet $(K, \varphi, \|.\|)$ has the absorbent property if for any closed subring L of $\widetilde{\overline{K}}_{\|.\|}$, $K \subset L$, and for any $\alpha \in \overline{K} \setminus K$ with

$$dist_{\|.\|^{\tilde{}}}(L, \alpha) < c_* \omega(\alpha)$$

one has that $\alpha \in L$. Here

$$dist_{\|.\|^{\tilde{}}}(L, \alpha) = \inf_{y \in L} \left\{ \|y - \alpha\|^{\tilde{}} \right\}$$

and $c_* = 1$ or $c_* = \frac{1}{2}$ whenever φ is non-Archimedean or Archimedean respectively.

For instance, if (K, φ) is complete then, relative to the unique extension $\overline{\varphi}$ of φ to $\overline{K}$ the triplet $(K, \varphi, \overline{\varphi})$ has the absorbent property (see [6] and Theorem 4).

Definition 3. Let us preserve the above notation and hypotheses. We say that the triplet $(K, \varphi, \|.\|)$ verifies Krasner's Lemma if for any $\alpha \in \overline{K} \setminus K$ and $y \in \widetilde{\overline{K}}_{\|.\|}$ with $\|y - \alpha\|^{\tilde{}} < c_* \omega(\alpha)$ one has that $\alpha \in \widetilde{K(y)}$.

Theorem 5. *The triplet $(K, \varphi, \|.\|)$ has the absorbent property if and only if it verifies Krasner's Lemma.*

Proof. a) Assume that $(K, \varphi, \|.\|)$ has the absorbent property. Let $\alpha \in \overline{K} \setminus K$ and $y \in \widetilde{\overline{K}}_{\|.\|}$ with $\|y - \alpha\|^{\tilde{}} < c_* \omega(\alpha)$. Since

$$dist_{\|.\|^{\tilde{}}}(\widetilde{K(y)}, \alpha) \leq \|y - \alpha\|^{\tilde{}} < c_* \omega(\alpha)$$

and since $(K, \varphi, \|.\|)$ has the absorbent property, one obtain that $\alpha \in \widetilde{K(y)}$.

b) Conversely, we suppose that $(K, \varphi, \|.\|)$ verifies Krasner's Lemma. Let L, $K \subset L \subset \widetilde{\overline{K}}_{\|.\|}$ be a closed subring in $\widetilde{\overline{K}}_{\|.\|}$, which contains K. Let $\alpha \in \overline{K} \setminus K$ be such that $dist_{\|.\|^{\tilde{}}}(L, \alpha) < c_* \omega(\alpha)$. Then there exists at least one $\beta \in L$ with

$$\|\beta - \alpha\|^{\tilde{}} < c_* \omega(\alpha).$$

Since $(K, \varphi, \|.\|)$ verifies Krasner's Lemma we get that $\alpha \in \widetilde{K(\beta)} \subset L$, because L is closed, i.e., $\alpha \in L$, so $(K, \varphi, \|.\|)$ has the absorbent property and the proof is completed. □

Definition 4. With the above notation and hypotheses, we say that the triplet $(K, \varphi, \|.\|)$ is an appropriate triplet if for any closed subring L, $K \subset L \subset \widetilde{\overline{K}}_{\|.\|}$ one has that $\widetilde{L \cap \overline{K}} = L$.

For instance, if (K, φ) is complete and if $\|x\| = \overline{\varphi}(x)$ for any $x \in \overline{K}$, where $\overline{\varphi}$ is the unique extension of φ to $\overline{K}$, then the triplet $(K, \varphi, \overline{\varphi})$ is an appropriate triplet (see [6]).

It is not so difficult to prove the corresponding generalization of Theorem 4.

Theorem 6. *Let $(K, \varphi, \|.\|)$ be an appropriate triple. Then $(K, \varphi, \|.\|)$ has the absorbent property.*

Proof. Let $(K, \varphi, \|.\|)$ be an appropriate triple and let L a closed subring of $\widetilde{\overline{K}}_{\|.\|}$. Let α be in $\overline{K} \setminus K$ such that $dist_{\|.\|^{\tilde{}}}(L, \alpha) < c_* \omega(\alpha)$. Since $L = \widetilde{L \cap \overline{K}}$ one has that

$$dist_{\|.\|^{\tilde{}}}(L, \alpha) = dist_{\|.\|^{\tilde{}}}(L \cap \overline{K}, \alpha) < c_* \omega(\alpha).$$

From Theorem 1 we get that $\alpha \in L \cap \overline{K} \subset L$, i.e., $(K, \varphi, \|.\|)$ has the absorbent property. □

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