On the subordination under Bernardi operator

By Janusz SOKÓŁ[*)] and Mamoru NUNOKAWA[**)]

(Communicated by Kenji FUKAYA, M.J.A., Dec. 12, 2012)

Abstract: Let $\mathcal{H}$ denote the class of analytic functions in the unit disc on the complex plane $\mathbf{C}$. Let $\mathcal{E}$ be a subclass of $\mathcal{H}$. If the operator $I:\mathcal{E}\to\mathcal{H}$ satisfies

$$f(z)\prec g(z)\Rightarrow I[f](z)\prec I[g](z)$$

for all $f,g\in\mathcal{E}$, then it is called subordination-preserving operator on the class $\mathcal{E}$. In this work we consider the convexity of the Bernardi operator. We prove also that the Bernardi is the subordination-preserving operator on the class of starlike functions. The applications of main results are also presented.

Key words: Convex functions; Hadamard product; Bernardi operator; Libera operator; preserving operator; subordination.

1. Introduction. Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $\mathbf{U}=\{z:\ |z|<1\}$ on the complex plane $\mathbf{C}$. For $a\in\mathbf{C}$ and $n\in\mathbf{N}$ we denote by

$$\mathcal{H}[a,n]=\{f\in\mathcal{H}:f(z)=a+a_nz^n+\cdots\}$$

and

$$\mathcal{A}_n=\{f\in\mathcal{H}:f(z)=z+a_{n+1}z^{n+1}+\cdots\},$$

so $\mathcal{A}=\mathcal{A}_1$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ whose members are univalent in $\mathbf{U}$.

The class $\mathcal{S}^*_\alpha$ of starlike functions of order $\alpha<1$ may be defined as

$$\mathcal{S}^*_\alpha=\left\{f\in\mathcal{A}:\ \mathfrak{Re}\frac{zf'(z)}{f(z)}>\alpha,\ z\in\mathbf{U}\right\}.$$

The class $\mathcal{S}^*_\alpha$ and the class $\mathcal{K}_\alpha$ of convex functions of order $\alpha<1$

$$\begin{aligned}\mathcal{K}_\alpha:&=\left\{f\in\mathcal{A}:\ \mathfrak{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha,\ z\in\mathbf{U}\right\}\\&=\{f\in\mathcal{A}:\ zf'\in\mathcal{S}^*_\alpha\}\end{aligned}$$

were introduced by Robertson in [7]. If $\alpha\in[0,1)$, then a function in either of these sets is univalent, if $\alpha<0$ it may fail to be univalent. In particular we denote $\mathcal{S}^*_0=\mathcal{S}^*,\mathcal{K}_0=\mathcal{K}$, the classes of starlike and convex functions, respectively. Recall that $f\in\mathcal{A}$ is said to be in the class $\mathcal{C}_\alpha$, [3], of close-to-convex functions of order α, $\alpha<1$, if and only if there exist $g\in\mathcal{S}^*_\alpha$, $\varphi\in\mathbf{R}$, such that

$$\mathfrak{Re}\,e^{i\varphi}\frac{zf'(z)}{g(z)}>0,\quad z\in\mathbf{U}.$$

For $f(z)=a_0+a_1z+a_2z^2+\cdots$ and $g(z)=b_0+b_1z+b_2z^2+\cdots$ the Hadamard product (or convolution) is defined by $(f*g)(z)=a_0b_0+a_1b_1z+a_2b_2z^2+\cdots$. If $X,Y\subset\mathcal{H}$ we also use the notation

$$X*Y:=\{f*g:f\in X,\ g\in Y\}.$$

The convolution has the algebraic properties of ordinary multiplication. The class $\mathcal{A}$ of analytic functions is closed under convolution, that is $\mathcal{A}*\mathcal{A}=\mathcal{A}$. In 1973, Rusheweyh and Sheil-Small [10] proved the Pòlya-Schoenberg conjecture that the class of convex functions is preserved under convolution: $\mathcal{K}*\mathcal{K}=\mathcal{K}$. Many other convolution problems were studied by St. Rusheweyh in [9] and have found many applications in various fields.

We say that the $f\in\mathcal{H}$ is subordinate to $g\in\mathcal{H}$ in the unit disc $\mathbf{U}$, written $f\prec g$ if and only if there exits an analytic function $w\in\mathcal{H}$ such that $w(0)=0,|w(z)|<1$ and $f(z)=g[w(z)]$ for $z\in\mathbf{U}$. Therefore, $f\prec g$ in $\mathbf{U}$ implies $f(\mathbf{U})\subset g(\mathbf{U})$. In particular if g is univalent in $\mathbf{U}$, then

$$f\prec g\ \Leftrightarrow\ [f(0)=g(0)\text{ and }f(\mathbf{U})\subset g(\mathbf{U})]. \tag{1.1}$$

2. Main result. The Alexander integral operator is defined by

2010 Mathematics Subject Classification. Primary 30C45; Secondary 30C80.

*) Department of Mathematics, Rzeszów University of Technology, Al. Powstańców Warszawy 12, 35-959 Rzeszów, Poland.

**) University of Gunma, Hoshikuki-cho 798-8, Chuou-Ward, Chiba 260-0808, Japan.

doi: 10.3792/pjaa.89.11

$$\mathrm{A} : \mathcal{A}_n \to \mathcal{A}_n, \quad \mathrm{A}[f](z) = \int_0^z \frac{f(t)}{t}\,\mathrm{d}t,$$

while

$$\mathrm{L} : \mathcal{H} \to \mathcal{H}, \quad \mathrm{L}[f](z) = \frac{2}{z}\int_0^z f(t)\,\mathrm{d}t$$

is the Libera operator [5]. The above operators A and L are the special cases of the Bernardi operator [1] which is defined for $k = 0$ and for $k \in \mathbf{C}$, $\mathfrak{Re}\{k\} > 0$, by

$$\mathrm{L}_k : \mathcal{H} \to \mathcal{H}, \quad \mathrm{L}_k[f](z) = \frac{1+k}{z^k}\int_0^z f(t)t^{k-1}\,\mathrm{d}t.$$

It is easy to see that

$$\mathrm{L}_k : \mathcal{A}_n \to \mathcal{A}_n, \quad \mathrm{L}_k : \mathcal{H}[a,n] \to \mathcal{H}[a(1+k)/k, n].$$

Using the convolution we can write for $f \in \mathcal{H}[a,n]$

$$\mathrm{L}_k[f](z) = f(z) * \sum_{n=0}^{\infty} \frac{k+1}{k+n} z^n. \tag{2.1}$$

The classes $\mathcal{S}^*$ and $\mathcal{K}$ are preserved under each of these operators whenever $\mathfrak{Re}\{k\} > 0$, Ruscheweyh [8] (earlier Bernardi [1] if k is a positive integer), i.e.: $\mathrm{L}_k[\mathcal{K}] \subset \mathcal{K}$, $\mathrm{L}_k[\mathcal{S}^*] \subset \mathcal{S}^*$.

We shall need the following lemma.

Lemma 2.1 ([6, p. 35]). *Suppose that the function $\Psi : \mathbf{C}^2 \times \mathbf{U} \to \mathbf{C}$ satisfies the condition $\mathfrak{Re}\{\Psi(i\varrho, \sigma)\} \le \delta$ for real $\varrho, \sigma \le -n(1+\varrho^2)/2$ and all $z \in \mathbf{U}$. If $q(z) = 1 + a_n z^n + \ldots$ is analytic in $\mathbf{U}$ and*

$$\mathfrak{Re}\{\Psi(q(z), zq'(z))\} > \delta$$

for $z \in \mathcal{U}y$, then $\mathfrak{Re}\{q(z)\} > 0$ in $\mathbf{U}$.

We note that Lemma 2.1 is a corollary of the fundamental result in theory of differential subordinations deeply developed by Miller and Mocanu [6]. The function Ψ is called admissible function.

Theorem 2.2. *Let f be in the class $\mathcal{A}_n$ and k be a non-negative real number. If*

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \delta_0(k) = \begin{cases} -nk/2 & \text{for } 0 \le k \le 1, \\ -n/(2k) & \text{for } k > 1, \end{cases} \tag{2.2}$$

for $z \in \mathbf{U}$, then $\mathrm{L}_k[f]$ is convex univalent function.

Proof. After some calculation we obtain

$$q(z) + \frac{zq'(z)}{k + q(z)} = 1 + \frac{zf''(z)}{f'(z)}, \tag{2.3}$$

where

$$q(z) = 1 + \frac{z(\mathrm{L}_k[f](z))''}{(\mathrm{L}_k[f](z))'}. \tag{2.4}$$

It is known that $\mathrm{L}_k : \mathcal{A}_n \to \mathcal{A}_n$, thus $\mathrm{L}_k[f]$ is of the form $\mathrm{L}_k[f](z) = z + a_{n+1}z^{n+1} + \cdots$. If q is of the form $q(z) = 1 + c_1 z + c_2 z^2 + \cdots$, then differentiating

$$z(\mathrm{L}_k[f](z))'' = (q(z) - 1)(\mathrm{L}_k[f](z))'$$

and comparing the coefficients of both sides we obtain one after the other

$$c_1 = c_2 = \ldots = c_{n-1} = 0, \quad c_n = n(n+1)a_{n+1}, \ldots.$$

Therefore, $q(z) = 1 + n(n+1)a_{n+1}z^n + \cdots$. To make use of Lemma 2.1 we consider the function

$$\Psi(r, s) = r + \frac{s}{k + r}$$

and $\delta = \delta_0(k)$. Then by (2.2), (2.3) we have $\mathfrak{Re}\{\Psi(q(z), zq'(z))\} > \delta$, furthermore

$$\mathfrak{Re}\{\Psi(i\varrho, \sigma)\} = \mathfrak{Re}\left(i\varrho + \frac{\sigma}{k + i\varrho}\right) = \frac{k\sigma}{k^2 + \varrho^2}.$$

If $\sigma \le -n(1+\varrho^2)/2$, then

$$\frac{k\sigma}{k^2+\varrho^2} \le -\frac{nk(1+\varrho^2)}{2(k^2+\varrho^2)} \le \delta_0(k).$$

Applying Lemma 2.1 with we obtain that $\mathfrak{Re}\{q(z)\} > 0$ for $z \in \mathbf{U}$, hence trough (2.4) we see that $\mathrm{L}_k[f]$ is the convex univalent function whenever f satisfies (2.2). □

The above theorem is a generalization of the following one which is obtained from Theorem 2.2 with $k = n = 1$.

Corollary 2.3 ([6, p. 66]). *Let f be in the class $\mathcal{A}$. If*

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > -\frac{1}{2}$$

for $z \in \mathbf{U}$, then the function

$$\mathrm{L}[f](z) = \frac{2}{z}\int_0^z f(t)\,\mathrm{d}t$$

is in the class $\mathcal{K}$ of convex univalent functions.

The above property of the Libera operator L extends an earlier result in [5] that $\mathrm{L}[\mathcal{K}] \subset \mathcal{K}$. Note that the operator L is well defined in the whole class $\mathcal{H}$.

Corollary 2.4. *Let f be in the class $\mathcal{A}_n$ and let k be a non-negative real number. Assume also* *that f satisfies condition* (2.2)*. Then we have*

(i) *If $g \in \mathcal{C}_\alpha$ then $\mathrm{L}_k[g * f] \in \mathcal{C}_\alpha$,*

(ii) *If $g \in \mathcal{S}^*_\alpha$ then $\mathrm{L}_k[g * f] \in \mathcal{S}^*_\alpha$,*

(iii) *If $g \in \mathcal{K}_\alpha$ then $\mathrm{L}_k[g * f] \in \mathcal{K}_\alpha$.*

Proof. It is known [10], that the classes $\mathcal{C}_\alpha$, $\mathcal{S}^*_\alpha$ and $\mathcal{K}_\alpha$ are closed under convolution with convex univalent and normalized functions. Because $\mathrm{L}_k[g * f] = g * \mathrm{L}_k[f]$ and by Theorem 2.2 $\mathrm{L}_k[f] \in \mathcal{K}$ the results (i)–(iii) becomes obvious. □

Corollary 2.5. *Let f be in the class $\mathcal{S}$ and let k be a non-negative real number. If $r > 0$ satisfies*

$$\frac{r^2 - 4r + 1}{1 - r^2} \geq \delta_0(k),$$

with $\delta_0(k)$ given in (2.2)*, then $\mathrm{L}_k[f]$ is convex univalent in the disc $|z| < r$.*

Proof. It is known that $f \in \mathcal{S}$, then for $z = re^{it}$

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{r^2 - 4r + 1}{1 - r^2}.$$

Therefore, by Theorem 2.2 the function $\mathrm{L}_k[f]$ is convex univalent in the disc $|z| < r$. □

We have $r^2 - 4r + 1 > 0$ for $0 \leq r < 2 - \sqrt{3} \approx 0.2679$, while $\delta_0(k) \leq 0$. Therefore, if $f \in \mathcal{S}$, k is a non-negative real number, and $0 \leq r < 2 - \sqrt{3}$, then $\mathrm{L}_k[f]$ is convex univalent in the disc $|z| < r$. The above corollary for the Koebe function $f(z) = z/(1 - z)^2$ and $k = 1$ becomes the following one.

Corollary 2.6. *The function*

$$\mathrm{L}_1[z/(1 - z)^2] = 2\left\{\frac{1}{1 - z} + \frac{1}{z}\log(1 - z)\right\} = \sum_{n=1}^{\infty} \frac{2n}{n + 1} z^n$$

is convex univalent in the disc $|z| < 4 - \sqrt{13} \approx 0.39$.

Corollary 2.7. *Let h be in the class $\mathcal{A}_n$ and k be a non-negative real number. Assume that*

$$\mathfrak{Re}\left\{1 + \frac{zh''(z)}{h'(z)}\right\} > \delta_0(k) = \begin{cases} -nk/2 & \textit{for } 0 \leq k \leq 1, \\ -n/(2k) & \textit{for } k > 1, \end{cases} \tag{2.5}$$

for $z \in \mathbf{U}$. Assume also that $g(z) = a + b_n z^n + b_{n+1} z^{n+1} + \cdots$ is analytic in $\mathbf{U}$. If

$$g(z) + \frac{zg'(z)}{c} \prec \mathrm{L}_k[h] \quad (z \in \mathbf{U}) \tag{2.6}$$

for $\mathfrak{Re}[c] \geq 0$, $c \neq 0$, then

$$g(z) \prec q_n(z) \prec \mathrm{L}_k[h] \quad (z \in \mathbf{U}), \tag{2.7}$$

where $q_n(z) = \frac{c}{nz^{c/n}} \int_0^z t^{c/n - 1} \mathrm{L}_k[h](t)\mathrm{d}t$. Moreover, the function $q_n(z)$ is convex univalent and is the best dominant of (2.6) *in the sense that $g \prec q_n$ for all g satisfying* (2.6)*, and if there exists q such that $g \prec q$ for all g satisfying* (2.6)*, then $q_n \prec q$.*

Proof. It is known [2] that the subordination (2.6) with convex univalent right-hand side is sufficient for (2.7) with the best dominant $q_n(z)$. By Theorem 2.2 the function $\mathrm{L}_k[h]$ is convex univalent in the unit disc and we get the result. □

Notice that the function $q_n(z)$ is the Bernardi integral operator on the function $\mathrm{L}_k[h]$:

$$q_n(z) = \frac{1}{1 + n} \mathrm{L}_{c/n}[\mathrm{L}_k[h] - a](z) + a.$$

Theorem 2.8. *Assume that k is a complex number with $\mathfrak{Re}\{k\} > 0$, or $k = 0$. If $g \in \mathcal{H}$ and f is in the class $\mathcal{S}^*$ of starlike functions, then*

$$g \prec f \Rightarrow \mathrm{L}_k[g] \prec \mathrm{L}_k[f]. \tag{2.8}$$

Proof. The class $\mathcal{S}^*$ is preserved under the operator L_k whenever $k = 0$ or $\mathfrak{Re}\{k\} > 0$, Ruscheweyh [8], i.e.: $\mathrm{L}_k[\mathcal{S}^*] \subset \mathcal{S}^*$. This fact was proved in [4] too. Note that if $f \in \mathcal{S}$ only, then $\mathrm{L}_k[f]$ may be infinite-valent in the unit disc. Because $\mathrm{L}_k[f]$ is univalent, then there exists a function w, $w(0) = 0$, such that in a disc $|z| < r_0 \leq 1$

$$\mathrm{L}_k[g](z) = \mathrm{L}_k[f](w(z)). \tag{2.9}$$

If $\mathrm{L}_k[g] \not\prec \mathrm{L}_k[f]$, then there exists a $z_0 \in \mathcal{U}$, such that $|w(z_0)| = 1$.

From (2.9) we have

$$z^k \mathrm{L}_k[g](z) = z^k \mathrm{L}_k[f](w(z)),$$

hence by (2.1)

$$z^k g(z) * \sum_{n=1}^{\infty} \frac{k + 1}{k + n} z^{k+n} = z^k f(w(z)) * \sum_{n=1}^{\infty} \frac{k + 1}{k + n} z^{k+n}. \tag{2.10}$$

The property $z(p(z) * q(z))' = p(z) * zq'(z)$ used in (2.10) yields

$$z^k g(z) * \sum_{n=1}^{\infty} (k + 1) z^{k+n} = z^k f(w(z)) * \sum_{n=1}^{\infty} (k + 1) z^{k+n}, \tag{2.11}$$

or, equivalently

$$g(z) = f(w(z)) \tag{2.12}$$

Because f is starlike univalent and there exists a $z_0 \in \mathbf{U}$, such that $|w(z_0)| = 1$, we obtain a contradiction with $g \prec f$. □

Finally, we give the two applications of Theorem 2.2. If we consider for $a \in [1, 2]$ the function

$$p_a(z) = \frac{1}{a}\left\{\frac{1}{(1-z)^a} - 1\right\} = z + \frac{a+1}{2!}z^2 + \cdots = \sum_{n=1}^{\infty}\frac{(a)_n}{n!a}z^n \quad z \in \mathbf{U}, \tag{2.13}$$

then $p_a \in \mathcal{A}_1$ and it satisfies

$$\mathfrak{Re}\left(1 + \frac{zp_a''(z)}{p_a'(z)}\right) = \mathfrak{Re}\,\frac{1+az}{1-z} > -\frac{a-1}{2} \quad z \in \mathbf{U},$$

thus p_a satisfies condition (2.2) with $k = a-1$ such that $0 \le k \le 1$. Therefore, in this case, by Theorem 2.2 and by (2.1) the function

$$\mathrm{L}_{a-1}[p_a](z) = p_a(z) * \sum_{n=0}^{\infty}\frac{a}{a-1+n}z^n = \sum_{n=1}^{\infty}\frac{(a)_n}{(a-1+n)n!}z^n$$

is convex univalent function.

Secondly, considering for $l \in [1, 2]$ the function

$$r_l(z) = \frac{z}{(1+z^l)^{1/l}} = z\left(\sum_{n=0}^{\infty}\frac{(1/l)_n}{n!}z^{ln}\right) \quad z \in \mathbf{U},$$

it is easy to check that $r_l \in \mathcal{A}_1$ and

$$\mathfrak{Re}\left(1 + \frac{zr_l''(z)}{r_l'(z)}\right) = \frac{1-lz^l}{1+z^l} > -\frac{l-1}{2} \quad z \in \mathbf{U}.$$

Therefore, r_l satisfies condition (2.2) with $k = l-1$ such that $0 \le k \le 1$. By Theorem 2.2 the function

$$\mathrm{L}_{l-1}[r_l](z) = r_l(z) * \sum_{n=0}^{\infty}\frac{l}{l-1+n}z^n = \sum_{n=0}^{\infty}\frac{l(1/l)_n}{(l-1+ln+1)n!}z^{ln+1} = \sum_{n=0}^{\infty}\frac{(1/l)_n}{(1+n)n!}z^{ln+1}$$

is convex univalent function.

Acknowledgment. The authors would like to express their sincerest thanks to the referees for a careful reading and various suggestions made for the improvement of the paper.

References

[1] S. D. Bernardi, Convex and starlike univalent functions, Trans. Am. Math. Soc. **135** (1969), 429–446.

[2] D. J. Hallenbeck and St. Ruscheweyh, Subordination by convex functions, Proc. Am. Math. Soc. **52** (1975), 191–195.

[3] W. Kaplan, Close-to-convex schlicht functions, Michigan Math. J. **1** (1952), issue 2, 169–185.

[4] Z. Lewandowski, S. Miller and E. Złotkiewicz, Generating functions for some classes of univalent functions, Proc. Am. Math. Soc. **56** (1976), 111–117.

[5] R. J. Libera, Some classes of regular univalent functions, Proc. Am. Math. Soc. **16** (1965), 755–758.

[6] S. S. Miller and P. T. Mocanu, *Differential subordinations*, Monographs and Textbooks in Pure and Applied Mathematics, 225, Dekker, New York, 2000.

[7] M. I. S. Robertson, On the theory of univalent functions, Ann. of Math. (2) **37** (1936), no. 2, 374–408.

[8] St. Ruscheweyh, New criteria for univalent functions, Proc. Am. Math. Soc. **49** (1975), 109–115.

[9] St. Ruscheweyh, *Convolutions in geometric function theory*, Séminaire de Mathématiques Supérieures, 83, Presses Univ. Montréal, Montreal, QC, 1982.

[10] St. Ruscheweyh and T. Sheil-Small, Hadamard products of Schlicht functions and the Pólya-Schoenberg conjecture, Comment. Math. Helv. **48** (1973), 119–135.