Note on non-discrete complex hyperbolic triangle groups of type $(n, n, \infty; k)$

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(Communicated by Kenji FUKAYA, M.J.A., Sept. 12, 2013)

Abstract: A complex hyperbolic triangle group is a group generated by three complex involutions fixing complex lines in complex hyperbolic space. In a previous paper [3] we discussed complex hyperbolic triangle groups of type $(n, n, \infty; k)$ and proved that for $n \geq 29$ these groups are not discrete. In this paper we show that if $n \geq 22$, then complex hyperbolic triangle groups of type $(n, n, \infty; k)$ are not discrete and give a new list of non-discrete groups of type $(n, n, \infty; k)$.

Key words: Complex hyperbolic triangle group; complex involution.

1. Introduction. Let n and k be positive integers. Let I_1, I_2, I_3 be the following matrices:

$$I_1 = \begin{bmatrix} 1 & \rho & \overline{\tau} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \; I_2 = \begin{bmatrix} -1 & 0 & 0 \\ \overline{\rho} & 1 & \sigma \\ 0 & 0 & -1 \end{bmatrix},$$

$$I_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \tau & \overline{\sigma} & 1 \end{bmatrix}.$$

Assume that ρ, σ, τ satisfy the conditions $|\rho| = 2\cos(\pi/n)$, $|\sigma| = 2$, $|\tau| = 2\cos(\pi/n)$, $|\rho\tau - \overline{\sigma}| = 2\cos(\pi/k)$. Then we have that $I_1^2 = I_2^2 = I_3^2 = (I_1I_2)^n = (I_3I_1)^n = (I_1I_2I_1I_3)^k = E$ (the identity matrix) and I_2I_3 is a parabolic element. We call the group generated by I_1, I_2 and I_3 a *complex hyperbolic triangle group of type* $(n, n, \infty; k)$ and denote it by $\Gamma(n, n, \infty; k)$. Up to conjugation, there is a one-parameter family of these groups parametrized by $k(\geq [n/2] + 1)$.

It is interesting to ask which values of the parameter correspond to discrete groups as mentioned in [5]. This will be done in a case by case fashion, so we try to restrict the range to search for discrete groups by finding non-discrete groups. In this paper we improve our previous result in [3] and give a new list of non-discrete groups of type $(n, n, \infty; k)$.

2. Non-discrete groups. To show a group of type $(n, n, \infty; k)$ to be non-discrete, we use the following version of Jørgensen's inequality.

Lemma 2.1 ([4, Corollary 2.3]). *Let Γ be a complex hyperbolic triangle group of type $(n, n, \infty; k)$. If $n \geq 7$ and*

$$(\rho\sigma\tau + \overline{\rho}\,\overline{\sigma}\,\overline{\tau} - 2|\sigma|^2 - 2|\tau|^2 - |\rho|^2 + 4)^2 < 4 - |\rho|^2,$$

then Γ is not discrete.

From the conditions on ρ, σ, τ, we have

$$\rho\sigma\tau + \overline{\rho}\,\overline{\sigma}\,\overline{\tau} = 16\cos^4\frac{\pi}{n} - 2\cos\frac{2\pi}{k} + 2.$$

Hence, we can rewrite the inequality in Lemma 2.1 as

$$\left(\cos\frac{2\pi}{k} + 1 - 8\cos^4\frac{\pi}{n} + 6\cos^2\frac{\pi}{n}\right)^2 < \sin^2\frac{\pi}{n}.$$

Let $a_n = -1 + 8\cos^4(\pi/n) - 6\cos^2(\pi/n) - \sin(\pi/n)$ and $b_n = -1 + 8\cos^4(\pi/n) - 6\cos^2(\pi/n) + \sin(\pi/n)$. We immediately obtain.

Lemma 2.2. *Let Γ be a complex hyperbolic triangle group of type $(n, n, \infty; k)$ with $n \geq 7$. If $a_n < \cos(2\pi/k) < b_n$, then Γ is not discrete.*

We are ready to state our theorem, which shows a new list of non-discrete groups of type $(n, n, \infty; k)$. In particular, we prove that $\Gamma(n, n, \infty; k)$ for $n \geq 22$ and $k \geq [n/2] + 1$ is not discrete.

Theorem 2.3. *Let $\Gamma = < I_1, I_2, I_3 >$ be a complex hyperbolic triangle group of type $(n, n, \infty; k)$ with $k \geq [n/2] + 1$. The following groups are non-discrete.*

(1) $\Gamma(5, 5, \infty; 3)$.
(2) $\Gamma(6, 6, \infty; 5)$.

2010 Mathematics Subject Classification. Primary 22E40, 32Q45, 51M10.

doi: 10.3792/pjaa.89.100

(3) $\Gamma(7,7,\infty;4)$, $\Gamma(7,7,\infty;6)$.
(4) $\Gamma(8,8,\infty;5)$.
(5) $\Gamma(9,9,\infty;5)$, $\Gamma(9,9,\infty;6)$.
(6) $\Gamma(10,10,\infty;6)$, $\Gamma(10,10,\infty;9)$.
(7) $\Gamma(11,11,\infty;6)$, $\Gamma(11,11,\infty;10)$, $\Gamma(11,11,\infty;11)$.
(8) $\Gamma(12,12,\infty;7)$, *and* $\Gamma(12,12,\infty;k)$ *for* $11 \le k \le 16$.
(9) $\Gamma(13,13,\infty;7)$, *and* $\Gamma(13,13,\infty;k)$ *for* $12 \le k \le 38$.
(10) $\Gamma(14,14,\infty;8)$, *and* $\Gamma(14,14,\infty;k)$ *for* $k \ge 12$.
(11) $\Gamma(15,15,\infty;8)$, $\Gamma(15,15,\infty;9)$, $\Gamma(15,15,\infty;10)$, *and* $\Gamma(15,15,\infty;k)$ *for* $k \ge 13$.
(12) $\Gamma(16,16,\infty;9)$, $\Gamma(16,16,\infty;10)$, *and* $\Gamma(16,16,\infty;k)$ *for* $k \ge 14$.
(13) $\Gamma(17,17,\infty;9)$, $\Gamma(17,17,\infty;10)$, $\Gamma(17,17,\infty;11)$, *and* $\Gamma(17,17,\infty;k)$ *for* $k \ge 15$.
(14) $\Gamma(18,18,\infty;10)$, $\Gamma(18,18,\infty;11)$, $\Gamma(18,18,\infty;12)$, *and* $\Gamma(18,18,\infty;k)$ *for* $k \ge 16$.
(15) $\Gamma(19,19,\infty;10)$, $\Gamma(19,19,\infty;11)$, $\Gamma(19,19,\infty;12)$, $\Gamma(19,19,\infty;13)$, *and* $\Gamma(19,19,\infty;k)$ *for* $k \ge 17$.
(16) $\Gamma(20,20,\infty;11)$, $\Gamma(20,20,\infty;12)$, $\Gamma(20,20,\infty;13)$, $\Gamma(20,20,\infty;14)$, $\Gamma(20,20,\infty;15)$ *and* $\Gamma(20,20,\infty;k)$ *for* $k \ge 18$.
(17) $\Gamma(21,21,\infty;11)$, $\Gamma(21,21,\infty;12)$, $\Gamma(21,21,\infty;13)$, $\Gamma(21,21,\infty;14)$, $\Gamma(21,21,\infty;15)$, $\Gamma(21,21,\infty;16)$, *and* $\Gamma(21,21,\infty;k)$ *for* $k \ge 19$.
(18) $\Gamma(22,22,\infty;k)$ *for any* k (≥ 12).
(19) $\Gamma(n,n,\infty;k)$ *for any* n (> 22).

Proof. First, from Lemma 2.2, it is readily verified that the following are not discrete:

1) $\Gamma(11,11,\infty;6)$;
2) $\Gamma(12,12,\infty;7)$;
3) $\Gamma(13,13,\infty;7)$;
4) $\Gamma(14,14,\infty;8)$;
5) $\Gamma(15,15,\infty;8)$ and $\Gamma(15,15,\infty;9)$;
6) $\Gamma(16,16,\infty;9)$ and $\Gamma(16,16,\infty;10)$;
7) $\Gamma(17,17,\infty;9)$, $\Gamma(17,17,\infty;10)$, and $\Gamma(17,17,\infty;11)$;
8) $\Gamma(18,18,\infty;10)$, $\Gamma(18,18,\infty;11)$, and $\Gamma(18,18,\infty;12)$;
9) $\Gamma(19,19,\infty;k)$ for $10 \le k \le 13$;
10) $\Gamma(20,20,\infty;k)$ for $11 \le k \le 15$;
11) $\Gamma(21,21,\infty;k)$ for $11 \le k \le 16$;
12) $\Gamma(22,22,\infty;k)$ for $12 \le k \le 18$;
13) $\Gamma(23,23,\infty;k)$ for $12 \le k \le 20$;
14) $\Gamma(24,24,\infty;k)$ for $13 \le k \le 22$;
15) $\Gamma(25,25,\infty;k)$ for $13 \le k \le 25$;
16) $\Gamma(26,26,\infty;k)$ for $14 \le k \le 29$;
17) $\Gamma(27,27,\infty;k)$ for $14 \le k \le 33$;
18) $\Gamma(28,28,\infty;k)$ for $15 \le k \le 40$.

The list above includes 36 new non-discrete groups, which were not shown in [3, Theorem 4.2].

Next we show another 6 new non-discrete groups, which were not shown in [3, Theorem 4.2]. Before this, we recall the following two equalities in [1, Theorem 7]:

$$(\alpha) \qquad \cos\frac{\pi}{5} - \cos\frac{2\pi}{5} = \frac{1}{2}\,;$$

$$(\beta) \qquad -\cos\frac{\pi}{15} + \cos\frac{2\pi}{15} + \cos\frac{4\pi}{15} - \cos\frac{7\pi}{15} = \frac{1}{2}\,.$$

Consider the elements $I_1I_2I_1I_2I_3I_2$ and $I_3I_1I_3I_1I_2I_1$, which are written as $I_{(12)^2 32}$ and $I_{(31)^2 21}$, respectively. It is seen that

$$\begin{aligned} &\mathit{trace}(I_{(12)^2 32}) \\ &\quad = (\rho\sigma\tau + \overline{\rho}\,\overline{\sigma}\,\overline{\tau})(1-|\rho|^2) - 1 + |\sigma|^2 \\ &\qquad + |\rho|^2(-2|\sigma|^2 + |\tau|^2 + |\rho|^2|\sigma|^2) \\ &\quad = 2\cos\frac{2\pi}{k} + 4\cos\frac{2\pi}{k}\cos\frac{2\pi}{n} - 2\cos\frac{6\pi}{n} \\ &\qquad - 2\cos\frac{2\pi}{n} + 1 \end{aligned}$$

and

$$\begin{aligned} &\mathit{trace}(I_{(31)^2 21}) \\ &\quad = (\rho\sigma\tau + \overline{\rho}\,\overline{\sigma}\,\overline{\tau})(1-|\tau|^2) - 1 + |\rho|^2 \\ &\qquad + |\tau|^2(-2|\rho|^2 + |\sigma|^2 + |\rho|^2|\tau|^2) \\ &\quad = 2\cos\frac{2\pi}{k} + 4\cos\frac{2\pi}{k}\cos\frac{2\pi}{n} - 2\cos\frac{4\pi}{n} \\ &\qquad - 2\cos\frac{2\pi}{n} + 1. \end{aligned}$$

Now, consider the case $n = 8$ and $k = 5$. Here, it is numerically checked that the value of $\mathit{trace}(I_{(12)^2 32})$ is about 2.4920. It follows from [2, Theorem 6.24] that the element $I_{(12)^2 32}$ is regular elliptic. Then there is a real number ϕ such that

$$\begin{aligned} &2\cos\frac{2\pi}{5} + 4\cos\frac{2\pi}{5}\cos\frac{2\pi}{8} - 2\cos\frac{6\pi}{8} - 2\cos\frac{2\pi}{8} + 1 \\ &\quad = 1 + 2\cos\phi\pi. \end{aligned}$$

By using the above equality (α), we have

$$\cos\frac{\pi}{5} - \cos\frac{7\pi}{20} + \cos\frac{3\pi}{20} - \cos\phi\pi = \frac{1}{2}\,.$$

Since $1 + 2\cos\phi\pi = 2.4920\cdots$, $\cos(\pi/3) < \cos\phi\pi < \cos(\pi/6)$. Assume that $\cos(\pi/5) = \cos\phi\pi$. Then it is seen that

$$-\cos\frac{7\pi}{20}+\cos\frac{3\pi}{20}=2\sin\frac{\pi}{4}\sin\frac{\pi}{10}$$
$$=\sqrt{2}\left(\frac{\sqrt{5}-1}{4}\right)<\frac{1}{2},$$

which is a contradiction. Therefore $\cos(\pi/5)\neq\cos\phi\pi$. Suppose that ϕ is a rational number. Then $\pi/5, 7\pi/20, 3\pi/20$ and $|\phi|\pi$ are four distinct rational multiples of π lying strictly between 0 and $\pi/2$. For any non-empty subset S of $\{\pi/5, 7\pi/20, 3\pi/20, |\phi|\pi\}$, any rational linear combination of the cosines of elements in S does not appear in the list of [1, Theorem 7]. Hence, we have a contradiction. Therefore ϕ is not rational, which implies that $I_{(12)^2 32}$ has infinite order. Thus $\Gamma(8,8,\infty;5)$ is not discrete. Next, consider the following cases: $n=6$ and $k=5$, $n=7$ and $k=6$, $n=9$ and $k=6$, $n=10$ and $k=6$, and $n=15$ and $k=10$. In each case, it is seen that the element $I_{(31)^2 21}$ is regular elliptic. It implies that there is a real number ψ such that

$$(*)\ 2\cos\frac{2\pi}{k}+4\cos\frac{2\pi}{k}\cos\frac{2\pi}{n}-2\cos\frac{4\pi}{n}-2\cos\frac{2\pi}{n}+1$$
$$=1+2\cos\psi\pi.$$

In the case where $n=6$ and $k=5$, using (α), we obtain

$$(a)\qquad \cos\frac{\pi}{5}+\cos\frac{2\pi}{5}-\cos\psi\pi=\frac{1}{2}.$$

Similarly, for cases $n=7$ and $k=6$, $n=9$ and $k=6$, and $n=10$ and $k=6$, we obtain the equations,

$$(b)\qquad -\cos\frac{3\pi}{7}+\cos\psi\pi=\frac{1}{2},$$

$$(c)\qquad \cos\frac{4\pi}{9}+\cos\psi\pi=\frac{1}{2},$$

$$(d)\qquad \cos\frac{2\pi}{5}+\cos\psi\pi=\frac{1}{2},$$

respectively. For the case $n=15$ and $k=10$, using (α) and (β), we have

$$(e)\qquad -\cos\frac{2\pi}{5}+\cos\frac{7\pi}{15}+\cos\psi\pi=\frac{1}{2}.$$

As before, [1, Theorem 7] tells us that there are no rational solutions to the above equations (a), (b), (c), (d) and (e), so $I_{(31)^2 21}$ is of infinite order in each case. Hence the groups $\Gamma(6,6,\infty;5)$, $\Gamma(7,7,\infty;6)$, $\Gamma(9,9,\infty;6)$, $\Gamma(10,10,\infty;6)$ and $\Gamma(15,15,\infty;10)$ are not discrete.

Combining the above with [3, Theorem 4.2], we have our desired result. □

Remark 2.4. In $\Gamma(10,10,\infty;6)$, $I_{(12)^2 32}$ is a regular elliptic element of order 10, while in $\Gamma(6,6,\infty;5)$, $\Gamma(7,7,\infty;6)$, $\Gamma(9,9,\infty;6)$ or $\Gamma(15,15,\infty;10)$, $I_{(12)^2 32}$ is not elliptic.

Acknowledgements. The author would like to thank Prof. Masahiro Ishizuka for helpful discussions. He is also grateful to the referee for many useful comments and suggestions. This research was supported by Grant-in-Aid for Scientific Research (C) (No. 22540212), Japan Society for the Promotion of Science.

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