Notes on parameters of quiver Hecke algebras

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Abstract: Varagnolo-Vasserot and Rouquier proved that, in a symmetric generalized Cartan matrix case, the simple modules over the quiver Hecke algebra with a special parameter correspond to the upper global basis. In this note we show that the simple modules over the quiver Hecke algebras with a *generic* parameter also correspond to the upper global basis in a symmetric generalized Cartan matrix case.

Key words: Global basis; Khovanov-Lauda-Rouquier algebras; categorification.

1. Introduction. Lascoux-Leclerc-Thibon ([8]) conjectured that the irreducible representations of Hecke algebras of type A are controlled by the upper global basis ([4,5]) (or dual canonical basis ([10])) of the basic representation of the affine quantum group $U_q(A_\ell^{(1)})$. Then, S. Ariki ([1]) proved this conjecture by generalizing it to cyclotomic Hecke algebras. The crucial ingredient in his proof was the fact that the cyclotomic Hecke algebras categorify the irreducible highest weight representations of $U(A_\ell^{(1)})$. Because of the lack of grading on the cyclotomic Hecke algebras, these algebras do not categorify the representation of the quantum group.

Then Khovanov-Lauda and Rouquier introduced independently a new family of *graded* algebras, a generalization of affine Hecke algebras of type A, in order to categorify arbitrary quantum groups ([6,7,11]). These algebras are called *Khovanov-Lauda-Rouquier algebras* or *quiver Hecke algebras*.

Let $U_q(\mathfrak{g})$ be the quantum group associated with a symmetrizable Cartan datum and let $\{R(\beta)\}_{\beta\in\mathsf{Q}^+}$ be the corresponding quiver Hecke algebras. Then it was shown in [6,7] that there exists an algebra isomorphism

$$U_{\mathbf{A}}^-(\mathfrak{g}) \simeq \bigoplus_{\beta\in\mathsf{Q}^+} \mathrm{K}(R(\beta)\text{-proj}),$$

where $U_{\mathbf{A}}^-(\mathfrak{g})$ is the $\mathbf{A}$-form of the half $U_q^-(\mathfrak{g})$ of the quantum group $U_q(\mathfrak{g})$ with $\mathbf{A} = \mathbf{Z}[q, q^{-1}]$, and $\mathrm{K}(R(\beta)\text{-proj})$ is the Grothendieck group of the category $R(\beta)$-proj of finitely generated projective graded $R(\beta)$-modules. The positive root lattice is denoted by Q^+. By the duality, we have

$$U_{\mathbf{A}}^-(\mathfrak{g})^* \simeq \bigoplus_{\beta\in\mathsf{Q}^+} \mathrm{K}(R(\beta)\text{-gmod}), \tag{1.1}$$

where $U_{\mathbf{A}}^-(\mathfrak{g})^*$ is the direct sum of the dual of the weight space $U_{\mathbf{A}}^-(\mathfrak{g})_{-\beta}$ of $U_{\mathbf{A}}^-(\mathfrak{g})$, and $R(\beta)$-gmod is the abelian category of graded $R(\beta)$-modules which are finite-dimensional over the base field $\mathbf{k}$.

When the generalized Cartan matrix is a symmetric matrix, Varagnolo and Vasserot ([13]) and Rouquier ([12]) proved that the *upper global basis* introduced by the author or Lusztig's *dual canonical basis* corresponds to the isomorphism classes of simple $R(\beta)$-modules via the isomorphism (1.1).

However, for a given generalized Cartan matrix, associated quiver Hecke algebras are not unique and depend on the parameters c. Varagnolo-Vasserot and Rouquier have proved the above results for a very special choice c_0 of parameters (see (2.5)). Let us denote by $R(\beta)_{c_0}$ the quiver Hecke algebra with the choice c_0, and by $R(\beta)_{\mathrm{c_{gen}}}$ the quiver Hecke algebra with a generic choice $\mathrm{c_{gen}}$ of parameters. When a simple $R(\beta)_{\mathrm{c_{gen}}}$-module is specialized at the special parameter c_0, it may be a reducible $R(\beta)_{c_0}$-module. The purpose of this note is to prove that the specialization of any simple $R(\beta)_{\mathrm{c_{gen}}}$-module at c_0 remains a simple $R(\beta)_{c_0}$-module. In other words, the set of isomorphism classes of simple $R(\beta)_{\mathrm{c_{gen}}}$-modules also corresponds to the upper global basis.

2. Review on global bases and quiver Hecke algebras.

2.1. Global bases. Let I be a finite index set. An integral square matrix $A = (a_{i,j})_{i,j\in I}$ is called a *symmetrizable generalized Cartan matrix* if it

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satisfies (i) $a_{i,i}=2$ $(i\in I)$, (ii) $a_{i,j}\le 0$ $(i\ne j)$, (iii) $a_{i,j}=0$ if $a_{j,i}=0$ $(i,j\in I)$, (iv) there is a diagonal matrix $D=\mathrm{diag}(d_i\in\mathbf{Z}_{>0}\mid i\in I)$ such that DA is symmetric.

A *Cartan datum* $(A,P,\Pi,P^\vee,\Pi^\vee)$ consists of

(1) a symmetrizable generalized Cartan matrix A,
(2) a free abelian group P of finite rank, called the *weight lattice*,
(3) $P^\vee:=\mathrm{Hom}(P,\mathbf{Z})$, called the *co-weight lattice*,
(4) $\Pi=\{\alpha_i\mid i\in I\}\subset P$, called the set of *simple roots*,
(5) $\Pi^\vee=\{h_i\mid i\in I\}\subset P^\vee$, called the set of *simple coroots*,

satisfying the condition: $\langle h_i,\alpha_j\rangle=a_{i,j}$ for all $i,j\in I$.

Since A is symmetrizable, there is a symmetric bilinear form $(\ |\ )$ on P satisfying

$$(\alpha_i|\alpha_j)=d_ia_{i,j}\quad\text{and}\quad(\alpha_i|\lambda)=d_i\langle h_i,\lambda\rangle$$

for all $i,j\in I$, $\lambda\in P$. The free abelian group $\mathsf{Q}=\bigoplus_{i\in I}\mathbf{Z}\alpha_i$ is called the *root lattice*. Set $\mathsf{Q}^+=\sum_{i\in I}\mathbf{Z}_{\ge0}\alpha_i\subset\mathsf{Q}$ and $\mathsf{Q}^-=\sum_{i\in I}\mathbf{Z}_{\le0}\alpha_i\subset\mathsf{Q}$. For $\beta=\sum_{i\in I}m_i\alpha_i\in\mathsf{Q}$, we set $\mathrm{ht}(\beta)=\sum_{i\in I}|m_i|$.

Let q be an indeterminate. Set $q_i=q^{d_i}$ for $i\in I$ and we define $[n]_i=(q_i^n-q_i^{-n})(q_i-q_i^{-1})^{-1}$ and $[n]_i!=\prod_{k=1}^n[k]_i$ for $n\in\mathbf{Z}_{\ge0}$.

Definition 2.1. The *quantum algebra* $U_q(\mathfrak{g})$ associated with a Cartan datum $(A,P,\Pi,\Pi^\vee)$ is the algebra over $\mathbf{Q}(q)$ generated by e_i,f_i $(i\in I)$ and q^h $(h\in P^\vee)$ satisfying following relations:

(i) $q^0=1$, $q^hq^{h'}=q^{h+h'}$ for $h,h'\in P^\vee$,
(ii) $q^he_iq^{-h}=q^{\langle h,\alpha_i\rangle}e_i$, $\quad q^hf_iq^{-h}=q^{-\langle h,\alpha_i\rangle}f_i$ for $h\in P^\vee, i\in I$,
(iii) $e_if_j-f_je_i=\delta_{i,j}\dfrac{K_i-K_i^{-1}}{q_i-q_i^{-1}}$, where $K_i=q_i^{h_i}$,
(iv) $\displaystyle\sum_{r=0}^{1-a_{i,j}}(-1)^re_i^{(1-a_{i,j}-r)}e_je_i^{(r)}=0$ if $i\ne j$, where $e_i^{(n)}=e_i^n/[n]_i!$,
(v) $\displaystyle\sum_{r=0}^{1-a_{i,j}}(-1)^rf_i^{(1-a_{i,j}-r)}f_jf_i^{(r)}=0$ if $i\ne j$, where $f_i^{(n)}=f_i^n/[n]_i!$.

Let $U_q^-(\mathfrak{g})$ be the $\mathbf{Q}(q)$-subalgebra of $U_q(\mathfrak{g})$ generated by the elements f_i. We define the endomorphisms e_i' and e_i'' of $U_q^-(\mathfrak{g})$ by

$$[e_i,a]=(q_i-q_i^{-1})^{-1}(K_ie_i''a-K_i^{-1}e_i'a)\text{ for }a\in U_q^-(\mathfrak{g}).$$

Then e_i' and the left multiplication of f_j satisfy the q-boson commutation relations

$$e_i'f_j-q_i^{-a_{i,j}}f_je_i'=\delta_{i,j}.$$

Set $\mathbf{A}=\mathbf{Z}[q,q^{-1}]$ and let $U_{\mathbf{A}}^-(\mathfrak{g})$ be the $\mathbf{A}$-subalgebra of $U_q^-(\mathfrak{g})$ generated by the elements $f_i^{(n)}$. Then $U_{\mathbf{A}}^-(\mathfrak{g})$ has a weight decomposition $U_{\mathbf{A}}^-(\mathfrak{g})=\bigoplus_{\beta\in\mathsf{Q}^-}U_{\mathbf{A}}^-(\mathfrak{g})_\beta$ where $U_{\mathbf{A}}^-(\mathfrak{g})_\beta:=\{a\in U_{\mathbf{A}}^-(\mathfrak{g})\mid q^haq^{-h}=q^{\langle h_i,\beta\rangle}a\}$. Set $U_{\mathbf{A}}^-(\mathfrak{g})^*=\bigoplus_{\beta\in\mathsf{Q}^-}\mathrm{Hom}_{\mathbf{A}}(U_{\mathbf{A}}^-(\mathfrak{g})_\beta,\mathbf{A})$ and let e_i, $f_i'\in\mathrm{End}_{\mathbf{A}}(U_{\mathbf{A}}^-(\mathfrak{g})^*)$ be the transposes of $f_i,e_i'\in\mathrm{End}_{\mathbf{A}}(U_{\mathbf{A}}^-(\mathfrak{g}))$, respectively. Note that $U_{\mathbf{A}}^-(\mathfrak{g})_0$ is a free $\mathbf{A}$-module with a basis 1, and hence $U_{\mathbf{A}}^-(\mathfrak{g})_0^*$ is a free $\mathbf{A}$-module generated by the dual basis of 1, which is denoted by ϕ.

Proposition 2.2 ([4,5]). *There exists a unique basis* $\{\mathrm{G^{up}}(b)\}_{b\in B}$ *of the* $\mathbf{A}$*-module* $U_{\mathbf{A}}^-(\mathfrak{g})^*$*, called* the upper global basis*, which satisfies the following conditions*:

(i) $\phi\in\{\mathrm{G^{up}}(b)\mid b\in B\}$,
(ii) *for any* $b\in B$, $G(b)$ *belongs to* $(U_{\mathbf{A}}^-(\mathfrak{g})_\beta)^*$ *for some* $\beta\in\mathsf{Q}^-$*, which is denoted by* $\mathrm{wt}(b)$,
(iii) *Set* $\varepsilon_i(b)=\max\{n\in\mathbf{Z}_{\ge0}\mid e_i^n\mathrm{G^{up}}(b)\ne0\}$*. Then for any* $b\in B$ *and* $i\in I$*, there exists* $\tilde f_ib\in B$ *such that, when writing*

$$f_i'\mathrm{G^{up}}(b)=\sum_{b'\in B}F^i_{b,b'}\mathrm{G^{up}}(b')\quad\text{with }F^i_{b,b'}\in\mathbf{A},$$

we have
(a) $F^i_{b,\tilde f_ib}=q_i^{-\varepsilon_i(b)}$,
(b) $\varepsilon_i(\tilde f_ib)=\varepsilon_i(b)+1$,
(c) $F^i_{b,b'}=0$ *if* $b'\ne\tilde f_ib$ *and* $\varepsilon_i(b')\ge\varepsilon_i(b)+1$,
(d) $F^i_{b,b'}\in qq_i^{-\varepsilon_i(b)}\mathbf{Z}[q]$ *for* $b'\ne\tilde f_ib$.

(iv) *for* $b\in B$ *such that* $\varepsilon_i(b)>0$*, there exists* $\tilde e_ib\in B$ *such that, when writing*

$$e_i\mathrm{G^{up}}(b)=\sum_{b'\in B}E^i_{b,b'}\mathrm{G^{up}}(b')\quad\text{with }E^i_{b,b'}\in\mathbf{A},$$

we have
(a) $E^i_{b,\tilde e_ib}=[\varepsilon_i(b)]_i$,
(b) $\varepsilon_i(\tilde e_ib)=\varepsilon_i(b)-1$,
(c) $E^i_{b,b'}=0$ *if* $b'\ne\tilde e_ib$ *and* $\varepsilon_i(b')\ge\varepsilon_i(b)-1$,
(d) *any* $E^i_{b,b'}$ *is invariant under the automorphism* $q\mapsto q^{-1}$,
(e) $E^i_{b,b'}\in qq_i^{1-\varepsilon_i(b)}\mathbf{Z}[q]$ *for* $b'\ne\tilde e_ib$.

(v) $\tilde f_i\tilde e_ib=b$ *if* $\varepsilon_i(b)>0$*, and* $\tilde e_i\tilde f_ib=b$.

Note that B has the weight decomposition

$$B=\bigsqcup_{\beta\in\mathsf{Q}^-}B_\beta\quad\text{with }B_\beta:=\{b\in B\mid\mathrm{wt}(b)=\beta\}.$$

There exists a unique involution (called the *bar involution*) $-\colon U_{\mathbf{A}}^-(\mathfrak{g})^*\to U_{\mathbf{A}}^-(\mathfrak{g})^*$ such that

$$(2.1)\quad \begin{aligned} &\text{(a) } (qu)^- = q^{-1}\overline{u} \quad \text{for any } u \in U_{\mathbf{A}}^-(\mathfrak{g})^*,\\ &\text{(b) } -\circ e_i = e_i \circ - \quad \text{for any } i,\\ &\text{(c) } \overline{\phi} = \phi. \end{aligned}$$

We have

$$\overline{\mathrm{G}^{\mathrm{up}}(b)} = \mathrm{G}^{\mathrm{up}}(b) \quad \text{for any } b \in B.$$

2.2. Quiver Hecke algebras. In this subsection, we recall the construction of the quiver Hecke algebras associated with a Cartan datum $(A, P, \Pi, P^\vee, \Pi^\vee)$. For $i, j \in I$ such that $i \neq j$, set

$$S_{i,j} = \{(p,q) \in \mathbf{Z}_{\geq 0}^2 \mid d_i p + d_j q = -(\alpha_i, \alpha_j)\}.$$

Let $\mathbf{k}(A)$ be the commutative $\mathbf{Z}$-algebra generated by indeterminates $\{t_{i,j;p,q}\}$ and the inverse of $t_{i,j;-a_{i,j},0}$ where $i, j \in I$ such that $i \neq j$ and $(p,q) \in S_{i,j}$. They are subject to the defining relations: $t_{i,j;p,q} = t_{j,i;q,p}$.

Let us define the polynomials $(Q_{i,j})_{i,j \in I}$ in $\mathbf{k}(A)[u,v]$ by

$$Q_{i,j}(u,v) = \begin{cases} 0 & \text{if } i = j,\\ \displaystyle\sum_{(p,q)\in S_{i,j}} t_{i,j;p,q} u^p v^q & \text{if } i \neq j. \end{cases}$$

They satisfy $Q_{i,j}(u,v) = Q_{j,i}(v,u)$.

We denote by $S_n = \langle s_1, \dots, s_{n-1} \rangle$ the symmetric group on n letters, where $s_i := (i, i+1)$ is the transposition of i and $i+1$. Then S_n acts on I^n.

Definition 2.3 ([6,11])**.** The *quiver Hecke algebra* $R(n)$ of degree n associated with a Cartan datum $(A, P, \Pi, P^\vee, \Pi^\vee)$ is the $\mathbf{Z}$-graded algebra over $\mathbf{k}(A)$ generated by $e(\nu)$ $(\nu \in I^n)$, x_k $(1 \leq k \leq n)$, τ_l $(1 \leq l \leq n-1)$ satisfying the following defining relations:

$$\begin{aligned}
&e(\nu)e(\nu') = \delta_{\nu,\nu'} e(\nu),\ \textstyle\sum_{\nu \in I^n} e(\nu) = 1,\\
&x_k x_l = x_l x_k,\ x_k e(\nu) = e(\nu) x_k,\\
&\tau_l e(\nu) = e(s_l(\nu))\tau_l,\ \tau_k\tau_l = \tau_l\tau_k \quad \text{if } |k-l| > 1,\\
&\tau_k^2 e(\nu) = Q_{\nu_k,\nu_{k+1}}(x_k, x_{k+1}) e(\nu),\\
&(\tau_k x_l - x_{s_k(l)}\tau_k)e(\nu)\\
&\quad = \begin{cases} -e(\nu) & \text{if } l = k,\ \nu_k = \nu_{k+1},\\ e(\nu) & \text{if } l = k+1,\ \nu_k = \nu_{k+1},\\ 0 & \text{otherwise,} \end{cases}\\
&(\tau_{k+1}\tau_k\tau_{k+1} - \tau_k\tau_{k+1}\tau_k)e(\nu)\\
&\quad = \begin{cases} \overline{Q}_{\nu_k,\nu_{k+1}}(x_k, x_{k+1}, x_{k+2})e(\nu) & \text{if } \nu_k = \nu_{k+2},\\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Here $\overline{Q}_{i,j}(u,v,w) = (Q_{i,j}(u,v) - Q_{i,j}(w,v))(u-w)^{-1}$. The $\mathbf{Z}$-grading on $R(n)$ is given by assigning $\deg e(\nu) = 0$, $\deg x_k e(\nu) = (\alpha_{\nu_k}|\alpha_{\nu_k})$, $\deg \tau_l e(\nu) = -(\alpha_{\nu_l}|\alpha_{\nu_{l+1}})$.

Note that $R(n)$ has an anti-involution ψ that fixes the generators x_k, τ_l and $e(\nu)$.

For $n \in \mathbf{Z}_{\geq 0}$ and $\beta \in \mathsf{Q}^+$ such that $\mathrm{ht}(\beta) = n$, we set

$$I^\beta = \{\nu = (\nu_1, \dots, \nu_n) \in I^n \mid \alpha_{\nu_1} + \cdots + \alpha_{\nu_n} = \beta\}.$$

We define

$$(2.2)\quad \begin{aligned} &e(\beta) = \textstyle\sum_{\nu \in I^\beta} e(\nu),\\ &R(\beta) = R(n)e(\beta) = \displaystyle\bigoplus_{\nu \in I^\beta} R(n)e(\nu). \end{aligned}$$

The algebra $R(\beta)$ is called the *quiver Hecke algebra at* β.

Similarly, for $\beta, \gamma \in \mathsf{Q}^+$ with $m = \mathrm{ht}(\beta)$ and $n = \mathrm{ht}(\gamma)$

$$e(\beta,\gamma) = \textstyle\sum_\nu e(\nu) \in R(m+n)$$

where ν ranges over the set of $\nu \in I^{m+n}$ such that $\sum_{k=1}^m \alpha_{\nu_k} = \beta$ and $\sum_{k=m+1}^{m+n} \alpha_{\nu_k} = \gamma$.

Then $R(m+n)e(\beta,\gamma)$ is a graded $(R(\beta+\gamma), R(\beta) \otimes R(\gamma))$-bimodule. For a graded $R(\beta)$-module M and a graded $R(\gamma)$-module N, we define their *convolution* $M \circ N$ by

$$M \circ N = R(\beta+\gamma)e(\beta,\gamma) \underset{R(\beta)\otimes R(\gamma)}{\otimes} (M \otimes N).$$

For $\ell \in \mathbf{Z}_{\geq 0}$, we define the graded $R(\ell\alpha_i)$-module $L(i^\ell)$ by

$$L(i^\ell) = q_i^{\ell(\ell-1)/2}\left(R(\ell\alpha_i) \Big/ \left(\sum_{k=1}^{\ell} R(\ell\alpha_i) x_k\right)\right).$$

Here $q\colon \mathrm{Mod}(R(\beta)) \to \mathrm{Mod}(R(\beta))$ is the grade-shift functor:

$$(2.3)\quad (qM)_k = M_{k-1},$$

and $q_i = q^{(\alpha_i|\alpha_i)/2}$.

For a commutative ring $\mathbf{k}$ and a ring homomorphism $c\colon \mathbf{k}(A) \to \mathbf{k}$, we denote by $R(\beta)_{\mathbf{k}}$ the algebra $\mathbf{k} \otimes_{\mathbf{k}(A)} R(\beta)$.

Let us denote by $X(A)$ the scheme $\mathrm{Spec}(\mathbf{k}(A))$. For $x \in X(A)$, let us denote by $\mathbf{k}(x)$ the residue field of the local ring $(\mathscr{O}_{X(A)})_x$ and denote by $R(\beta)_x$ the $\mathbf{k}(x)$-algebra $\mathbf{k}(x) \otimes_{\mathbf{k}(A)} R(\beta)$.

Let us take a commutative field $\mathbf{k}$ and a homomorphism $\mathbf{k}(x) \to \mathbf{k}$. For $\beta \in \mathsf{Q}^+$, let us denote by $R(\beta)_{\mathbf{k}}$-gmod the abelian category of graded $R(\beta)_{\mathbf{k}}$-modules finite-dimensional over $\mathbf{k}$. Then the set of isomorphism classes of simple objects in

$R(\beta)_{\mathbf{k}}$-gmod is isomorphic to the one for $R(\beta)_x$-gmod by $S \mapsto \mathbf{k} \otimes_{\mathbf{k}(x)} S$ (see [6, Corollary 3.19]).

For $i \in I$ and $x \in X(A)$ we have functors

$$R(\beta)_x\text{-gmod} \underset{E_i}{\overset{F_i}{\rightleftarrows}} R(\beta+\alpha_i)_x\text{-gmod}.$$

Here these functors are defined by

$$F_i M = M \circ L(i),$$
$$E_i N = e(\beta, \alpha_i) N$$

for $M \in R(\beta)_x$-gmod and $N \in R(\beta+\alpha_i)_x$-gmod. Then we have

$$0 \to \mathrm{id} \to E_i F_i \to q_i^{-2} F_i E_i \to 0,$$
$$E_i F_j \simeq q^{-(\alpha_i, \alpha_j)} F_j E_i \quad \text{for } i \neq j,$$

which immediately follows from [3, Theorem 3.6].

Let $\mathrm{K}(R(\beta)_x\text{-gmod})$ denote the Grothendieck group of the abelian category $R(\beta)_x$-gmod. Then, it has a structure of a $\mathbf{Z}[q, q^{-1}]$-module induced by the grade-shift functor on $R(\beta)_x$-gmod.

Then the following theorem holds.

Theorem 2.4 ([6])**.** *There exists a unique $\mathbf{Z}[q, q^{-1}]$-linear isomorphism*

$$\bigoplus_{\beta \in \mathsf{Q}^+} \mathrm{K}(R(\beta)_x\text{-gmod}) \xrightarrow{\sim} U_{\mathbf{A}}^-(\mathfrak{g})^* \tag{2.4}$$

such that

(i) *the induced actions $[E_i]$ and $[F_i]$ by E_i and F_i correspond to e_i and f_i',*

(ii) *$\phi \in U_{\mathbf{A}}^-(\mathfrak{g})^*$ corresponds to the regular representation of $R(0)_x$.*

Let $\mathrm{D}\colon R(\beta)_x\text{-gmod} \to (R(\beta)_x\text{-gmod})^{\mathrm{opp}}$ be the duality functor $M \mapsto M^*$ induced by the antiautomorphism ψ of $R(\beta)_x$. We can easily see by the characterization (2.1) of the bar involution that the induced endomorphism $[\mathrm{D}]$ of $\bigoplus_{\beta \in \mathsf{Q}^+} \mathrm{K}(R(\beta)_x\text{-gmod})$ corresponds to the bar involution $^-$ of $U_{\mathbf{A}}^-(\mathfrak{g})^*$.

The Grothendieck group $\mathrm{K}(R(\beta)_x\text{-gmod})$ is a free $\mathbf{Z}$-module with the basis consisting of $[S]$ where S ranges over the set of isomorphism classes of simple graded $R(\beta)_x$-modules. Khovanov-Lauda ([6]) proved that for any simple graded $R(\beta)_x$-module S, there exists $r \in \mathbf{Z}$ such that $\mathrm{D}(q^r S) \simeq q^r S$. Let $\overline{\mathrm{Irr}}(R(\beta)_x)$ be the set of isomorphism classes of simple graded $R(\beta)_x$-modules S such that $\mathrm{D}(S) \simeq S$. Then $\mathrm{K}(R(\beta)_x\text{-gmod})$ is a free $\mathbf{Z}[q, q^{-1}]$-module with $\{[S] \mid S \in \overline{\mathrm{Irr}}(R(\beta)_x)\}$ as a basis.

For a simple graded module S, let us denote by $\varepsilon_i(S)$ the largest integer k such that $E_i^k S \neq 0$. Recall that q denotes the shift-functor and $q_i = q^{(\alpha_i|\alpha_i)/2}$.

Proposition 2.5 ([6,9])**.** *Let $x \in X(A)$, $\beta \in \mathsf{Q}^+$ and S a simple graded $R(\beta)_x$-module.*

(i) *The cosocle of $F_i S$ is a simple module. Its image under $q_i^{\varepsilon_i(S)}$ is denoted by $\widetilde{F}_i S$.*

(ii) *If $\varepsilon_i(S) > 0$ then the socle of $E_i S$ is simple. Its image under $q_i^{1-\varepsilon_i(S)}$ is denoted by $\widetilde{E}_i S$.*

(iii) *$\widetilde{F}_i \widetilde{E}_i S \simeq S$ if $\varepsilon_i(S) > 0$, and $\widetilde{E}_i \widetilde{F}_i S \simeq S$.*

(iv) *If S is invariant by the duality D, then so are $\widetilde{F}_i S$ and $\widetilde{E}_i S$.*

(v) *The set $\bigsqcup_{\beta \in \mathsf{Q}^+} \overline{\mathrm{Irr}}(R(\beta)_x)$ is isomorphic to B, and $\widetilde{E}_i$ and $\widetilde{F}_i$ correspond to $\tilde{e}_i$ and $\tilde{f}_i$ by this isomorphism.*

Hence, the cosocle of $F_i S$ is isomorphic to $q_i^{-\varepsilon_i(S)} \widetilde{F}_i S$, the socle of $E_i S$ is isomorphic to $q_i^{\varepsilon_i(S)-1} \widetilde{E}_i S$ and the cosocle of $E_i S$ is isomorphic to $q_i^{-\varepsilon_i(S)+1} \widetilde{E}_i S$.

For $b \in B_{-\beta}$, let us denote by $L_x(b)$ the corresponding simple graded $R(\beta)_x$-module in $\overline{\mathrm{Irr}}(R(\beta)_x)$.

Now assume that A is symmetric and consider a $\mathbf{k}$-valued point c_0 of $X(A)$ given by

$$\begin{aligned} &Q_{i,j}(u,v) = b_{i,j}(u-v)^{-a_{i,j}} \text{ for } i \neq j \text{ where} \\ &\mathbf{k} \text{ is a field of characteristic 0 and } b_{i,j} \in \mathbf{k}^\times. \end{aligned} \tag{2.5}$$

Then the following theorem is proved by Varagnolo-Vasserot ([13]) and Rouquier ([12]).

Theorem 2.6. *Assume that the generalized Cartan matrix A is symmetric. Then the basis $\{[L_{c_0}(b)]\}_{b \in B}$ corresponds to the upper global basis $\{\mathrm{G}^{\mathrm{up}}(b)\}_{b \in B}$ by the isomorphism $\bigoplus_{\beta \in \mathsf{Q}^+} \mathrm{K}(R(\beta)_{c_0}\text{-gmod}) \xrightarrow{\sim} U_{\mathbf{A}}^-(\mathfrak{g})^*$.*

For $M \in R(\beta)_x$-gmod, let us define its character $\mathrm{ch}(M)$ by

$$\mathrm{ch}(M) = \sum_{\nu \in I^\beta,\, k \in \mathbf{Z}} \dim(e(\nu)M)_k q^k e(\nu)$$

as an element of $\bigoplus_{\nu \in I^\beta} \mathbf{Z}[q, q^{-1}] e(\nu)$. Then we have

$$\mathrm{ch}(L_{c_0}(b)) = \sum_{\nu \in I^\beta} (e_{\nu_1} \cdots e_{\nu_n} \mathrm{G}^{\mathrm{up}}(b)) e(\nu) \tag{2.6}$$

for $b \in B_{-\beta}$.

3. Main results. Let $\mathrm{c}_{\mathrm{gen}}$ be the generic point of $X(A)$. For $\beta \in \mathsf{Q}^+$ and $b \in B_{-\beta}$, let us consider the simple graded $R(\beta)_{\mathrm{c}_{\mathrm{gen}}}$-module $L_{\mathrm{c}_{\mathrm{gen}}}(b)$.

Proposition 3.1. *The set*

$$U_b := \{x \in X(A) \mid \mathrm{ch}(L_x(b)) = \mathrm{ch}(L_{\mathrm{c}_{\mathrm{gen}}}(b))\}$$

is a Zariski open subset of $X(A)$ and there exists a graded $\mathscr{O}_{U_b} \otimes_{\mathbf{k}(A)} R(\beta)$-module $\mathcal{L}(b)$ defined on U_b such that it is locally free as an $\mathscr{O}_{U_b}$-module and the stalk of $\mathcal{L}(b)$ at any $x \in U_b$ is isomorphic to $L_x(b)$.

Proof. We shall prove it by induction on $\mathrm{ht}(\beta)$. We may assume $\beta \neq 0$. Take an $i \in I$ such that $\ell := \varepsilon_i(b) \neq 0$. Set $\beta' = \beta - \ell\alpha_i$ and $b' = \tilde{e}_i^{\ell} b$. For any $x \in X(A)$, the graded $R(\beta)_x$-module $L_x(b)$ is a simple cosocle of $L_x(b') \circ L(i^\ell)$. Moreover the kernel of $L_x(b') \circ L(i^\ell) \twoheadrightarrow L_x(b)$ is $\{s \in L_x(b') \circ L(i^\ell) \mid e(\beta', \ell\alpha_i)R(\beta)s = 0\}$.

By the induction hypothesis, there exists an $\mathscr{O}_{U_{b'}} \otimes_{\mathbf{k}(A)} R(\beta')$-module $\mathcal{L}(b')$ as above. Set $\mathcal{R} = \mathscr{O}_{U_{b'}} \otimes_{\mathbf{k}(A)} R(\beta)$ and we shall denote by $\mathcal{M}$ the $\mathcal{R}$-module $\mathcal{L}(b') \circ L(i^\ell)$. Let f be the composition

$$\begin{aligned} \mathcal{M} &\to \mathscr{H}om_{\mathscr{O}_{X(A)}|_{U_{b'}}}(\mathcal{R}, \mathcal{M}) \\ &\to \mathscr{H}om_{\mathscr{O}_{X(A)}|_{U_{b'}}}(\mathcal{R}, \mathcal{M}/(1 - e(\beta', \ell\alpha_i))\mathcal{M}). \end{aligned}$$

Then the kernel of f coincides with the sheaf

$$\{u \in \mathcal{M} \mid e(\beta', \ell\alpha_i)\mathcal{R}u = 0\}.$$

The homomorphism f factors through

$$\begin{aligned} \mathcal{M} &\xrightarrow{\overline{f}} \mathscr{H}om_{\mathscr{O}_{U_{b'}}}(\mathcal{R}/\mathcal{R}_{\geq m}, \mathcal{M}/(1 - e(\beta', \ell\alpha_i))\mathcal{M}) \\ &\rightarrowtail \mathscr{H}om_{\mathscr{O}_{U_{b'}}}(\mathcal{R}, \mathcal{M}/(1 - e(\beta', \ell\alpha_i))\mathcal{M}) \end{aligned}$$

for a sufficient large integer m. Here $\mathcal{R}_{\geq m} = \bigoplus_{k \geq m} \mathcal{R}_k$. Therefore $\overline{f}$ is a morphism of vector bundles on $U_{b'}$. On the other hand, U_b is the set of $x \in U_{b'}$ such that the rank of $\overline{f}$ at x is equal to its rank at the generic point. Hence U_b is an open subset of $X(A)$ and the image of $\bar{f}|_{U_b}$ satisfies the condition for $\mathcal{L}(b)$. $\square$

For $x \in X(A)$ and $b \in B$, let us consider the condition

(3.1) $L_x(b)$ corresponds to the upper global basis $\mathrm{G}^{\mathrm{up}}(b)$ by the isomorphism (2.4).

We shall prove the following theorem.

Theorem 3.2. *Let c_0 be a point of $X(A)$ satisfying* (3.1) *for any $b \in B$. Then c_0 belongs to U_b for any $b \in B$. Hence* (3.1) *holds also for any $x \in U_b$.*

Proof. It is enough to show that $\mathrm{c_{gen}}$ satisfies (3.1). W shall take a triple $(K, \mathscr{O}, \mathbf{k})$ such that $K = \mathbf{k}(\mathrm{c_{gen}})$, $\mathcal{O}$ is a discrete valuation ring, K coincides with the fraction field of $\mathcal{O}$, $\mathbf{k}$ is the residue field of $\mathcal{O}$, $(\mathscr{O}_{X(A)})_{x_0} \subset \mathcal{O}$ and $(\mathscr{O}_{X(A)})_{x_0} \subset \mathcal{O} \to \mathbf{k}$ factors through $\mathbf{k}(x_0)$. Such a triple exists (see [2, (7.1.7)]).

We have the reduction map

$$\mathrm{Red}_{K,\mathbf{k}}\colon \mathrm{K}(R(\beta)_K) \longrightarrow \mathrm{K}(R(\beta)_{\mathbf{k}})$$

by assigning $[K \otimes_{\mathcal{O}} L] \in \mathrm{K}(R(\beta)_K)$ to $[\mathbf{k} \otimes_{\mathscr{O}} L] \in \mathrm{K}(\mathcal{R}(\beta)_{\mathbf{k}})$ for a graded $R(\beta)_{\mathcal{O}}$-module L that is finitely generated and torsion-free as an $\mathcal{O}$-module. The homomorphism $\mathrm{Red}_{K,\mathbf{k}}$ commutes with the duality D. Also it is compatible with the correspondence (2.4), namely we have a commutative diagram:

$$\begin{array}{ccc} \displaystyle\bigoplus_{\beta \in \mathsf{Q}^+} \mathrm{K}\big(R(\beta)_K\text{-gmod}\big) & \xrightarrow{\mathrm{Red}_{K,\mathbf{k}}} & \displaystyle\bigoplus_{\beta \in \mathsf{Q}^+} \mathrm{K}\big(R(\beta)_{\mathbf{k}}\text{-gmod}\big) \\ & \searrow^{\sim} & \downarrow{\sim} \\ & & U_{\mathbf{A}}^-(\mathfrak{g})^* \end{array}$$

For $b \in B$, set $L(b)_K := L_{\mathrm{c_{gen}}}(b)$ and $L(b)_{\mathbf{k}} := \mathbf{k} \otimes_{\mathbf{k}(c_0)} L_{c_0}(b)$. Take $b \in B_{-\beta}$, and let $L(b)_{\mathcal{O}}$ be an $R(\beta)_{\mathcal{O}}$-lattice of $L(b)_K$, i.e., a finitely generated graded $R(\beta)_{\mathcal{O}}$-submodule $L(\beta)_{\mathcal{O}}$ of $L(b)_K$ such that $K \otimes_{\mathcal{O}} L(b)_{\mathcal{O}} = L(b)_K$. In order to see the theorem, it is enough to show that $\mathbf{k} \otimes_{\mathcal{O}} L(b)_{\mathcal{O}} \simeq L(b)_{\mathbf{k}}$.

We shall prove it by induction on $\mathrm{ht}(\beta)$. Take an $i \in I$ such that $\varepsilon_i(b) > 0$ and set $b' = \tilde{e}_i b$. Then $[L(b')_K]$ corresponds to $\mathrm{G}^{\mathrm{up}}(b')$ by the induction hypothesis. We take an $R(\beta')_{\mathcal{O}}$-lattice $L(b')_{\mathcal{O}}$ of $L(b')_K$. Then by the induction hypothesis, we have $L(b')_{\mathbf{k}} \simeq \mathbf{k} \otimes_{\mathcal{O}} L(b')_{\mathcal{O}}$. The image of $q_i^{\varepsilon_i(b')} F_i L(b')_{\mathcal{O}}$ by $q_i^{\varepsilon_i(b')} F_i L(b')_K \twoheadrightarrow L(b)_K$ is an $R(\beta)_{\mathcal{O}}$-lattice of $L(b)_K$, and we can take it as $L(b)_{\mathcal{O}}$. Since $q_i^{\varepsilon_i(b')} F_i L(b')_{\mathbf{k}} \simeq q_i^{\varepsilon_i(b')} \mathbf{k} \otimes_{\mathcal{O}} F_i L(b')_{\mathcal{O}} \twoheadrightarrow \mathbf{k} \otimes_{\mathcal{O}} L(b)_{\mathcal{O}}$, the simples in a Jordan-Holder series of $\mathbf{k} \otimes_{\mathcal{O}} L(b)_{\mathcal{O}}$ appears in the one of $q_i^{\varepsilon_i(b')} F_i L(b')_{\mathbf{k}}$.

Now assume that $q^r L(b_1)_{\mathbf{k}}$ appears in $\mathrm{Red}_{K,\mathbf{k}} L(b)_K = [\mathbf{k} \otimes_{\mathcal{O}} L(b)_{\mathcal{O}}]$ for $r \in \mathbf{Z}$ and $b_1 \in B_{-\beta}$. Then $q^r \mathrm{G}^{\mathrm{up}}(b_1)$ appears in $q_i^{\varepsilon_i(b')} f_i' \mathrm{G}^{\mathrm{up}}(b')$ by the assumption that c_0 satisfies (3.1). In particular, $L(b)_{\mathbf{k}}$ appears in $[\mathbf{k} \otimes_{\mathcal{O}} L(b)_{\mathcal{O}}]$ exactly once by (iiia) in Proposition 2.2. Now assume that $(r, b_1) \neq (0, b)$. Then (iiia) and (iiid) in Proposition 2.2 imply that $r > 0$. Since $L(b)_K$ is stable by the duality functor D, $q^{-r} L(b_1)_{\mathbf{k}} \simeq \mathrm{D}(q^r L(b_1)_{\mathbf{k}})$ also appears in $\mathrm{Red}_{K,\mathbf{k}} L(b)_K$. Hence $-r > 0$. It is a contradiction. This shows the desired result: $\mathbf{k} \otimes_{\mathcal{O}} L(b)_{\mathcal{O}} \simeq L(b)_{\mathbf{k}}$. This completes the proof of Theorem 3.2. $\square$

Example 3.3. Let us give an example of a simple $R(\beta)$-module which does not correspond to any element in the upper global basis. Let $\mathfrak{g} = A_1^{(1)}$ with $I = \{0, 1\}$, $(\alpha_0|\alpha_0) = (\alpha_1|\alpha_1) = -(\alpha_0|\alpha_1) = 2$, and $Q_{0,1}(u, v) = u^2 + auv + v^2$. Here $\mathbf{k}$ is an arbitrary field and $a \in \mathbf{k}$. Set $\delta = \alpha_0 + \alpha_1$, $b' = \tilde{f}_1 \tilde{f}_0 \phi$ and $N = L(b')$. Then $N = \mathbf{k}v$ with $x_1 v = x_2 v = \tau_1 v = 0$ and $v = e(01)v$. Set $M = N \circ N$, and $u = v \otimes v \in M$. Then $\mathrm{ch}(M) = 2e(0101) + [2]^2 e(0011)$. Here $e(0101)M = \mathbf{k}u \oplus \mathbf{k}w$ with $w := \tau_2\tau_3\tau_1\tau_2 u$. By the weight consideration, $\tau_k e(0101)M = 0$ for $k = 1, 3$

and $x_k e(0101)M = 0$ for $1 \le k \le 4$. Easy calculations show that $\tau_2 w = -a\tau_2 u$. Hence $y := w + au$ is annihilated by all x_k's and τ_k's and $\mathbf{k}y$ is an $R(2\delta)$-submodule of M. Set $M_0 = M/\mathbf{k}y$. Then $[M_0]$ corresponds to $\mathrm{G}^{\mathrm{up}}(b)$ with $b := \tilde{f}_1^2 \tilde{f}_0^2 \phi$. It is easy to see that M_0 is a simple $R(2\delta)$-module if $a \neq 0$. When $a = 0$, $e(0011)M_0$ is a simple $R(2\delta)$-submodule of M_0 and $L(b) = e(0011)M_0$. Note that the case (2.5) is when $a = \pm 2$.

Example 3.4. Let us give another example of a simple $R(\beta)$-module which does not correspond to any element in the upper global basis. Let $\mathfrak{g} = A_2^{(1)}$ with $I = \mathbf{Z}/3\mathbf{Z} = \{0, 1, 2\}$ with $(\alpha_i|\alpha_i) = 2$ and $(\alpha_i|\alpha_j) = -1$ for $i \neq j$ and $Q_{i,i+1}(u, v) = a_i u + b_{i+1} v$ $(i \in I)$ with $a_i, b_i \in \mathbf{k}^\times$, where $\mathbf{k}$ is an arbitrary field. Set $\delta = \alpha_0 + \alpha_1 + \alpha_2$, $b' = \tilde{f}_2 \tilde{f}_1 \tilde{f}_0 \phi$ and $N = L(b')$. Then $N = \mathbf{k}v$ with $x_k v = \tau_\ell v = 0$ and $v = e(012)v$. Set $M = N \circ N$ and $u = v \otimes v \in M$. Then $\mathrm{ch}(M) = 2e(012012) + [2]^3 e(001122) + [2]^2 e(001212) + [2]^2 e(010122) + [2]e(010212)$. Here $e(012012)M = \mathbf{k}u \oplus \mathbf{k}w$ with $w := \tau_3\tau_4\tau_5\tau_2\tau_3\tau_4\tau_1\tau_2\tau_3 u$. By the weight consideration $\tau_k e(012012)M = 0$ for $k \neq 3$ and $x_k e(012012)M = 0$ for $1 \le k \le 6$. By calculations, we have $\tau_3 w = -\gamma\tau_3 u$ where $\gamma = a_0 a_1 a_2 - b_0 b_1 b_2$. Hence $y := w + \gamma u$ is annihilated by all x_k's and τ_k's and $\mathbf{k}y$ is an $R(2\delta)$-submodule of M. Set $M_0 = M/\mathbf{k}y$. Then $[M_0]$ corresponds to $\mathrm{G}^{\mathrm{up}}(b)$ with $b := \tilde{f}_2^2 \tilde{f}_1^2 \tilde{f}_0^2 \phi$. It is easy to see that M_0 is a simple $R(2\delta)$-module if $\gamma \neq 0$. When $\gamma = 0$, $S := (1 - e(012012))M_0 = R(2\delta)\tau_3 u$ is a simple $R(2\delta)$-module and $L(b) = S$ and $\mathrm{ch}(M_0/S) = e(012012)$. Note that the case (2.5) corresponds to $a_0 a_1 a_2 + b_0 b_1 b_2 = 0$.

Remark 3.5. If we assume

(3.2) the simple modules of $R(\beta)_{\mathrm{c_{gen}}}$ correspond to the upper global basis,

then $\mathrm{G}^{\mathrm{up}}(b) \in \sum_{S \in \overline{\mathrm{Irr}}(R(\beta)_x)} \mathbf{Z}_{\ge 0}[q, q^{-1}][S]$ for any $x \in X(A)$ and $b \in B$. We can ask if this positivity assertion still holds without the assumption (3.2).

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References

[1] S. Ariki, On the decomposition numbers of the Hecke algebra of $G(m, 1, n)$, J. Math. Kyoto Univ. **36** (1996), no. 4, 789–808.
[2] A. Grothendieck, Éléments de géométrie algébrique II, Étude globale élémentaire de quelques classes de morphismes, Inst. Hautes Études Sci. Publ. Math. **8** (1961), 5-222.
[3] S.-J. Kang and M. Kashiwara, Categorification of Highest Weight Modules via Khovanov-Lauda-Rouquier Algebras, arXiv:1102.4677.
[4] M. Kashiwara, On crystal bases of the Q-analogue of universal enveloping algebras, Duke Math. J. **63** (1991), no. 2, 465–516.
[5] M. Kashiwara, Global crystal bases of quantum groups, Duke Math. J. **69** (1993), no. 2, 455–485.
[6] M. Khovanov and A. D. Lauda, A diagrammatic approach to categorification of quantum groups. I, Represent. Theory **13** (2009), 309–347.
[7] M. Khovanov and A. D. Lauda, A diagrammatic approach to categorification of quantum groups II, Trans. Amer. Math. Soc. **363** (2011), no. 5, 2685–2700.
[8] A. Lascoux, B. Leclerc and J.-Y. Thibon, Hecke algebras at roots of unity and crystal bases of quantum affine algebras, Comm. Math. Phys. **181** (1996), no. 1, 205–263.
[9] A. D. Lauda and M. Vazirani, Crystals from categorified quantum groups, Adv. Math. **228** (2011), no. 2, 803–861.
[10] G. Lusztig, *Introduction to quantum groups*, Progress in Mathematics, 110, Birkhäuser Boston, Boston, MA, 1993.
[11] R. Rouquier, 2-Kac-Moody algebras, arXiv:0812.5023.
[12] R. Rouquier, Quiver Hecke algebras and 2-Lie algebras, arXiv:1112.3619v1.
[13] M. Varagnolo and E. Vasserot, Canonical bases and KLR-algebras, J. Reine Angew. Math. **659** (2011), 67–100.