Classification of visible actions on flag varieties

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Abstract: We give a complete classification of the pairs (L, H) of Levi subgroups of compact simple Lie groups G such that the L-action on a generalized flag variety G/H is strongly visible (or equivalently, the H-action on G/L or the diagonal G-action on $(G \times G)/(L \times H)$). The notion of visible actions on complex manifolds was introduced by T. Kobayashi, and a classification was accomplished by himself for the type A groups [J. Math. Soc. Japan, 2007]. A key step is to classify the pairs (L, H) for which the multiplication mapping $L \times G^{\sigma} \times H \to G$ is surjective, where σ is a Chevalley–Weyl involution of G. We then see that strongly visible actions, multiplicity-free restrictions of representations (c.f. Littelmann, Stembridge), the decomposition $G = LG^{\sigma}H$ and spherical actions are all equivalent in our setting.

Key words: Multiplicity-free representation; semisimple Lie group; flag variety; visible action; Cartan decomposition; herringbone stitch.

1. Introduction.

1.1. Classification problem of visible actions. Let G be a connected compact Lie group and L, H Levi subgroups of G. Then, the homogeneous spaces G/H and G/L are generalized flag varieties. In this article, we give a classification of triples (G, L, H) such that the following three group-actions are strongly visible:

$$
\begin{aligned}
&L \curvearrowright G/H,\\
&H \curvearrowright G/L,\\
&G \curvearrowright (G \times G)/(L \times H).
\end{aligned}
$$

Here, a holomorphic action of a group H on a complex manifold D is called strongly visible ([Ko2]) if the following two conditions are satisfied:

- There exists a real submanifold S (*slice*) such that $D' := H \cdot S$ is an open subset of D.
- There exists an anti-holomorphic diffeomorphism σ of D' such that $\sigma|_S = \mathrm{id}_S$ and $\sigma(H \cdot x) = H \cdot x$ for any $x \in S$.

Classification problem of visible actions was discussed previously in some other settings, see [Ko2, Ko4, Ko5, Sa] for example.

1.2. Multiplicity-free representations. The above problem is closely related to the multiplicity-freeness property of finite dimensional representations. Various examples of multiplicity-free representations have been obtained by many people. For finite dimensional cases, typical approaches are:

(a) (Sphericity) Verify the existence of an open orbit of a Borel subgroup.

(b) (Combinatorics) Using character formulas.

A new approach has been introduced by T. Kobayashi, i.e., the propagation theorem of multiplicity-freeness property ([Ko3, Theorem 4.3]) using the notion of visible actions on complex manifolds ([Ko1]). The advantage of this approach is that not only finite dimensional cases but also infinite dimensional (both discrete and continuous spectra) cases can be applied by this method (c.f. [Ko2, Ko6]).

1.3. Relation between a normalization theory of matrices and visible actions. A theory of normal forms is often connected with a decomposition theory of groups. A prototype is the diagonalization of symmetric matrices by orthogonal groups, which is equivalent to the Cartan decomposition $G = KAK$ for $G = \mathrm{GL}(n, \mathbf{R})$. A similar type of the decomposition theorem of the form $G = KBH$ or its variants has been well-established by the work of É. Cartan, M. Flensted–Jensen [Fl], B. Hoogenboom [Ho] and T. Matsuki [Ma] under the assumption that both (G, H) and (G, L) are symmetric pairs. As explained below, we find an analogous decomposition in the

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strongly visible setting where (G, H) and (G, L) are not necessarily symmetric pairs.

1.4. Classification problem of a generalized Cartan decomposition. Suppose that Levi subgroups L and H contain the same maximal torus T of G, and let σ be a Chevalley–Weyl involution of G with respect to T in the sense that $\sigma(t) = t^{-1}$ for any $t \in T$. We consider the following

Problem 1. *Classify triples* (G, L, H) *such that the multiplication mapping*

$$\psi : L \times G^\sigma \times H \to G \tag{1}$$

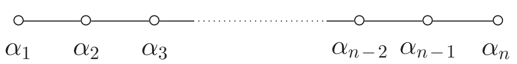
is surjective.

The Chevalley–Weyl involution σ induces an anti-holomorphic involution on the flag variety G/H, and $G^\sigma/(G^\sigma \cap H)$ is a totally real submanifold of the complex manifold G/H. Therefore if the multiplication mapping (1) is surjective then every L-orbits on G/H meets the totally real submanifold $G^\sigma/(G^\sigma \cap H)$, and hence the L-action on G/H is strongly visible. Likewise the other two group-actions $H \curvearrowright G/L$ and $G \curvearrowright (G \times G)/(L \times H)$ are strongly visible ([Ko1]). It is noteworthy that our classification results show that the converse is also true in the setting here (see Corollary 5.1).

2. Statement of the main result. Throughout this article, G is a connected compact simple Lie group, and we fix a simple system Π of the root system $\Delta(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$, and denote by L_j ($j = 1, 2$) the Levi subgroup whose root system is generated by a subset Π_j of Π. We say that (Π_1, Π_2) is of 'Hermitian type' when both G/L_1 and G/L_2 are Hermitian symmetric spaces, otherwise (Π_1, Π_2) is said to be of 'Non-Hermitian type'. We know as a special case of [Ko5] that $G = L_1 G^\sigma L_2$ if (Π_1, Π_2) is of Hermitian type. Here is a complete answer to Problem 1.

Main Theorem. *Let G be a connected compact simple Lie group, T a maximal torus of G, σ a Chevalley–Weyl involution and Π a simple system with respect to T. Then $G = L_1 G^\sigma L_2$ holds if and only if the pair (Π_1, Π_2) of proper subsets of Π is an entry of the tables below* (*we label the Dynkin diagrams following* [Bo]).

2.1. Classification for type $\mathbf{A}_n$ ([Ko4]).



Hermitian type:

I. $(\Pi_1)^c = \{\alpha_i\}, (\Pi_2)^c = \{\alpha_j\}$.

Non-Hermitian type:

I. $(\Pi_1)^c = \{\alpha_i, \alpha_j\}, (\Pi_2)^c = \{\alpha_k\}$. $\min_{p=i,j}\{p, n+1-p\} = 1$, *or* $i = j \pm 1$.

II. $(\Pi_1)^c = \{\alpha_i, \alpha_j\}, (\Pi_2)^c = \{\alpha_k\}$, $\min\{k, n+1-k\} = 2$.

III. $(\Pi_1)^c = \{\alpha_i\}, \Pi_2$: anything, $i = 1$ *or* n.

Here, i, j, k satisfy $1 \le i, j, k \le n$.

2.2. Classification for type $\mathbf{B}_n$.

Hermitian type:

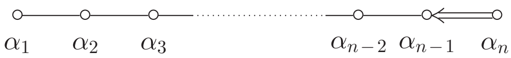
I. $(\Pi_1)^c = \{\alpha_1\}, (\Pi_2)^c = \{\alpha_1\}$.

Non-Hermitian type:

I. $(\Pi_1)^c = \{\alpha_n\}, (\Pi_2)^c = \{\alpha_n\}$.

II. $(\Pi_1)^c = \{\alpha_1\}, (\Pi_2)^c = \{\alpha_i\}$, $2 \le i \le n$.

2.3. Classification for type $\mathbf{C}_n$.



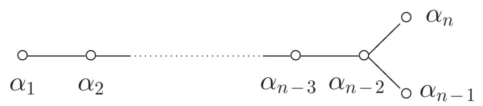
Hermitian type:

I. $(\Pi_1)^c = \{\alpha_n\}, (\Pi_2)^c = \{\alpha_n\}$.

Non-Hermitian type:

I. $(\Pi_1)^c = \{\alpha_1\}, (\Pi_2)^c = \{\alpha_i\}, 1 \le i \le n$.

2.4. Classification for type $\mathbf{D}_n$.



Hermitian type:

I. $(\Pi_1)^c = \{\alpha_i\}, (\Pi_2)^c = \{\alpha_j\}$, $i, j \in \{1, n-1, n\}$.

Non-Hermitian type:

I. $(\Pi_1)^c = \{\alpha_1\}, (\Pi_2)^c = \{\alpha_j\}$, $j \ne 1, n-1, n$.

II. $(\Pi_1)^c = \{\alpha_i\}, (\Pi_2)^c = \{\alpha_j\}$, $i \in \{n-1, n\}, j \in \{2, 3\}$.

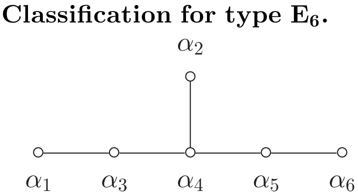
III. $(\Pi_1)^c = \{\alpha_i\}, (\Pi_2)^c = \{\alpha_j, \alpha_k\}$, $i \in \{n-1, n\}, j, k \in \{1, n-1, n\}$.

IV. $(\Pi_1)^c = \{\alpha_i\}, (\Pi_2)^c = \{\alpha_1, \alpha_2\}$, $i \in \{n-1, n\}$.

V. $(\Pi_1)^c = \{\alpha_1\}, (\Pi_2)^c = \{\alpha_j, \alpha_k\}$, $j \in \{n-1, n\}$ or $k \in \{n-1, n\}$.

VI. $(\Pi_1)^c = \{\alpha_i\}, (\Pi_2)^c = \{\alpha_2, \alpha_j\}$, $n = 4, (i, j) = (3, 4)$ or $(4, 3)$.

2.5. Classification for type $\mathbf{E}_6$.



Hermitian type:

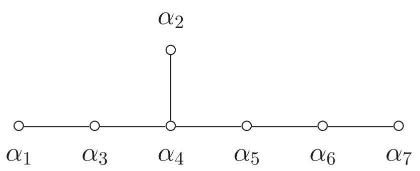
I. $(\Pi_1)^c = \{\alpha_i\}, (\Pi_2)^c = \{\alpha_j\}, i, j \in \{1, 6\}$.

Non-Hermitian type:

I. $(\Pi_1)^c = \{\alpha_i\}, (\Pi_2)^c = \{\alpha_1, \alpha_6\}, i = 1$ or 6.

II. $(\Pi_1)^c = \{\alpha_i\}, (\Pi_2)^c = \{\alpha_j\}$,
$i = 1$ or $6, j \neq 1, 4, 6$.

2.6. Classification for type $\mathbf{E_7}$.



Hermitian type:

I. $(\Pi_1)^c = \{\alpha_7\}, (\Pi_2)^c = \{\alpha_7\}$.

Non-Hermitian type:

I. $(\Pi_1)^c = \{\alpha_7\}, (\Pi_2)^c = \{\alpha_i\}, i = 1$ *or* 2.

2.7. Classification for type $\mathbf{E_8}$, $\mathbf{F_4}$, $\mathbf{G_2}$. *There is no pair* (Π_1, Π_2) *such that* $G = L_1 G^\sigma L_2$ *holds.*

For the convenience of the reader, we state an equivalent form of the classification of the triples $(\mathfrak{g}, \mathfrak{l}_1, \mathfrak{l}_2)$ of the Lie algebras in Main Theorem.

2.1′. Classification of $(\mathfrak{l}_1, \mathfrak{l}_2)$ for $\mathfrak{g} = \mathfrak{u}(n)$ ([Ko4]).

Hermitian type:

I. $(\mathbf{R}^2 \oplus \mathfrak{su}(i) \oplus \mathfrak{su}(n-i)$,
$\mathbf{R}^2 \oplus \mathfrak{su}(j) \oplus \mathfrak{su}(n-j))$.

Non-Hermitian type:

I. $(\mathbf{R}^3 \oplus \mathfrak{su}(i) \oplus \mathfrak{su}(n-i-1)$,
$\mathbf{R}^2 \oplus \mathfrak{su}(j) \oplus \mathfrak{su}(n-j))$.

II. $(\mathbf{R}^3 \oplus \mathfrak{su}(i) \oplus \mathfrak{su}(j) \oplus \mathfrak{su}(n-i-j)$,
$\mathbf{R}^2 \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(n-2))$.

III. $\mathfrak{l}_1 : \mathbf{R}^2 \oplus \mathfrak{su}(n-1)$, $\mathfrak{l}_2$: anything.

Here, i, j satisfy $1 \leq i, j \leq n$.

2.2′. Classification of $(\mathfrak{l}_1, \mathfrak{l}_2)$ for $\mathfrak{g} = \mathfrak{so}(2n+1)$.

Hermitian type:

I. $(\mathbf{R} \oplus \mathfrak{so}(2n-1)$, $\mathbf{R} \oplus \mathfrak{so}(2n-1)$.

Non-Hermitian type:

I. $(\mathfrak{u}(n), \mathfrak{u}(n))$.

II. $(\mathbf{R} \oplus \mathfrak{so}(2n-1)$,
$\mathbf{R} \oplus \mathfrak{su}(i) \oplus \mathfrak{so}(2n-2i+1))$,
$2 \leq i \leq n$.

2.3′. Classification of $(\mathfrak{l}_1, \mathfrak{l}_2)$ for $\mathfrak{g} = \mathfrak{sp}(n)$.

Hermitian type:

I. $(\mathfrak{u}(n), \mathfrak{u}(n))$.

Non-Hermitian type:

I. $(\mathbf{R} \oplus \mathfrak{sp}(n-1)$, $\mathbf{R} \oplus \mathfrak{su}(i) \oplus \mathfrak{sp}(n-i))$,
$1 \leq i \leq n$.

2.4′. Classification of $(\mathfrak{l}_1, \mathfrak{l}_2)$ for $\mathfrak{g} = \mathfrak{so}(2n)$.

Hermitian type:

I. $(\mathbf{R} \oplus \mathfrak{so}(2n-2)$, $\mathbf{R} \oplus \mathfrak{su}(n))$,
$(\mathbf{R} \oplus \mathfrak{so}(2n-2)$, $\mathbf{R} \oplus \mathfrak{so}(2n-2))$,
$(\mathbf{R} \oplus \mathfrak{su}(n)$, $\mathbf{R} \oplus \mathfrak{su}(n))$.

Non-Hermitian type:

I. $(\mathbf{R} \oplus \mathfrak{so}(2n-2)$, $\mathbf{R} \oplus \mathfrak{su}(j) \oplus \mathfrak{so}(2n-2j))$,
$j \neq 1, n$.

II. $(\mathbf{R} \oplus \mathfrak{su}(n)$, $\mathbf{R} \oplus \mathfrak{su}(j) \oplus \mathfrak{so}(2n-2j))$,
$j = 2$ or 3.

III. $(\mathbf{R} \oplus \mathfrak{su}(n)$, $\mathbf{R}^2 \oplus \mathfrak{su}(n-1))$.

IV. $(\mathbf{R} \oplus \mathfrak{su}(n)$, $\mathbf{R}^2 \oplus \mathfrak{so}(2n-4))$.

V. $(\mathbf{R} \oplus \mathfrak{so}(2n-2)$, $\mathbf{R}^2 \oplus \mathfrak{su}(j) \oplus \mathfrak{su}(n-j))$,
$1 \leq j \leq n-1$.

VI. $(\mathbf{R} \oplus \mathfrak{su}(4)$, $\mathbf{R}^2 \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2))$,
for $n = 4$ only.

2.5′. Classification of $(\mathfrak{l}_1, \mathfrak{l}_2)$ for $\mathfrak{g} = \mathfrak{e}_6$.

Hermitian type:

I. $(\mathbf{R} \oplus \mathfrak{so}(10)$, $\mathbf{R} \oplus \mathfrak{so}(10))$.

Non-Hermitian type:

I. $(\mathbf{R} \oplus \mathfrak{so}(10)$, $\mathbf{R}^2 \oplus \mathfrak{so}(8))$.

II. $(\mathbf{R} \oplus \mathfrak{so}(10)$, $\mathbf{R} \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(5))$,
$(\mathbf{R} \oplus \mathfrak{so}(10)$, $\mathbf{R} \oplus \mathfrak{su}(6))$.

2.6′. Classification of $(\mathfrak{l}_1, \mathfrak{l}_2)$ for $\mathfrak{g} = \mathfrak{e}_7$.

Hermitian type:

I. $(\mathbf{R} \oplus \mathfrak{e}_6$, $\mathbf{R} \oplus \mathfrak{e}_6)$.

Non-Hermitian type:

I. $(\mathbf{R} \oplus \mathfrak{e}_6$, $\mathbf{R} \oplus \mathfrak{so}(12))$,
$(\mathbf{R} \oplus \mathfrak{e}_6$, $\mathbf{R} \oplus \mathfrak{su}(7))$.

2.7′. Classification of $(\mathfrak{l}_1, \mathfrak{l}_2)$ for $\mathfrak{g} = \mathfrak{e}_8$, $\mathfrak{f}_4$ or $\mathfrak{g}_2$. There is no pair $(\mathfrak{l}_1, \mathfrak{l}_2)$ such that $G = L_1 G^\sigma L_2$ holds.

3. On the proof of sufficiency. For the proof of sufficiency of Main Theorem, we give a stronger result, namely, find an analogue of the Cartan decomposition $G = L_1 B L_2$ where B is some subset of G^σ. (In fact, this is the Cartan decomposition in the symmetric setting.) For this, we use the herringbone stitch method ([Ko4]) which reduces unknown decompositions to the known decomposition in the symmetric case. This method enables us to obtain a generalized Cartan decomposition $G = L_1 B L_2$ with $B \subset G^\sigma$ for almost all of the pairs (L_1, L_2) listed in the above. (The only exception is Case I of Non-Hermitian type for $\mathfrak{so}(2n+1)$, and in this case our proof is analogous to that of KAK decomposition for reductive groups.)

4. On the proof of necessity. In this section, we explain an idea of the proof of necessity of Main Theorem.

4.1. Classical cases. As in [Ko4], we can prove that $G \neq L_1 G^\sigma L_2$ for any pair (Π_1, Π_2) which is not in the tables in Main Theorem by using invariant theory for quivers associated to Levi subgroups. Let us explain briefly its idea.

Let G be a classical compact simple Lie group. We choose a suitable embedding of G into the unitary group $\mathrm{U}(N)(\subset \mathrm{M}(N,\mathbf{C}))$ so that G has a maximal torus consisting of diagonal matrices (we give an example of such embedding below). The point here is that it enables us to take the complex conjugation with respect to the real matrices $\mathrm{M}(N,\mathbf{R})$ as a Chevalley–Weyl involution of G, and Levi subgroups consisting of block diagonal matrices. Let σ be the complex conjugation and L, H Levi subgroups of G. Suppose that the subgroup H is of the form $H = \{x \in G : xJ = Jx\}$ for some $J \in \mathrm{M}(N,\mathbf{R})$. Then, we can find easily that $\mathrm{Ad}(G^\sigma H)J$ is contained in $\mathrm{M}(N,\mathbf{R})$. Therefore if there exists $g \in G$ such that

$$\mathrm{Ad}(Lg)J \cap \mathrm{M}(N,\mathbf{R}) = \varnothing, \tag{2}$$

then we obtain

$$G \neq LG^\sigma H. \tag{3}$$

In order to consider (1) infinitesimally, we fix a concrete realization of the classical compact Lie algebras as follows.

(A) $\mathfrak{u}(n) := \{X \in \mathfrak{gl}(n,\mathbf{C}) : {}^t\overline{X} + X = O\}$,
(B) $\mathfrak{so}(2n+1) := \{X \in \mathfrak{u}(2n+1) : {}^tXJ_{2n+1} + J_{2n+1}X = O\}$,
(C) $\mathfrak{sp}(n) := \{X \in \mathfrak{u}(2n) : {}^tXJ'_n + J'_nX = O\}$,
(D) $\mathfrak{so}(2n) := \{X \in \mathfrak{u}(2n) : {}^tXJ_{2n} + J_{2n}X = O\}$,

where J_m is the $m \times m$ anti-diagonal matrix whose non-zero entries consist only of 1, and J'_m is the $2m \times 2m$ anti-diagonal matrix whose non-zero entries in the upper half part consist only of 1, and that in the lower half part only of -1. Then, let $n = n_1 + \cdots + n_k$ be the partition with $n_i > 0$ $(i \neq k)$, $n_k \geq 0$, which corresponds to the Levi subgroup L, and X be an element of the Lie algebra of G. We express X in the block matrix form as follows.

(A) Write $X = (X_{ij})_{1\leq i,j\leq k}$ correspondingly to the partition $n = n_1 + \cdots + n_k$.
(B) Write $X = (X_{ij})_{1\leq i,j\leq 2k-1}$ correspondingly to the partition $2n+1 = n_1 + \cdots + n_{k-1} + (2n_k+1) + n_{k-1} + \cdots + n_1$.
(C) Write $X = (X_{ij})_{1\leq i,j\leq 2k-1}$ correspondingly to the partition $2n = n_1 + \cdots + n_{k-1} + 2n_k + n_{k-1} + \cdots + n_1$.
(D) Write $X = (X_{ij})_{1\leq i,j\leq 2k-1}$ correspondingly to the partition $2n = n_1 + \cdots + n_{k-1} + 2n_k + n_{k-1} + \cdots + n_1$.

Let $i_0 \to i_1 \to \cdots \to i_r = i_0$ be a loop with $i_s \in \{1,\dots,p\}$, $i_s \neq i_{s+1}$ $(0 \leq s \leq r-1)$ and $r \geq 2$, where p is the square root of the number of blocks of X (for instance, $p = k$ if G is of type A_{n-1}). Then we define a non-linear mapping $A_{i_0 i_1\cdots i_r}$ in the following way.

(A)

$$\mathrm{M}(n,\mathbf{C}) \xrightarrow{A_{i_0\cdots i_r}} \mathrm{M}(n_{i_0},\mathbf{C}),$$
$$P \longmapsto \tilde{P}_{i_0i_1}\tilde{P}_{i_1i_2}\cdots\tilde{P}_{i_{r-1}i_r},$$

where the matrix $\tilde{P}_{ij}$ is defined by

$$\tilde{P}_{ij} := \begin{cases} P_{ij} & (i<j), \\ {}^t\overline{P}_{ji} & (i>j). \end{cases}$$

(B)

$$\mathrm{M}(2n+1,\mathbf{C}) \xrightarrow{A_{i_0\cdots i_r}} \begin{cases} \mathrm{M}(n_{i_0},\mathbf{C}) & (i_0 \neq k), \\ \mathrm{M}(2n_{i_0}+1,\mathbf{C}) & (i_0 = k), \end{cases}$$
$$P \longmapsto \tilde{P}_{i_0i_1}\tilde{P}_{i_1i_2}\cdots\tilde{P}_{i_{r-1}i_r},$$

where the matrix $\tilde{P}_{ij}$ is defined by

$$\tilde{P}_{ij} := \begin{cases} P_{ij} & (i+j \leq 2k), \\ J_{n_i}{}^tP_{2k-j,2k-i}J_{n_j} & (i+j > 2k, i,j \neq k), \\ J_{2n_k+1}{}^tP_{2k-j,k}J_{n_j} & (i=k, j>k), \\ J_{n_i}{}^tP_{k,2k-i}J_{2n_k+1} & (i>k, j=k). \end{cases}$$

(C)

$$\mathrm{M}(2n,\mathbf{C}) \xrightarrow{A_{i_0\cdots i_r}} \begin{cases} \mathrm{M}(n_{i_0},\mathbf{C}) & (i_0 \neq k), \\ \mathrm{M}(2n_{i_0},\mathbf{C}) & (i_0 = k), \end{cases}$$
$$P \longmapsto \tilde{P}_{i_0i_1}\tilde{P}_{i_1i_2}\cdots\tilde{P}_{i_{r-1}i_r},$$

where the matrix $\tilde{P}_{ij}$ is defined by

$$\tilde{P}_{ij} := \begin{cases} P_{ij} & (i+j \leq 2k), \\ J_{n_i}{}^tP_{2k-j,2k-i}J_{n_j} & (i+j > 2k, i,j \neq k), \\ J'_{n_k}{}^tP_{2k-j,k}J_{n_j} & (i=k, j>k), \\ J_{n_i}{}^tP_{k,2k-i}J'_{n_k} & (i>k, j=k). \end{cases}$$

(D)

$$\mathrm{M}(2n,\mathbf{C}) \xrightarrow{A_{i_0\cdots i_r}} \begin{cases} \mathrm{M}(n_{i_0},\mathbf{C}) & (i_0 \neq k), \\ \mathrm{M}(2n_{i_0},\mathbf{C}) & (i_0 = k), \end{cases}$$
$$P \longmapsto \tilde{P}_{i_0i_1}\tilde{P}_{i_1i_2}\cdots\tilde{P}_{i_{r-1}i_r},$$

where the matrix $\tilde{P}_{ij}$ is defined by

$$\tilde{P}_{ij} := \begin{cases} P_{ij} & (i+j \leq 2k), \\ J_{n_i}{}^tP_{2k-j,2k-i}J_{n_j} & (i+j > 2k, i,j \neq k), \\ J_{2n_k}{}^tP_{2k-j,k}J_{n_j} & (i=k, j>k), \\ J_{n_i}{}^tP_{k,2k-i}J_{2n_k} & (i>k, j=k). \end{cases}$$

In any of these four cases, we have

$$A_{i_0\cdots i_r}(\mathrm{Ad}(l)X) = l_{i_0}A_{i_0\cdots i_r}(X)l_{i_0}^{-1}, \tag{4}$$

for any $l = (l_1, \dots, l_k) \in L = \mathrm{U}(n_1) \times \cdots \mathrm{U}(n_{k-1}) \times G(n_k)$. Here, $G(n_k)$ denotes $\mathrm{U}(n_k)$, $\mathrm{SO}(2n_k+1)$, $\mathrm{Sp}(n_k)$ or $\mathrm{SO}(2n_k)$. From the equality (4), we can find that the determinant is an invariant of the conjugation by L. Therefore, there exists an element $g \in G$ satisfying (2) if there are a loop $i_0 \to i_1 \to \cdots \to i_r = i_0$ and an element X of the Lie algebra of G such that the characteristic polynomial of the matrix $A_{i_0 \cdots i_r}([X, J])$ is not of real coefficients. The proof of the necessity part of Main Theorem (classical cases) can be carried out by finding such a loop and an element of the Lie algebra for each of pairs (Π_1, Π_2) which are not listed in the tables in Main Theorem.

4.2. Exceptional cases. For the proof in the exceptional cases, we use the propagation theorem of multiplicity-freeness property ([Ko3]) and Stembridge's classification of multiplicity-free tensor products ([St]).

5. Applications. Let G be a connected compact Lie group and L_j $(j = 1, 2)$ be the Levi subgroup corresponding to the subset Π_j of the simple system as in Section 2. In this section, we write $\mathcal{P}_j = G/L_j$ for the generalized flag variety corresponding to Π_j, and $(L_j)_{\mathbf{C}}$ for the complexification of L_j. We say an irreducible representation π of G belongs to a $\mathcal{P}_j$-series if the highest weight of π is given by a linear sum of fundamental weights corresponding to Π_j^c with coefficients in non-negative integers. In other words, π is obtained by a holomorphically induced representation from a unitary character of L_j.

Corollary 5.1. *Let G be a connected compact simple Lie group. The following eleven conditions on the pair of Levi subgroups L_1, L_2 are equivalent.*

(i) *The multiplication mapping $L_1 \times G^\sigma \times L_2 \to G$ is surjective.*
(ii) *The natural action $L_1 \curvearrowright \mathcal{P}_2$ is strongly visible.*
(iii) *The natural action $L_2 \curvearrowright \mathcal{P}_1$ is strongly visible.*
(iv) *The diagonal action $G \curvearrowright \mathcal{P}_1 \times \mathcal{P}_2$ is strongly visible.*
(v) *Any irreducible representation of G, which belongs to $\mathcal{P}_2$-series is multiplicity-free when restricted to L_1.*
(vi) *Any irreducible representation of G, which belongs to $\mathcal{P}_1$-series is multiplicity-free when restricted to L_2.*
(vii) *The tensor product of arbitrary two irreducible representations π_j $(j = 1, 2)$ of G belonging to $\mathcal{P}_j$-series is multiplicity-free.*
(viii) *$\mathcal{P}_2$ is a spherical variety of $(L_1)_{\mathbf{C}}$.*
(ix) *$\mathcal{P}_1$ is a spherical variety of $(L_2)_{\mathbf{C}}$.*
(x) *$\mathcal{P}_1 \times \mathcal{P}_2$ is a spherical variety of $(G)_{\mathbf{C}}$.*
(xi) *The pair (Π_1, Π_2) is one of the entries listed in Main Theorem up to switch of the factors.*

Proof. We prove that Main Theorem implies this corollary. The strategy of the proof is summarized in the below diagram.

$$
\begin{array}{ccccl}
 & \text{(vii)} & & \cdots\cdots & \text{multiplicity-free}\\
 & \Updownarrow & & & \\
 & \text{(xi)} & & \cdots\cdots & \text{classification of } (L_1,\ L_2)\\
 & \Updownarrow & & & \\
 & \text{(i)} & & \cdots\cdots & \text{Cartan decomposition}\\
\swarrow & \Downarrow & \searrow & & \\
\text{(ii)} & \text{(iv)} & \text{(iii)} & \cdots\cdots & \text{visible action}\\
\Downarrow & \Downarrow & \Downarrow & & \\
\text{(v)} \Leftrightarrow & \text{(vii)} & \Leftrightarrow \text{(vi)} & \cdots\cdots & \text{multiplicity-free}\\
\Updownarrow & \Updownarrow & \Updownarrow & & \\
\text{(viii)} & \text{(x)} & \text{(ix)} & \cdots\cdots & \text{spherical action.}
\end{array}
$$

The implication (vii) $\Rightarrow$ (xi) can be verified by comparing Stembridge's classification ([St]) with Main Theorem. The converse implication (xi) $\Rightarrow$ (vii) follows from the propagation theorem of multiplicity-freeness property ([Ko3, Theorem 4.3]). The equivalence (xi) $\Leftrightarrow$ (i) is our Main Theorem. The implications (i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (iv) are the *triunity of visibility* ([Ko1]). Each of the three implications (ii) $\Rightarrow$ (v), (iii) $\Rightarrow$ (vi) and (iv) $\Rightarrow$ (vii) is followed by the propagation theorem of multiplicity-freeness property. As in the proof of [Ko2, Corollary 15], we see that Vinberg and Kimel'fel'd [VK, Corollary 1] implies the three equivalences (v) $\Leftrightarrow$ (viii), (vi) $\Leftrightarrow$ (ix) and (vii) $\Leftrightarrow$ (x). The equivalence (v) $\Leftrightarrow$ (vii) $\Leftrightarrow$ (vi) on multiplicity-freeness of representations follows from Stembridge [St, Corollary 2.5]. This completes the proof of the corollary. □

Remark 5.2. For the pioneer work for Corollary 5.1, see [Ko4] for the type A case, and [Ko5] for the Hermitian case.

Remark 5.3. We note that L_j is maximal if and only if Π_j^c is a singleton. Under the assumption that both L_1 and L_2 are maximal, P. Littelmann [Li] proved (x) $\Leftrightarrow$ (xi). Our proof gives a new approach without the assumption of maximality.

The details of the proof of Main Theorem will appear elsewhere.

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