Some relations of Poncelet's porism for two ellipses

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Abstract: This paper shows the necessary and sufficient condition for bicentrical polygons which are circumscribed and inscribed by two ellipses using Jacobian elliptic functions. Moreover the formulae for a bicentrical triangle, quadrilateral and pentagon are presented and the fact that these formulae are the necessary and sufficient conditions for bicentrical polygons is presented.

Key words: Poncelet's closure theorem; bicentrical polygons; elliptic functions.

1. Introduction. The closure theorem of Poncelet is one of the most beautiful one in the mathematical science. The closure theorem had been proven by Jacobi [5] and Griffiths [4] as well as Poncelet himself [8]. Without saying Poncelet only considered conics in the real plane and proven it using projective geometry, but Jacobi and Griffiths did by different methods: using elliptic functions for pairs of circles in the real plane and using elliptic curves for smooth conics in complex projective space, respectively.

On the other hand, the relation between the radii and the line segment joining the centers of the circles of circumscription and inscription of a bicentrical polygon has been studied from of old. The relataion for a bicentrical triangle was given by Euler (sometimes called Chappele's formula [1]) as follows:

$$r^2 - d^2 = 2r\rho,$$

where r and ρ are radii and d is the distance between the centers of the circles of circumscription and inscription. The corresponding formula for a quadrilateral is

$$2\rho^2(r^2 + d^2) = (r^2 - d^2)^2,$$

which was given by Fuss [2]. Also Steiner [9] gave the formula for a pentagon as

$$\rho(r-d) = (r+d)\sqrt{(r-\rho+d)(r-\rho-d)} + (r+d)\sqrt{(r-\rho-d)2r}.$$

Some other fromulae were obtained for bicentrical higher polygons such as a hexagon, heptagon, etc.

This paper treats with bicentrical polygons for two ellipses [6]. We show the necessary and sufficient condition for bicentrical polygons which are circumscribed and inscribed by two ellipses. Moreover the formulae for a bicentrical triangle, quadrilateral and pentagon are presented and the fact that these formulae are the necessary and sufficient conditions for bicentrical polygons is shown.

This paper is constructed as follows: first we define some terminologies such as Poncelet's traverse and Poncelet's porism in the second section. In the third section we show the relations of the tangent on an inner ellipse and its intersections on an outer ellipse. The fourth section is for the theorem. In the final section the formulae for some bicentrical polygons are presented [7].

2. Poncelet's traverse and porism. We define *a Poncelet's traverse* and *a Poncelet's porism* in the following definitions:

Definition 2.1 (Poncelet's traverse). Let E_o and E_i be two ellipses in a plane. Suppose that the ellipse E_i is surrounded by the ellipse E_o. If from any point Q_1 on E_o we draw a tangent to E_i and extend the tangent so that it intersects E_o. Let Q_2 be the intersecting point on E_o. Again we draw a new tangent to E_i from Q_2 and extend this tangent similarly to intersect E_o. Let the intersecting point taken by this procedure be Q_3, which is different from Q_1. We continue in this way and obtain a series of points $Q_1Q_2Q_3\dots Q_iQ_{i+1}\dots$. We call this a series of points $Q_1Q_2Q_3\dots Q_iQ_{i+1}\dots$ a Poncelet's traverse. We obtain two different Poncelet's traverses depending on how to draw a tangent: clockwise or counter-clockwise. We don't change the direction of rotation of a series of points created on E_o.

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We can classify a Poncelet's traverse into the following two cases:
(1) $Q_i \neq Q_j, \forall i, j \in \mathbf{N}$.
(2) $\exists n \in \mathbf{N},\ Q_\ell = Q_{\ell+n},$ for $\forall \ell \in \mathbf{N}$ and $Q_i \neq Q_j$, $1 \leq i, j < n$.

Definition 2.2 (Poncelet's porism)**.** The case $n \geq 3$ of the above classified type (2) is called the Poncelet's porism and the created polygon $Q_1 Q_2 \dots Q_n$ is called the poristic n-gon.

The Poncelet's closure theorem says that in a poristic n-gon from any Q_1' on E_o a series of points $Q_1' Q_2' \dots Q_n' Q_{n+1}' Q_{n+2}' \dots$ becomes $Q_1' Q_2' \dots Q_n' Q_1' Q_2' \dots$, that is, a series of points creates the Poncelet's porism.

Remark 2.1. The shape of a poristic n-gon depends on the location of the initial point Q_1 but doesn't depend on the clockwise or counter-clockwise rotation of a Poncelet's traverse.

3. The relations of the tangent on an inner ellipse and its intersections on an outer ellipse. Let $A, B, a, b > 0$ and two ellipses E_o and E_i be written as

$$E_o = \left\{ (x, y) \;\middle|\; \frac{x^2}{A^2} + \frac{y^2}{B^2} - 1 = 0 \right\}, \tag{3.1}$$

$$E_i = \left\{ (x, y) \;\middle|\; \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \right\}. \tag{3.2}$$

Also we let

$$\tilde{E}_o = \left\{ (x, y) \;\middle|\; \frac{x^2}{A^2} + \frac{y^2}{B^2} - 1 \leq 0 \right\},$$
$$\tilde{E}_i = \left\{ (x, y) \;\middle|\; \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \leq 0 \right\}.$$

Now we prepare two lemmas without proofs.

Lemma 3.1. *$\tilde{E}_i \subset \tilde{E}_o$ if and only if $a < A$ and $b < B$.*

Lemma 3.2. *If $\tilde{E}_i \subset \tilde{E}_o$, then any tangent on E_i has two intersecting points with E_o.*

We let the two-dimensional plane except the origin be shown by one of the following polar coordinates:
(i) $(\rho \operatorname{cn}(u, k), \rho \operatorname{sn}(u, k))$ for a fixed k,
(ii) $(\rho \operatorname{sn}(u, k), \rho \operatorname{cn}(u, k))$ for a fixed k,
(iii) $(\rho \cos u, \rho \sin u)$ or $(\rho \sin u, \rho \cos u)$.
Here the modulus k of Jacobian elliptic functions is deternimed uniquely as $k^2 = \frac{A^2 b^2 - B^2 a^2}{b^2 (A^2 - a^2)}$ for the coordinate (i) and $k^2 = \frac{B^2 a^2 - A^2 b^2}{a^2 (B^2 - b^2)}$ for the coordinate (ii). The coordinate (i) and (ii) correspond to $\frac{b}{B} \geq \frac{a}{A}$ and $\frac{a}{A} \geq \frac{b}{B}$, respectively. Also The coordinate (iii) corresponds to $\frac{a}{A} = \frac{b}{B}$. To avoid unnecessary annoyance we shall omit the modulus hereinafter if nothing intervenes. Also $K(k)$ is the complete elliptic integral of the first kind.

Theorem 3.1. *) *Let $0 < a < A$ and $0 < b < B$ and two ellipses E_o and E_i be written as* (3.1) *and* (3.2)*. We suppose that $-K(k) < v_0 < K(k)$. The followings are equivalent.*
(1) *A tangent of E_i at $(a \operatorname{cn} u, b \operatorname{sn} u), u \in \mathbf{R}$ intersects at the following two points on E_o:*

$$\begin{aligned} &(A \operatorname{cn}(u - v_0), B \operatorname{sn}(u - v_0)), \\ &(A \operatorname{cn}(u + v_0), B \operatorname{sn}(u + v_0)). \end{aligned} \tag{3.3}$$

(2) *$\frac{a}{A}$ and $\frac{b}{B}$ are independent of u and written as follows:*

$$\frac{a}{A} = \operatorname{cn} v_0, \ \frac{b}{B} = \frac{\operatorname{cn} v_0}{\operatorname{dn} v_0}. \tag{3.4}$$

Proof. (1) $\Rightarrow$ (2)
First a tangent of E_i at a point $(a \operatorname{cn} u, b \operatorname{sn} u)$ is presented by

$$\frac{\operatorname{cn} u}{a} x + \frac{\operatorname{sn} u}{b} y = 1. \tag{3.5}$$

On the other hand, the line connecting two intersecting points (3.3) is written by

$$\begin{aligned} y = &\frac{B}{A} \frac{\operatorname{sn}(u + v_0) - \operatorname{sn}(u - v_0)}{\operatorname{cn}(u + v_0) - \operatorname{cn}(u - v_0)} x \\ &- B \frac{\operatorname{sn}(u + v_0) \operatorname{cn}(u - v_0) - \operatorname{sn}(u - v_0) \operatorname{cn}(u + v_0)}{\operatorname{cn}(u + v_0) - \operatorname{cn}(u - v_0)}. \end{aligned}$$

We shall write

$$\begin{aligned} &s_1 = \operatorname{sn} u, \ s_2 = \operatorname{sn} v_0, \ c_1 = \operatorname{cn} u, \\ &c_2 = \operatorname{cn} v_0, \ d_1 = \operatorname{dn} u, \ d_2 = \operatorname{dn} v_0 \end{aligned}$$

for short to avoid troublesome symbols hereafter. Using following addition theorems [10]:

$$\begin{aligned} \operatorname{sn}(u + v_0) \operatorname{cn}(u - v_0) &= \frac{s_1 c_1 d_2 + s_2 c_2 d_1}{1 - k^2 s_1^2 s_2^2}, \\ \operatorname{sn}(u - v_0) \operatorname{cn}(u + v_0) &= \frac{s_1 c_1 d_2 - s_2 c_2 d_1}{1 - k^2 s_1^2 s_2^2}, \\ \operatorname{sn}(u + v_0) - \operatorname{sn}(u - v_0) &= \frac{2 s_2 c_1 d_1}{1 - k^2 s_1^2 s_2^2}, \\ \operatorname{cn}(u + v_0) - \operatorname{cn}(u - v_0) &= -\frac{2 s_1 s_2 d_1 d_2}{1 - k^2 s_1^2 s_2^2}, \end{aligned}$$

we have

*) We find a similar problem of this theorem in the reference (Example 7, p141, [3]) but the problem wasn't solved completely.

$$y = -\frac{B}{A}\frac{c_1}{s_1 d_2}x + B\frac{c_2}{s_1 d_2}, \tag{3.6}$$

that is,

$$\frac{\operatorname{cn} u}{A \operatorname{cn} v_0}x + \frac{\operatorname{sn} u}{\frac{B \operatorname{cn} v_0}{\operatorname{dn} v_0}}y = 1. \tag{3.7}$$

We obtain (3.4) from (3.5) and (3.7).
(2) $\Rightarrow$ (1)
The fact that a tangent on E_i has two intersecting points on E_o is guaranteed by Lemma 3.2 since $a < A$ and $b < B$. The equation of a tangent is presented by

$$\frac{\operatorname{cn} u}{A \operatorname{cn} v_0}x + \frac{\operatorname{sn} u}{\frac{B \operatorname{cn} v_0}{\operatorname{dn} v_0}}y = 1$$

since we suppose (3.4). Therefore we obtain the intersecting points from the above equation and (3.1). Eliminating y yields the following quadratic equation:

$$(s_1^2 d_2^2 + c_1^2)x^2 - 2c_1 c_2 A x + (c_2^2 - s_1^2 d_2^2)A^2 = 0.$$

From this we obtain x-coordinates of the intersecting points as follows:

$$\begin{aligned} x &= \frac{c_1 c_2 A \pm A\sqrt{s_1^2 d_2^2 (c_1^2 - c_2^2 + s_1^2 d_2^2)}}{s_1^2 d_2^2 + c_1^2} \\ &= \frac{c_1 c_2 A \pm A\sqrt{s_1^2 d_2^2 s_2^2 d_1^2}}{1 - k^2 s_1^2 s_2^2} = A \operatorname{cn}(u \mp v_0) \end{aligned} \tag{3.8}$$

since $c_1^2 - c_2^2 + s_1^2 d_2^2 = 1 - s_1^2 - 1 + s_2^2 + s_1^2(1 - k^2 s_2^2) = s_2^2 d_1^2$ and $s_1^2 d_2^2 + c_1^2 = 1 - k^2 s_1^2 s_2^2$. On the other hand, y-coordinates of the intersecting points are easily obtained by the fact that (3.8) lie on E_o. $\square$

We show the following corollaries when we take the polar coordinates (ii) or (iii). The proofs are similar so that we omit them.

Corollary 3.1. *Let $0 < a < A$ and $0 < b < B$ and two ellipses E_o and E_i be written as* (3.1) *and* (3.2). *We suppose that $-K(k) < v_0 < K(k)$. The followings are equivalent.*
(1) *A tangent of E_i at $(a \operatorname{sn} u, b \operatorname{cn} u), u \in \mathbf{R}$ intersects at the following two points on E_o:*

$$(A \operatorname{sn}(u - v_0), B \operatorname{cn}(u - v_0)),$$
$$(A \operatorname{sn}(u + v_0), B \operatorname{cn}(u + v_0)).$$

(2) $\frac{a}{A}$ *and* $\frac{b}{B}$ *are independent of u and written as follows*:

$$\frac{a}{A} = \frac{\operatorname{cn} v_0}{\operatorname{dn} v_0}, \quad \frac{b}{B} = \operatorname{cn} v_0.$$

Corollary 3.2. *Let $0 < a < A$ and $0 < b < B$ and two ellipses E_o and E_i be written as* (3.1) *and* (3.2). *We suppose that $-\frac{\pi}{2} < v_0 < \frac{\pi}{2}$. The followings are equivalent.*
(1) *A tangent of E_i at $\left(a\begin{pmatrix}\cos\\ \sin\end{pmatrix}u, b\begin{pmatrix}\sin\\ \cos\end{pmatrix}u\right)$, $u \in \mathbf{R}$ intersects at the following two points on E_o:*

$$\left(A\begin{pmatrix}\cos\\ \sin\end{pmatrix}(u - v_0), B\begin{pmatrix}\sin\\ \cos\end{pmatrix}(u - v_0)\right),$$
$$\left(A\begin{pmatrix}\cos\\ \sin\end{pmatrix}(u + v_0), B\begin{pmatrix}\sin\\ \cos\end{pmatrix}(u + v_0)\right).$$

(2) $\frac{a}{A}$ *and* $\frac{b}{B}$ *are independent of u and written as follows*:

$$\frac{a}{A} = \frac{b}{B} = \cos v_0.$$

4. The necessary and sufficient condition for a poristic n-gon in two ellipses.

Now we write two ellipses E_o and E_i as follows:

$$E_o = \left\{(x, y) \,\middle|\, \frac{x^2}{A^2} + \frac{y^2}{B^2} - 1 = 0\right\}, \tag{4.1}$$

$$E_i = \left\{(x, y) \,\middle|\, \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0, \; a = A \operatorname{cn} v_0, b = \frac{B \operatorname{cn} v_0}{\operatorname{dn} v_0}\right\}. \tag{4.2}$$

Here $-K < v_0 < K$. In this situation, the intersecting points of E_o and a tangent of E_i at a point $(a \operatorname{cn} u, b \operatorname{sn} u)$ are presented by

$$\begin{aligned} &(A \operatorname{cn}(u - v_0), B \operatorname{sn}(u - v_0)), \\ &(A \operatorname{cn}(u + v_0), B \operatorname{sn}(u + v_0)) \end{aligned} \tag{4.3}$$

for $\forall u \in \mathbf{R}$.

Theorem 4.1. *The following two statements are equivalent for two ellipses E_o* (4.1) *and E_i* (4.2).
(1) *A Poncelet's traverse creates a poristic n-gon in the sense of Definition* 2.2.
(2) $v_0 = \frac{2m}{n}K, m \in \mathbf{Z} \setminus \{0\}, |2m| < n, n = 3, 4, \ldots$

Remark 4.1. We easily find that the theorem holds even when E_i is presented by

$$E_i = \left\{(x, y) \,\middle|\, \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0, \; a = \frac{B \operatorname{cn} v_0}{\operatorname{dn} v_0}, b = A \operatorname{cn} v_0\right\} \tag{4.4}$$

or

$$E_i = \left\{(x, y) \,\middle|\, \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0, \; a = A \cos v_0, \; b = B \cos v_0\right\}. \tag{4.5}$$

Remark 4.2. m presents the number of revolution of a series of points of Poncelet's traverse, that is, a series of points rotates m-revolutions on an outer ellipse. Also $\mathrm{sgn}(v_0)$ shows the direction of revolution.

Proof of Theorem 4.1. (2) $\Rightarrow$ (1).
The two intersecting points of E_o (4.1) and a tangent of E_i (4.2) at a point $(a\,\mathrm{cn}\,u, b\,\mathrm{sn}\,u)$ are presented by

$$(A\,\mathrm{cn}(u-v_0), B\,\mathrm{sn}(u-v_0)),$$
$$(A\,\mathrm{cn}(u+v_0), B\,\mathrm{sn}(u+v_0)).$$

By taking a point $(a\,\mathrm{cn}(u+2v_0), b\,\mathrm{sn}(u+2v_0))$ on E_i, the intersecting points become

$$(A\,\mathrm{cn}(u+v_0), B\,\mathrm{sn}(u+v_0)),$$
$$(A\,\mathrm{cn}(u+3v_0), B\,\mathrm{sn}(u+3v_0)).$$

Moreover by taking a point $(a\,\mathrm{cn}(u+4v_0),$ $b\,\mathrm{sn}(u+4v_0))$, the intersecting points become

$$(A\,\mathrm{cn}(u+3v_0), B\,\mathrm{sn}(u+3v_0)),$$
$$(A\,\mathrm{cn}(u+5v_0), B\,\mathrm{sn}(u+5v_0)).$$

In this way we obtain a series of points on E_o as follows:

(4.6)
$$(A\,\mathrm{cn}(u-v_0), B\,\mathrm{sn}(u-v_0)),$$
$$(A\,\mathrm{cn}(u+v_0), B\,\mathrm{sn}(u+v_0)), \ldots,$$
$$(A\,\mathrm{cn}(u+(2n-1)v_0), B\,\mathrm{sn}(u+(2n-1)v_0)), \ldots$$

Since $v_0 = \frac{2m}{n}K$,

(4.7)
$$\begin{aligned}
&\begin{pmatrix}\mathrm{cn}\\ \mathrm{sn}\end{pmatrix}(u-v_0) - \begin{pmatrix}\mathrm{cn}\\ \mathrm{sn}\end{pmatrix}(u+(2n-1)v_0)\\
&= \begin{pmatrix}\mathrm{cn}\\ \mathrm{sn}\end{pmatrix}\left(u - \frac{2m}{n}K\right) - \begin{pmatrix}\mathrm{cn}\\ \mathrm{sn}\end{pmatrix}\left(u + (2n-1)\frac{2m}{n}K\right)\\
&= \begin{pmatrix}\mathrm{cn}\\ \mathrm{sn}\end{pmatrix}\left(u - \frac{2m}{n}K\right) - \begin{pmatrix}\mathrm{cn}\\ \mathrm{sn}\end{pmatrix}\left(u + 4mK - \frac{2m}{n}K\right)\\
&= 0.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
(4.8)\qquad &\begin{pmatrix}\mathrm{cn}\\ \mathrm{sn}\end{pmatrix}\left(u + (2i-1)\frac{2m}{n}K\right)\\
&\quad - \begin{pmatrix}\mathrm{cn}\\ \mathrm{sn}\end{pmatrix}\left(u + (2j-1)\frac{2m}{n}K\right) \neq 0,\\
&\quad i \neq j, 1 \leq i, j < n
\end{aligned}$$

because letting the l.h.s. $= 0$ in (4.8) yields the difference of arguments must be $4qK, q \in \mathbf{Z}$ since the period of functions of cn and sn are $4K$ but the difference does not equal $4qK$. Therefore (4.7) and (4.8) show that the initial and n-th point of a series of points (4.6) overlap and simultaneously points less than the n-th point doesn't overlap each other. So a series of points (4.6) creates a poristic n-gon under the condition of (2).
(1) $\Rightarrow$ (2)
If a series of points (4.6) creates a poristic n-gon, then it follows

$$(4.9)\qquad \begin{pmatrix}\mathrm{cn}\\ \mathrm{sn}\end{pmatrix}(u-v_0) = \begin{pmatrix}\mathrm{cn}\\ \mathrm{sn}\end{pmatrix}(u+(2n-1)v_0).$$

From this we obtain $v_0 = \frac{2q}{n}K, q \in \mathbf{Z}$. Since $-K < v_0 < K$, the theorem follows. $\square$

5. The poristic relations between the quantities: semi-major and semi-minor axes of two ellipses [7]. The fact that the necessary and sufficient condition to create a poristic n-gon in E_o (4.1) and E_i (4.2) (or (4.4) or (4.5)) is $v_0 = \frac{2m}{n}K, m \in \mathbf{Z}\setminus\{0\}, |2m| < n, n = 3, 4, \ldots$ is stated at the previous section. In this section the relations between semi-major and semi-minor axes of two ellipses when a poristic n-gon is created are shown for $n = 3, 4, 5$, that is, triangle, quadrilateral and pentagon.

Theorem 5.1 (poristic triangle)**.** *A series of points of Poncelet's traverse creates a poristic triangle in* E_o (4.1) *and* E_i (4.2) (*or* (4.4) *or* (4.5)) *if and only if*

$$(5.1)\qquad \frac{a}{A} + \frac{b}{B} = 1.$$

Proof. We have only a case: $(n, m) = (3, 1)$ for a poristic triangle from Theorem 4.1. Then we only need to prove

$$v_0 = \pm\frac{2K}{3} \Longleftrightarrow \frac{a}{A} + \frac{b}{B} = 1.$$

($\Longrightarrow$) The necessary part is easy as follows:

$$\begin{aligned}
\frac{a}{A} + \frac{b}{B} &= \mathrm{cn}\left(\pm\frac{2K}{3}\right) + \frac{\mathrm{cn}(\pm\frac{2K}{3})}{\mathrm{dn}(\pm\frac{2K}{3})}\\
&= \mathrm{cn}\,\frac{2K}{3}\left(\frac{1 + \mathrm{dn}\,\frac{2K}{3}}{\mathrm{dn}\,\frac{2K}{3}}\right) = 1
\end{aligned}$$

since $\mathrm{cn}\,\frac{2K}{3} = \frac{\mathrm{dn}\frac{2K}{3}}{1+\mathrm{dn}\frac{2K}{3}}$.
($\Longleftarrow$) The sufficient part is performed by leading $v_0 = \pm\frac{2K}{3}$ from

$$\operatorname{cn} v_0 + \frac{\operatorname{cn} v_0}{\operatorname{dn} v_0} - 1 = 0. \tag{5.2}$$

Let the left-hand side of (5.2) be

$$f_3(v_0) = \operatorname{cn} v_0 + \frac{\operatorname{cn} v_0}{\operatorname{dn} v_0} - 1. \tag{5.3}$$

We check the shape of the function f_3 in $-K < v_0 < K$. Since f_3 is an even function so that it is enough to check it in $0 \le v_0 < K$. First we have

$$\begin{aligned} \frac{df_3}{dv_0} &= -\operatorname{sn} v_0 \operatorname{dn} v_0 - \operatorname{sn} v_0 + \frac{k^2 \operatorname{sn} v_0 \operatorname{cn}^2 v_0}{\operatorname{dn}^2 v_0} \\ &= (k^2 - 1 + \operatorname{dn} v_0(k^2 \operatorname{sn}^2 v_0 - 1)) \frac{\operatorname{sn} v_0}{\operatorname{dn}^2 v_0} \\ &\le 0, \text{ for } 0 \le v_0 < K. \end{aligned} \tag{5.4}$$

In the above inequality an equal sign holds only when $v_0 = 0$, so that the function f_3 is a monotone decreasing function in $0 \le v_0 < K$. Moreover there exists a unique point in $0 \le v_0 < K$ where v_0 satisfies $f_3(v_0) = 0$ since $f_3(0) = 1 > 0$ and $f_3(K) = -1 < 0$. Therefore we obtain $v_0 = \frac{2K}{3}$ from the proof of the necessary part. f_3 is a monotone increasing function in $-K < v_0 \le 0$ since f_3 is an even function, so that in a similar way we also obtain $v_0 = -\frac{2K}{3}$. Thus the theorem follows. □

We obtain the following corollary:

Corollary 5.1 (poristic triangle)**.** *In Theorem* 5.1 *when an outer ellipse changes into a circle, i.e.,* $A = B = R$ *(*R *is a radius of an outer circle), we have the following relation*:

$$a + b = R. \tag{5.5}$$

Also when an inner ellipse changes into a circle, i.e., $a = b = r$ *(*r *is a radius of an inner circle), we have the following relation*:

$$\frac{1}{A} + \frac{1}{B} = \frac{1}{r}. \tag{5.6}$$

Remark 5.1. When both ellipses change into circles, i.e., $A = B = R, a = b = r$, the well known relation: $\frac{r}{R} = \frac{1}{2} (= \cos\frac{\pi}{3})$ is obtained.

Theorem 5.2 (poristic quadrilateral)**.** *A series of points of Poncelet's traverse creates a poristic quadrilateral in* E_o (4.1) *and* E_i (4.2) (*or* (4.4) *or* (4.5)) *if and only if*

$$\frac{a^2}{A^2} + \frac{b^2}{B^2} = 1. \tag{5.7}$$

Proof. We have only a case: $(n, m) = (4, 1)$ for a poristic quadrilateral from Theorem 4.1. Then from Theorem 4.1 we only need to prove

$$v_0 = \pm\frac{K}{2} \Longleftrightarrow \frac{a^2}{A^2} + \frac{b^2}{B^2} = 1.$$

($\Longrightarrow$) Using the half-period formulae [10]: $\operatorname{cn}^2\frac{K}{2} = \frac{\operatorname{dn} K}{1+\operatorname{dn} K}$ and $\operatorname{dn}^2\frac{K}{2} = \operatorname{dn} K$, we obtain

$$\begin{aligned} \frac{a^2}{A^2} + \frac{b^2}{B^2} &= \operatorname{cn}^2\left(\pm\frac{K}{2}\right) + \frac{\operatorname{cn}^2(\pm\frac{K}{2})}{\operatorname{dn}^2(\pm\frac{K}{2})} \\ &= \operatorname{cn}^2\frac{K}{2}\left(\frac{1 + \operatorname{dn}^2\frac{K}{2}}{\operatorname{dn}^2\frac{K}{2}}\right) = 1. \end{aligned}$$

($\Longleftarrow$) The equation: $\frac{a^2}{A^2} + \frac{b^2}{B^2} - 1 = 0$ is equivalent to

$$\operatorname{cn}^2 v_0(1 + \operatorname{dn}^2 v_0) - \operatorname{dn}^2 v_0 = 0. \tag{5.8}$$

From this we only need to have $v_0 = \pm\frac{K}{2}$. In the same manner of the proof of a poristic triangle, let the left-hand side of (5.8) be

$$f_4(v_0) = \operatorname{cn}^2 v_0(1 + \operatorname{dn}^2 v_0) - \operatorname{dn}^2 v_0. \tag{5.9}$$

Since f_4 is an even function so that it is enough to check it in $0 \le v_0 < K$. First we have

$$\begin{aligned} \frac{df_4}{dv_0} &= -2\operatorname{cn} v_0 \operatorname{sn} v_0 \operatorname{dn} v_0(1 + \operatorname{dn}^2 v_0) \\ &\quad - 2k^2 \operatorname{sn} v_0 \operatorname{cn}^3 v_0 \operatorname{dn} v_0 \\ &\quad + 2k^2 \operatorname{sn} v_0 \operatorname{cn} v_0 \operatorname{dn} v_0 \\ &= 4 \operatorname{sn} v_0 \operatorname{cn} v_0 \operatorname{dn} v_0 (k^2 \operatorname{sn}^2 v_0 - 1) \\ &\le 0, \quad \text{for } 0 \le v_0 < K. \end{aligned} \tag{5.10}$$

In the above inequality an equal sign holds only when $v_0 = 0$, so that the function f_4 is a monotone decreasing function in $0 \le v_0 < K$. Moreover there exists a unique point in $0 \le v_0 < K$ where v_0 satisfies $f_4(v_0) = 0$ since $f_4(0) = 1 > 0$ and $f_4(K) = -k'^2 < 0$. Therefore we obtain $v_0 = \frac{K}{2}$ from the proof of the necessary part. f_4 is a monotone increasing function in $-K < v_0 \le 0$ since f_4 is an even function, so that in a similar way we also obtain $v_0 = -\frac{K}{2}$. Thus the theorem follows. □

We obtain the following corollary:

Corollary 5.2 (poristic quadrilateral)**.** *In Theorem* 5.2 *when an outer ellipse changes into a circle, we have the following relation*:

$$a^2 + b^2 = R^2. \tag{5.11}$$

Also when an inner ellipse changes into a circle, we have the following relation:

$$\frac{1}{A^2}+\frac{1}{B^2}=\frac{1}{r^2}. \tag{5.12}$$

Remark 5.2. When both ellipses change into circles, the well known relation: $\frac{r}{R}=\frac{1}{\sqrt{2}}\,(=\cos\frac{\pi}{4})$ is obtained.

There exist two cases: $(n,m)=(5,1)$ and $(5,2)$ when a series of points of Poncelet's traverse creates a poristic pentagon in E_o and E_i from Theorem 4.1. We call the former an ordinary porism and the latter a two-laps porism since $m=1$ and $m=2$ mean that a series of points of Poncelet's traverse rotates one-revolution and two-revolutions on an outer ellipse, respectively. The proof can be performed in the same manner so the necessary and sufficient conditons(relations) for ordinary and two-laps, poristic pentagons are only shown.

The relations for a poristic pentagon.

(a1) An ordinary, poristic pentagon which is circumscribed and inscribed by two ellipses

$$\frac{a^3}{A^3}+\frac{b^3}{B^3}+\left(\frac{a}{A}+\frac{b}{B}\right)^2=1+\left(\frac{a}{A}+\frac{b}{B}\right)\left(1+\frac{ab}{AB}\right).$$

(a2) An ordinary, poristic pentagon which is circumscribed by a circle and inscribed by an ellipse

$$a^3+b^3+R(a+b)^2=R^3+(a+b)(R^2+ab).$$

(a3) An ordinary, poristic pentagon which is circumscribed by an ellipse and inscribed by a circle

$$\frac{1}{A^3}+\frac{1}{B^3}+\frac{1}{r}\left(\frac{1}{A}+\frac{1}{B}\right)^2 =\frac{1}{r^3}+\left(\frac{1}{A}+\frac{1}{B}\right)\left(\frac{1}{r^2}+\frac{1}{AB}\right).$$

(a4) An ordinary, poristic pentagon which is circumscribed and inscribed by two circles

The well known relation: $\frac{r}{R}=\frac{1}{-1+\sqrt{5}}\,(=\cos\frac{\pi}{5})$ is obtained.

(b1) A two-laps, poristic pentagon which is circumscribed and inscribed by two ellipses

$$\frac{a^3}{A^3}+\frac{b^3}{B^3}+1=\left(\frac{a}{A}+\frac{b}{B}\right)\left(1+\frac{a}{A}+\frac{b}{B}+\frac{ab}{AB}\right).$$

(b2) A two-laps, poristic pentagon which is circumscribed by a circle and inscribed by an ellipse

$$a^3+b^3+R^3=(a+b)(R^2+(a+b)R+ab).$$

(b3) A two-laps, poristic pentagon which is circumscribed by an ellipse and inscribed by a circle

$$\frac{1}{A^3}+\frac{1}{B^3}+\frac{1}{r^3} =\left(\frac{1}{A}+\frac{1}{B}\right)\left(\frac{1}{r^2}+\left(\frac{1}{A}+\frac{1}{B}\right)\frac{1}{r}+\frac{1}{AB}\right)$$

(b4) A two-laps, poristic pentagon which is circumscribed and inscribed by two circles

$$\frac{r}{R}=\frac{1}{1+\sqrt{5}}\left(=\cos\frac{2}{5}\pi\right)$$

We can also obtain the relations for n-gon($n\geq 6$) in the same manner.

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