Dihedral G-Hilb via representations of the McKay quiver

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Abstract: For a given finite small binary dihedral group $G \subset \mathrm{GL}(2, \mathbf{C})$ we provide an explicit description of the minimal resolution Y of the singularity $\mathbf{C}^2/G$. The minimal resolution Y is known to be either the moduli space of G-clusters G-Hilb$(\mathbf{C}^2)$, or the equivalent $\mathcal{M}_\theta(Q, R)$, the moduli space of θ-stable quiver representations of the McKay quiver. We use both moduli approaches to give an explicit open cover of Y, by assigning to every distinguished G-graph Γ an open set $U_\Gamma \subset \mathcal{M}_\theta(Q, R)$, and calculating the explicit equation of U_Γ using the McKay quiver with relations (Q, R).

Key words: McKay correspondence; G-Hilbert scheme; quiver representations.

1. Introduction. The generalisation of the McKay correspondence [8], [12] to small finite subgroups $G \subset \mathrm{GL}(2, \mathbf{C})$ was established after Wunram [15] introduced the notion of special representation. The so-called "special" McKay correspondence relates the G-equivariant geometry of $\mathbf{C}^2$ and the minimal resolution Y of the quotient $\mathbf{C}^2/G$, establishing a one-to-one correspondence between the irreducible components of the exceptional divisor $E \subset Y$ and the special irreducible representations. This minimal resolution Y can be viewed as two equivalent moduli spaces: by a result of Ishii [4] it is known that $Y = G$-Hilb$(\mathbf{C}^2)$ the G-invariant Hilbert scheme introduced by Ito and Nakamura [5], and at the same time as $Y = \mathcal{M}_\theta(Q, R)$ the moduli space of θ-stable representations of the McKay quiver.

In the same spirit as [7] in this paper we treat the problem of describing G-Hilb$(\mathbf{C}^2)$ by giving an explicit affine open cover. In [9] Nakamura introduced the notion of G-graphs, providing a nice and friendly framework to describe G-Hilb$(\mathbf{C}^2)$ for finite abelian subgroups in $\mathrm{GL}(n, \mathbf{C})$. In this paper we consider the non-abelian analogue of a G-graph and provide an explicit method to interpret θ-stable representations of the McKay quiver from G-graphs and vice versa. By using the relations on the McKay quiver, this led us to describe explicitly an open cover $\mathcal{M}_\theta(Q, R)$ (hence for G-Hilb$(\mathbf{C}^2)$) for binary dihedral subgroups in $\mathrm{GL}(2, \mathbf{C})$ with the minimal number of open sets. Our method also recovers the ideals defining the G-clusters in G-Hilb$(\mathbf{C}^2)$.

An alternative description of an open cover for the minimal resolution Y has been discovered independently by Wemyss [13], [14] by using reconstruction algebras instead of the skew group ring.

2. Preliminaries.

2.1. Dihedral groups $\mathbf{BD}_{2n}(a)$ in $\mathbf{GL(2, C)}$. Let G be a finite small binary dihedral subgroup in $\mathrm{GL}(2, \mathbf{C})$. In terms of its action on the complex plane $\mathbf{C}^2$ we consider the representation of G, denoted by $\mathrm{BD}_{2n}(a)$, generated by $\alpha = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^a \end{pmatrix}$ and $\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ subject to relations:

$$(2n, a) = 1,\ a^2 \equiv 1 \pmod{2n},\ \gcd(a+1, 2n) \nmid n$$

where ε is a primitive $2n$-th root of unity. The group $\mathrm{BD}_{2n}(a)$ has order $4n$ and it contains the maximal normal index 2 cyclic subgroup $A := \langle \alpha \rangle \trianglelefteq G$, which we denote by $\frac{1}{2n}(1, a)$ (note that $\beta^2 \in A$). The condition $a^2 \equiv 1 \pmod{2n}$ is equivalent to the relation $\alpha\beta = \beta\alpha^a$, and $\gcd(a+1, 2n) \nmid n$ implies that the group is small (see [10], §3 for details).

Definition 2.1. Let $q := \frac{2n}{(a-1,2n)}$, and k such that $n = kq$.

The group $\mathrm{BD}_{2n}(a)$ has $4k$ irreducible 1-dimensional representations ρ_j^+ and ρ_j^- of the form

$$\rho_j^\pm(\alpha) = \varepsilon^j,\ \rho_j^\pm(\beta) = \begin{cases} \pm i & \text{if } n, j \text{ odd} \\ \pm 1 & \text{otherwise} \end{cases}$$

where ε is a $2n$-th primitive root of unity and j is such that $j \equiv aj \pmod{2n}$. The values r for which $r \not\equiv ar \pmod{2n}$ form in pairs the $n - k$ irreducible

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2-dimensional representations V_r of the form

$$V_r(\alpha) = \begin{pmatrix} \varepsilon^r & 0 \\ 0 & \varepsilon^{ar} \end{pmatrix},\ V_r(\beta) = \begin{pmatrix} 0 & 1 \\ (-1)^r & 0 \end{pmatrix}$$

By definition, the natural representation is V_1.

In what follows we take the notation as in [16] §10. Let $V(=V_1)$ a vector space with basis $\{x, y\}$ where G acts naturally. Define $S = \operatorname{Sym} V := \mathbf{C}[V^*]$ the polynomial ring in the variables x and y. Then the action of G extends to S by $g \cdot f(x, y) := f(g(x), g(y))$ for $f \in S$, $g \in G$.

Definition 2.2. Let $G = \mathrm{BD}_{2n}(a)$, $f \in S$.

$$f \in \rho_j^{\pm} :\Longleftrightarrow \alpha(f) = \varepsilon^j f, \beta(f) = \begin{cases} \pm if & \text{if } n, j \text{ odd} \\ \pm f & \text{otherwise} \end{cases}$$

$$(f, \beta(f)) \in V_k :\Longleftrightarrow \alpha(f, \beta(f)) = (\varepsilon^k f, \varepsilon^{ak}\beta(f))$$

Let $S_\rho := \{f \in \mathbf{C}[x, y] : f \in \rho\}$ the S^G-module of ρ-invariants. Note that these are precisely the Cohen Macaulay S^G-modules $S_\rho = (S \otimes \rho^*)^G$ where G acts on S as above and G acts on a representation ρ by the inverse transpose.

2.2. *G*-Hilb and *G*-graphs. Let $G = \mathrm{BD}_{2n}(a) \subset \mathrm{GL}(2, \mathbf{C})$ be a binary dihedral subgroup.

Definition 2.3. A *G-cluster* is a G-invariant zero dimensional subscheme $\mathcal{Z} \subset \mathbf{C}^2$ such that $\mathcal{O}_\mathcal{Z} \cong \mathbf{C}[G]$ the regular representation as G-modules. The *G-Hilbert scheme* G-$\mathrm{Hilb}(\mathbf{C}^2)$ is the moduli space parametrising G-clusters.

Recall that $\mathbf{C}[G] = \bigoplus_{\rho \in \mathrm{Irr}\, G} (\rho)^{\dim \rho}$, where every irreducible representation ρ appears $\dim \rho$ times in the sum. Thus, as a vector space, $\mathcal{O}_\mathcal{Z}$ has in its basis $\dim \rho$ elements in each ρ. To describe a distinguished basis of $\mathcal{O}_\mathcal{Z}$ with this property, it is convenient to use the notion of G-graph.

Definition 2.4. Let $G = \mathrm{BD}_{2n}(a)$. A *G-graph* is a subset $\Gamma \subset \mathbf{C}[x, y]$ satisfying the following:

(a) It contains $\dim \rho$ number of elements in each irreducible representation ρ.

(b) If a monomial $x^{\lambda_1} y^{\lambda_2}$ is a summand of a polynomial $P \in \Gamma$, then for every $0 \le \mu_j \le \lambda_j$, the monomial $x^{\mu_1} y^{\mu_2}$ must be a summand of some polynomial $Q_{\mu_1, \mu_2} \in \Gamma$.

For any G-graph Γ there exists an open set $U_\Gamma \subset$ G-$\mathrm{Hilb}(\mathbf{C}^2)$ consisting of all G-clusters $\mathcal{Z}$ such that $\mathcal{O}_\mathcal{Z}$ admits Γ for basis as a vector space. It is proved in [11] (see Theorem 3.4) that given the set of all possible G-graphs $\{\Gamma_i\}$, their union covers G-$\mathrm{Hilb}(\mathbf{C}^2)$.

Example 2.5. $\Gamma = \{1, x, x^2, y, xy\}$ is a $\frac{1}{5}(1, 3)$-graph. For the non-abelian binary dihedral group $D_4 = \langle \frac{1}{4}(1, 3), \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \rangle \subset \mathrm{SL}(2, \mathbf{C})$, $\Lambda = \{1, x, y, x^2 + y^2, x^2 - y^2, y^3, -x^3, x^4 - y^4\}$ is a D_4-graph (note that $(x, y), (y^3, -x^3) \in V_1$).

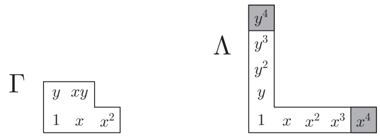

Fig. 1. Representation of the G-graphs Γ and Λ.

We say that an ideal I *represents* a G-graph Γ, and we write I_Γ, if $\mathbf{C}[x, y]/I$ admits Γ as basis.

In Example 1, $I_\Gamma = (x^3, x^2y, y^2)$ and similarly $I_\Lambda = (xy, x^4 + y^4)$. The pictorial description of Γ and Λ is shown in Fig. 1. Notice that for Λ the elements $x^2 + y^2 \in \rho_2^+$ and $x^2 - y^2 \in \rho_2^-$ are described by x^2 and y^2 respectively, and the relation $x^4 + y^4 = 0$ identifies x^4 and y^4 in $\mathbf{C}[x, y]/I_\Lambda$.

3. *G*-graphs for $\mathbf{BD}_{2n}(a)$ groups. Let $G = \mathrm{BD}_{2n}(a)$. The minimal resolution Y of $\mathbf{C}^2/G$ is obtained as follows (see [5] §1.2): First act with A on $\mathbf{C}^2$ and consider A-$\mathrm{Hilb}(\mathbf{C}^2)$ as the minimal resolution of $\mathbf{C}^2/A$. To complete the action of G act with $G/A \cong \langle \bar{\beta} \rangle$ on A-$\mathrm{Hilb}(\mathbf{C}^2)$. The conditions $a^2 \equiv 1 \pmod{2n}$ and $\gcd(a + 1, 2n) \nmid n$ imply that the continued fraction $\frac{2n}{a}$ is symmetric with respect to the middle entry. Then the coordinates along the exceptional divisor $E = \bigcup_{i=1}^{2m-1} E_i$ are symmetric with respect to the middle curve E_m. The action of G/A identifies the rational curves on E pairwise except in E_m where we have an involution. Thus the quotient $\widetilde{Y} = A$-$\mathrm{Hilb}(\mathbf{C}^2)/(G/A)$ has two A_1 singularities, and the blow-up of these two points gives G-$\mathrm{Hilb}(\mathbf{C}^2)$ by the uniqueness of minimal models of surfaces.

Let us now translate this construction into graphs. Any G/A-orbit in A-$\mathrm{Hilb}(\mathbf{C}^2)$ consists of two A-clusters $\mathcal{Z}$ and $\beta(\mathcal{Z})$, with symmetric A-graphs Γ and $\beta(\Gamma)$ respectively. They are represented by $I_\Gamma = (x^s, y^u, x^{s-v}y^{u-r})$ and $I_{\beta(\Gamma)} = (y^s, x^u, x^{u-r}y^{s-v})$, where $e_i = \frac{1}{2n}(r, s)$, $e_{i+1} = \frac{1}{2n}(u, v)$ are two consecutive lattice points in the boundary of the Newton polygon of the lattice $L := \mathbf{Z}^2 + \frac{1}{2n}(1, a) \cdot \mathbf{Z}$.

If we denote by $\mathcal{Y}$ the corresponding G-cluster, then it is clear that $\mathcal{Z} \cup \beta(\mathcal{Z}) \subset \mathcal{Y}$. Thus $I_\mathcal{Y} \subset I_\mathcal{Z} \cap I_{\beta(\mathcal{Z})}$ which implies that $\widetilde{\Gamma} := \Gamma \cup \beta(\Gamma) \subset \Gamma_\mathcal{Y}$. Note that the representation of $\widetilde{\Gamma}$ in the lattice of monomials is symmetric with respect to the diagonal, and the inclusion $\widetilde{\Gamma} \subset \Gamma_\mathcal{Y}$ is never an equality since Γ and $\beta(\Gamma)$ always share a common subset of elements $R \subset \widetilde{\Gamma}$. The subset $\widetilde{\Gamma}$ is called *qG-graph.*

Thus, to obtain a G-graph from $\widetilde{\Gamma}$ we must add $\sharp R$ elements to $\widetilde{\Gamma}$ preserving the representation spaces contained in R according to Definition 2.4. It is shown in [11] that the extension from a qG-graph $\widetilde{\Gamma}$ to a G-graph Γ is unique. The following theorem resumes the classification of G-graphs for $\mathrm{BD}_{2n}(a)$ groups describing their defining ideals in each case.

Theorem 3.1 ([11]). *Let $G = \mathrm{BD}_{2n}(a)$ be a small binary dihedral group and let Γ_i be the A-graph corresponding to the two consecutive lattice points $e_i = \frac{1}{2n}(r,s)$, $e_{i+1} = \frac{1}{2n}(u,v)$ of the Newton polygon of the lattice L. Denote by $\Gamma := \Gamma(r,s;u,v)$ the G-graph corresponding to the qG-graph $\Gamma_i \cup \beta(\Gamma_i)$. Then we have the following possibilities*:

- *If $u < s - v$ then Γ is of type A and it is represented by the ideal $I_A = (x^u y^u,\ x^{s-v}y^{u-r} + (-1)^{u-r}x^{u-r}y^{s-v}, x^{r+s} + (-1)^r y^{r+s})$.*
- *If $u - r = s - v := m$ then Γ is of type B and*

(a) If $u < 2m$ then Γ is of type B_1 and it is represented by the ideal $I_{B_1} = (x^{r+s} + (-1)^r y^{r+s}$, $x^{m+s}y^{m-r} + (-1)^{m-r}x^{m-r}y^{m+s}$, $x^u y^m$, $x^m y^u)$.

(b) If $u \geq 2m$ then Γ is of type B_2 and $I_{B_2} = (x^{2m}y^{2m}$, x^{s+m}, y^{s+m}, $x^u y^m$, $x^m y^u)$.

In addition, when $u = v = q := \frac{2n}{(a-1,2n)}$ we have four G-graphs of types C^+, C^-, D^+ and D^-.

- *The G-graphs of types $D^\pm$ are represented by the ideals $I_{D^\pm} = (x^q \pm (-i)^q y^q, x^{s-r}y^{s-r})$.*
- *For G-graphs of types $C^\pm$ we have two cases*:

(a) If $2q < s$, and we call $m_1 := s - q$ and $m_2 := q - r$, they are represented by the ideals $I_{C_A^\pm} = ((x^q \pm (-i)^q y^q)^2, x^s y^{m_2} \pm (-1)^r i^q x^{m_2} y^s, x^{m_1}y^{m_2} \pm (-1)^{m_2}x^{m_2}y^{m_1})$.

(b) If $2q = r + s$ then $I_{C_B^\pm} = (y^m(x^q \pm (-i)^q y^q)$, $x^m(x^q \pm (-i)^q y^q)$, $x^{s-r}y^{s-r}$, $x^s y^m, x^m y^s)$.

Remark 3.2. The list of ideals in Theorem 3.1 define in G-$\mathrm{Hilb}(\mathbf{C}^2)$ the intersection points of two of the exceptional curves plus the strict transform of the coordinate axis in $\mathbf{C}^2$.

Example 3.3. Consider the $\frac{1}{12}(1,7)$-graphs given by $I_\Gamma = (x^7, y^2, x^5 y)$ and $I_{\beta(\Gamma)} = (y^7, x^2, xy^5)$, with $r = 1$, $s = 7$, $u = 2$, $v = 2$. The overlap subset is $R = \{1, x, y, xy\}$ where $1 \in \rho_0^+$, $xy \in \rho_8^-$ and $(x,y) \in V_1$. Then we must add the elements $x^5y - xy^5 \in \rho_0^+$, $x^8 - y^8 \in \rho_8^-$ and $(y^7, -x^7) \in V_1$. The graph is represented by $(x^2y^2, x^5y - xy^5, x^8 - y^8)$.

Theorem 3.4 ([11]). *Let $G = \mathrm{BD}_{2n}(a)$ be small and let $P \in G$-$\mathrm{Hilb}(\mathbf{C}^2)$ be defined by the ideal I. Then we can always choose a basis for $\mathbf{C}[x,y]/I$ from the list Γ_A, Γ_B, Γ_{C^+}, Γ_{C^-}, Γ_{D^+}, Γ_{D^-}. Moreover, if $\Gamma_0, \ldots, \Gamma_{m-1}$, Γ_{C^+}, Γ_{C^-}, $\Gamma_{D^+}, \Gamma_{D^-}$ is the list of G-graphs, then an open cover of G-$\mathrm{Hilb}(\mathbf{C}^2)$ is given by $U_{\Gamma_0}, \ldots, U_{\Gamma_{m-1}}, U_{\Gamma_{C^+}}, U_{\Gamma_{C^-}}, U_{\Gamma_{D^+}}, U_{\Gamma_{D^-}}$.*

4. $\mathcal{M}_\theta(Q,R)$ and orbifold McKay quiver.

Let $G = \mathrm{BD}_{2n}(a)$ and let $A = \frac{1}{2n}(1,a) \trianglelefteq G$. Denote by $\mathrm{Irr}\, G$ the set of irreducible representations of G. For the background material on quivers refer to [1]. We consider left modules (and actions), and by a path pq we mean p followed by q. Let (Q,R) a quiver with relations, fix $\mathbf{d} = (d_i)_{i\in Q_0}$ the dimension vector of the representations of (Q,R), and let $\mathbf{V}(I_R) \subset \mathbf{A}^N \cong \bigoplus_{a\in Q_1} \mathrm{Mat}_{d_{t(a)}\times d_{h(a)}}$ the representation space subject to the ideal of relations I_R. For θ generic we define $\mathcal{M}_\theta := \mathcal{M}_\theta(Q,R) = \mathbf{V}(I_R) /\!\!/_\theta \prod \mathrm{GL}(d_i)$ the moduli space of θ-stable representations of (Q,R) (see [6], [3]). Taking Q to be the McKay quiver and a particular choice of generic θ (see §5) it is well known that $\mathcal{M}_\theta \cong G$-$\mathrm{Hilb}(\mathbf{C}^2)$.

The McKay quiver of G, denoted by $\mathrm{McKayQ}(G)$, is defined by having one vertex for every $\rho \in \mathrm{Irr}\, G$ and by the number of arrows from ρ to σ to be $\dim_{\mathbf{C}} \mathrm{Hom}_{\mathbf{C}G}(\rho \otimes V, \sigma)$. Equivalently, due to Auslander it is known that $\mathrm{McKayQ}(G)$ is the underlying quiver of the algebra $\mathrm{End}_{S^G}(\bigoplus_{\rho\in\mathrm{Irr}\, G} S_\rho)$ where $S_\rho = (S \otimes \rho^*)^G$ as in 2.2 (see [16] for a proof in dimension 2).

The $\mathrm{McKayQ}(A)$ can be drawn on a torus as follows: Let $M \cong \mathbf{Z}^2$ be the lattice of monomials and $M_{\mathrm{inv}} \cong \mathbf{Z}^2$ the sublattice of invariant monomials by A. If we take $M_{\mathbf{R}} = M \otimes_{\mathbf{Z}} \mathbf{R}$ we can consider the torus $T := M_{\mathbf{R}}/M_{\mathrm{inv}}$. The vertices in $\mathrm{McKayQ}(A)$ are precisely $Q_0 = M \cap T$, and the arrows between vertices are the natural multiplications by x and y in M. It is easy to see that we can always choose a fundamental domain $\mathcal{D}$ for T to be the parallelogram with vertices 0, (k,k), $(2q,0)$ and $(k+2q,k)$ where the opposite sides are identified.

Proposition 4.1. (*i*) *The McKay quiver Q of $\mathrm{BD}_{2n}(a)$ is the $\mathbf{Z}/2$-orbifold quotient of the McKay quiver for the Abelian subgroup A (see Fig. 2).*
(*ii*) *The relations R on the Q which gives the identification between G-Hilb$(\mathbf{C}^2)$ and $\mathcal{M}_\theta$ are $a_ib_{i+1} = 0, f_ie_{i+1} = 0, b_ia_i + d_ic_i = r_{i,1}u_{i,1}$, $c_id_{i+1} = 0$, $h_ig_{i+1} = 0$, $e_if_i + g_ih_i = u_{i,q-2}r_{i+1,q-2}$, $u_{i,j}r_{i+1,j} = r_{i,j+1}u_{i,j+1}$, considering subindices modulo k.*

Notation 4.2. The source and target for $r_{i,j}$ and $u_{i,j}$ are $r_{i,j} : S_{V_{\overline{(i-1)(a+1)+j}}} \to S_{V_{\overline{(i-1)(a+1)+j+1}}}$, and $u_{i,j} : S_{V_{\overline{(i-1)(a+1)}+j+1}} \to S_{V_{i(a+1)+j}}$ with $i \in [0, k-1]$, $j \in [1, q-2]$, where $\overline{i}$ denotes $i \mod k$.

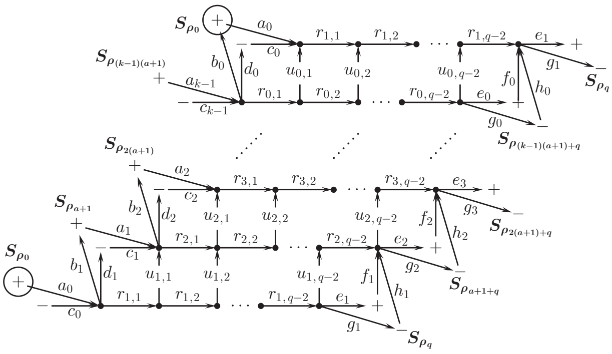

Fig. 2. McKay quiver for $\mathrm{BD}_{2n}(a)$ groups.

Remark 4.3. In the case $q = 2$ the relations are $a_i b_{i+1} = 0$, $f_i e_{i+1} = 0$, $c_i d_{i+1} = 0$, $h_i g_{i+1} = 0$ and $b_i a_i + d_i c_i = e_i f_i + g_i h_i$.

Proof. (i) Let $\operatorname{Irr} A = \{\rho_0, \dots, \rho_{2n-1}\}$. The group G acts on A by conjugation, which induces an action of $G/A \cong \mathbf{Z}/2$ on A by $\beta \cdot h := \beta h \beta^{-1}$, for any $h \in A$. Therefore G/A acts on $\operatorname{Irr} A$ by $\beta \cdot \rho_k := \rho_{ak}$, for $\rho_k \in \operatorname{Irr} A$. The free orbits are $\{\rho_i, \rho_{ai}\}$ with $ai \not\equiv i \bmod 2n$, producing the 2-dimensional representations V_i in McKayQ(G). Every fixed point ρ_j with $aj \equiv j \pmod{2n}$ splits into the two 1-dimensional representations ρ_j^+ and ρ_j^- in McKayQ(G).

Note that McKayQ(G) is now drawn on a cylinder where only the top and bottom sides are identified. The arrows of McKayQ(A) going in and out fixed representations split into two different arrows, while for the rest we have a 1 to 1 correspondence between arrows in McKayQ(A) and McKayQ(G).

(ii) Let $\mathbf{k}Q$ be the path algebra, $S = \mathbf{C}[x, y]$ and V^* the natural representation. Tensoring with $\det_{V^*} := \bigwedge^2 V^*$ induces a permutation τ on Q_0 by $e_i = \tau(e_j) \iff \rho_i = \rho_j \otimes \det_{V^*}$. Now consider an arrow $a : e_i \to e_j$ as an element $\psi_a \in \operatorname{Hom}_{\mathbf{C}G}(\rho_i, \rho_j \otimes V^*)$. Then for any path $p = a_1 a_2$ of length 2 we can consider the G-module homomorphism $\rho_{t(p)} \xrightarrow{\psi_p} \rho_{h(p)} \otimes V^{*\otimes r} \xrightarrow{\mathrm{id}_{\rho_{h(p)}} \otimes \gamma} \rho_{h(p)} \otimes \det_{V^*}$ where ψ_p is the composition of the maps ψ_{a_1} and $\psi_{a_2} \otimes \mathrm{id}_V^*$, and $\gamma : V^{*\otimes 2} \to \bigwedge^2 V^*$ sends $v_1 \otimes v_2 \mapsto v_1 \wedge v_2$. By Schur's Lemma the composition of the maps above is zero if $\tau(h(p)) \neq t(p)$, a scalar c_p otherwise. It is known by [2] that for a finite small $G \subset \mathrm{GL}(2, \mathbf{C})$ (and more generally for any small finite subgroup $G \subset \mathrm{GL}(r, \mathbf{C})$) the skew group algebra $S * G$ is Morita equivalent to the algebra $\mathbf{k}Q/\langle \partial_p \Phi : |p| = 0 \rangle$, where $\Phi := \sum_{|p|=2} (c_p \dim h(p)) p$ and ∂_p are derivations with respect to paths of length 0, i.e. vertices e_i. Since the θ-stable $S * G$-modules (θ as in §5) are precisely the G-clusters, which gives $\mathcal{M}_\theta \cong G\text{-Hilb}(\mathbf{C}^2)$.

For $G = \mathrm{BD}_{2n}(a)$, $\det_{V^*} = \rho_{a+1}^+$ so τ translates McKayQ(G) one step diagonally up (see Fig. 2). Only paths of length 2 joining two vertices identified by τ appear in Φ, giving the relations R by derivations with respect to the vertices of Q. $\square$

5. Explicit calculation of G-Hilb($\mathbf{C}^2$). Let $G = \mathrm{BD}_{2n}(a)$ and (Q, R) be the McKay quiver as in 4. Denote the arrows by $\mathbf{a} = (a, A)$, $\mathbf{a} = \binom{a}{A}$, or $\mathbf{a} = \begin{pmatrix} a & A \\ a' & A' \end{pmatrix}$ depending on the dimensions at the source and target of $\mathbf{a}$. We take representations of Q with dimension vector $\mathbf{d} = (\dim \rho_i)_{i \in Q_0}$ and the generic stability condition $\theta = (1 - \sum_{\rho_i \in \operatorname{Irr} G} \dim \rho_i, 1 \dots, 1)$, which imply $G\text{-Hilb}(\mathbf{C}^2) \cong \mathcal{M}_\theta$.

Claim 5.1. A representation of Q is stable if and only if there exist $\dim \rho_i$ linearly independent maps from the distinguished source, chosen to be $\rho_0 \in Q_0$, to every other vertex ρ_i in Q.

Indeed, a representation W is not θ-stable iff $\exists\, W' \subset W$ proper with $\theta(W') < 0$. Since the only nonzero entry of θ corresponds to ρ_0^+ and $\mathbf{d}(W')$ has to be strictly smaller than $\mathbf{d}(W)$, this is equivalent to say that there are strictly less linearly independent paths from ρ_0^+ to ρ_i than $\dim \rho_i$, for $i \neq 0$.

By making a correspondence between elements of a G-graph and paths in Q, an open cover of $\mathcal{M}_\theta$ is given by the ones corresponding to the G-graphs. The G-graphs predetermine the choices of linear independent paths, thus giving a covering of $\mathcal{M}_\theta$ with the minimal number of open sets.

Theorem 5.2. *Let $G = \mathrm{BD}_{2n}(a)$ and let $\Gamma = \Gamma(r, s; u, v)$ be a G-graph with $U_\Gamma \subset \mathcal{M}_\theta$. Then,*

- *If Γ is of type A then U_{Γ_A} is given by:*
$a_0, D_0, F_0 \neq 0$, and e_i, g_i, $r_{i,j}$, $U'_{i,j} \neq 0$ for all i, j.
$a_i, H_i \neq 0$ for i even, and $c_i, F_i \neq 0$ for i odd.
For $0 < i < u$ set $b_i, D_i \neq 0$ if i is even, and $B_i, d_i \neq 0$ if i is odd. For $i \geq u$ set $B_i, D_i \neq 0$.
- *If Γ is of type B then U_{Γ_B} is given by:*
$a_0, d_0, H_0 \neq 0$, $e_i \neq 0$, $g_i \neq 0$, $r_{i,j} \neq 0$ for all i, j.
$a_i, b_i, D_i, H_i \neq 0$ for i even, and $B_i, c_i, d_i, F_i \neq 0$ for i odd. If $r > 1$ also $C_0, R'_{1,1}, \dots, R'_{1,r-2} \neq 0$.

If Γ is of type B_1 then also set $R'_{r+1,1}, \dots, R'_{r+1,u-r-2} \neq 0$ and $U'_{i,j} \neq 0$, $\forall i \neq 0, r$ and $\forall j$. For $i = 0, r$ we have $U'_{0,r}, \dots, U'_{0,q-2} \neq 0$ and $U'_{r,u-r}, \dots U'_{r,q-2} \neq 0$.
Also if $q > 2$, $C_r \neq 0$ if r even, or $A_r \neq 0$ if r odd.

If Γ is of type B_2 then also set $U'_{i,j} \neq 0$, $\forall i > 0$ and $\forall j$ and $U'_{0,r}, \dots U'_{0,q-2} \neq 0$.

- *If Γ is of type C then*

(*a*) *The conditions for* $U_{\Gamma_{C^-}} \subset \mathcal{M}_\theta$ *are the same as the ones for* $\Gamma_i(r,s;q,q)$ *with* $i = A$ *or* B, *and the condition* $F_0 \neq 0$ *instead of* $H_0 \neq 0$.

(*b*) *The open conditions for the case* Γ_{C^+} *are the same as those for* Γ_{C^-} *but swapping the conditions for* F_i *for* H_i *and vice versa.*

• *If* Γ *is of type D then* $U_{\Gamma_{D^\pm}}$ *is defined by:*
$a_0, C_0, d_0 \neq 0$, *and* $a_i, b_i, D_i \neq 0$ *for* i *even,* $B_i, c_i, d_i \neq 0$ *for* i *odd.*
$U'_{i,j} \neq 0$ *for all* $i > 0$ *and all* j, $U'_{0,r}, \dots U'_{0,q-2} \neq 0$.
$r_{i,j} \neq 0$ *for all* i, j *except for* $r_{i,q-i}$, $i = \in [2, k-1]$.
$R'_{1,1}, \dots, R'_{1,r-2} \neq 0$, $u_{i,q-i} \neq 0$ *for* $i \in [2, k-1]$.

If Γ *is a G-graph of type* D^+ *then we also set* e_0, H_0, G_0, E_1, f_1, g_1, $H_1 \neq 0$. *If* i *is even then* E_i, g_i, $F_i \neq 0$, *and if* i *is odd then* e_i, G_i, $H_i \neq 0$.
If Γ *is a G-graph of type* D^- *we set* E_0, F_0, g_0, e_1, F_1, G_1, $h_1 \neq 0$. *If* i *is even then* e_i, G_i, $H_i \neq 0$, *and if* i *is odd then* E_i, g_i, $F_i \neq 0$ *with* $i \in [0, k-1]$.

Proof. An open set in $\mathcal{M}_\theta$ is obtained by making open conditions in the parameter space $\mathbf{V}(I_R) \subset \mathbf{A}^N$. We can change basis at every vertex to take 1 as basis for every 1-dimensional vertex, and $(1,0)$ and $(0,1)$ for every 2-dimensional. Thus, by 5.1 the element $1 \in \rho_0^+$ generates the whole representation with this basis. For instance, we always choose $a_0 = (1,0)$.

Given any G-graph Γ the corresponding open set $U_\Gamma \subset \mathcal{M}_\theta$ is obtained by taking the open conditions according to the elements of Γ. This is done by considering Q to be given (see 4) by the S^G-modules S_ρ as vertices, and the irreducible maps between them to be the arrows. See Fig. 3 for the case n even, where the segment is repeated throughout the quiver. When n is odd replace e_i by $\left(\begin{smallmatrix} x \\ -iy \end{smallmatrix}\right)$, f_i by $(y, -ix)$, g_i by $\left(\begin{smallmatrix} x \\ iy \end{smallmatrix}\right)$ and h_i by (y, ix) (to verify the relations in 4.1 we have to multiply $r_{i,j}$ and $u_{i,j}$ by $\sqrt{2}$ for every i, j). By Claim 5.1 these irreducible maps send $1 \in S_{\rho_0^+}$ once to every other $S_{\rho^\pm}$ and twice to every other S_{V_k} linear independently. Denote the polynomials obtained by $f_{\rho^\pm}$ and (g_V, g'_V), (h_V, h'_V) respectively. In this way, for any stable representation all modules S_ρ have assigned basis polynomials. Thus, if we take the open conditions such that basis elements generated from $1 \in S_{\rho_0^+}$ form the G-graph Γ, we obtain the desired open set $U_\Gamma \in \mathcal{M}_\theta$.

If $f \in S_{\rho^\pm}$ and $f \neq f_\rho^\pm$ (i.e. $f \notin \Gamma$), then $\exists\, c \in \mathbf{C}$ such that $f = c f_\rho^\pm$ where c is the path in the representation connecting 1 and f (similarly $(f, f') = c_1(g_V, g'_V) + c_2(h_V, h'_V)$ for $c_1, c_2 \in \mathbf{C}$, $(f, f') \in S_V$). Then U_Γ parametrises every G-cluster with Γ as G-graph, so the union of U_Γ covers $\mathcal{M}_\theta$. We prove the

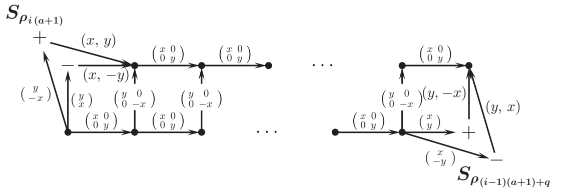

Fig. 3. Segment i of the quiver between the modules S_{ρ_i}.

result case by case. It is worth mentioning that for $\mathrm{BD}_{2n}(a)$ groups we have $(k,q) = 1$ (see [10] §3.3.1).

Case A: We start to generate the representation from $\mathbf{1} \in \rho_0^+$ and $\mathbf{a_0} = (1,0)$. We choose to obtain the basis element $(1,0)$ at every 2-dimensional vertex with horizontal arrows taking $\mathbf{r_{i,j}} = \left(\begin{smallmatrix} 1 & 0 \\ r'_{i,j} & R'_{i,j} \end{smallmatrix}\right)$ $\forall i, j$, $\mathbf{a_i} = (1,0)$ for i even, $\mathbf{c_i} = (1,0)$ for i odd. The open conditions needed are $r_{i,j} \neq 0\ \forall i, j$, $a_i \neq 0$ for i even, $c_i \neq 0$ and i odd. Similarly, we choose to reach $(0,1)$ at every 2-dimensional vertex with vertical arrows taking $\mathbf{u_{i,j}} = \left(\begin{smallmatrix} u_{i,j} & U_{i,j} \\ 0 & 1 \end{smallmatrix}\right)$, $\mathbf{h_i} = (0,1)$ for i even, $\mathbf{f_i} = (0,1)$ for i odd, by the open conditions $U'_{i,j} \neq 0\ \forall i, j$, $H_i \neq 0$ for i even, $F_i \neq 0$ for i odd (including $i = 0$).

For the 1-dimensional representations on the right hand side we take $e_i = g_i = 1$ for all i, with the open conditions $e_i, g_i \neq 0$. On the left hand side, since the G-graph is of type $\Gamma_A(r,s;u,v)$ we have $x^u y^u \notin \Gamma_A$ but $x^i y^i \in \Gamma_A$ for $i < u$. In fact, $x^i y^i \in \rho_{i(a+1)}^{(-1)^i}$. Thus, we need to reach $\rho_{i(a+1)}^{(-1)^i}$ with a nonzero map for $0 < i < u$ with a composition of maps of length i. We can achieve such a map by taking $d_1 = b_2 = d_3 = 1$, … until $d_{u-1} = 1$ if u is even, or $b_{u-1} = 1$ if u is odd. The condition $x^u y^u \notin \Gamma_A$ is given by $B_u = 1$ if u is even, or $D_u = 1$ if u is odd. Finally, from row u to the top row the choices are always $B_i, D_i \neq 0$, $i \neq 0$ and $D_0 \neq 0$.

Case B: In this case $x^u y^k, x^k y^u \notin \Gamma_B$, which implies that $x^i y^i \in \Gamma_B$ for $i < u$. This explains the choices at the left hand side of the quiver, while on the right hand side remain the same as before. Since $x^u y^m, x^m y^u \in V_r$, the conditions $x^u y^m, x^m y^u \notin \Gamma_B$ are expressed with choices $C_0, R_{1,1}, R_{1,2}, \dots, R_{1,r-2} \neq 0$. If $r \leq k$ we have a G-graph of type B_1, otherwise we have a type B_2.

Case C: If the G-graph $\Gamma(r,s;q,q)$ is of type B, then the open conditions are made at the special representation V_r. The difference between the C^+ and C^- is given by $(+)^2 \notin \Gamma_{C^+}$ and $(-)^2 \notin \Gamma_{C^-}$ which are the choices on the vertical arrows in the right side of Q.

Case D: In this case $(+) \notin \Gamma_{D^+}$ (or $(-) \notin \Gamma_{D^-}$). The open condition is made at the special representation ρ_q^+ (or at ρ_q^- respectively). For instance, in the D^+

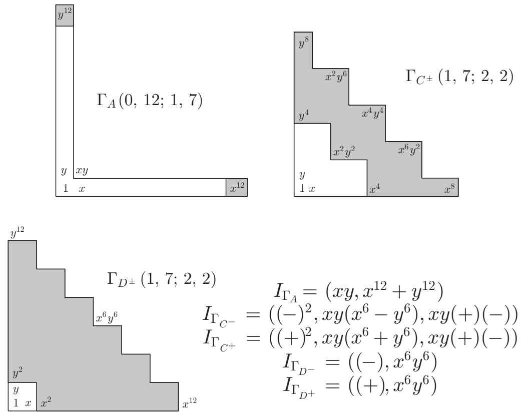

Fig. 4. The $\mathrm{BD}_{12}(7)$-graphs and the representation ideals, where $(+) := x^2+y^2$ and $(-) := x^2-y^2$.

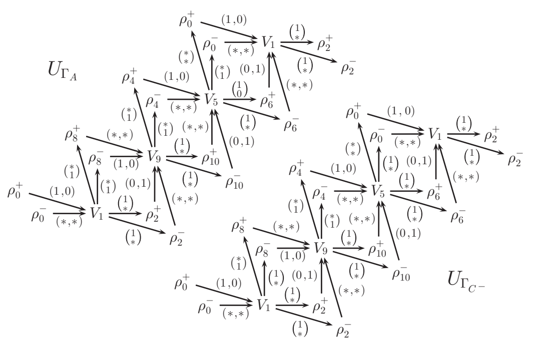

Fig. 5. Open sets U_{Γ_A}, $U_{\Gamma_{C^-}} \subset \mathcal{M}_\theta$ for $\mathrm{BD}_{12}(7)$.

Table I. Basis elements of the G-graph $\Gamma_{D^-}(1,7;2,2)$

ρ_0^+	1	ρ_6^+	$(+)^3$	V_1	$(1,0)=(x,\,y)$
ρ_0^-	$2xy(+)^2$	ρ_6^-	$2xy(+)^5$		$(0,1)=(y\,(+)^3,\,x(+)^3)$
ρ_2^+	$(+)$	ρ_8^+	$(+)^4$	V_5	$(1,0)=(x\,(+)^2,\,y(+)^2)$
ρ_2^-	$2xy(+)^3$	ρ_8^-	$2xy$		$(0,1)=(y\,(+)^5,\,-x(+)^5)$
ρ_4^+	$(+)^2$	ρ_{10}^+	$(+)^5$	V_9	$(1,0)=(x\,(+)^4,\,y(+)^4)$
ρ_4^-	$2xy(+)^4$	ρ_{10}^-	$2xy(+)$		$(0,1)=(y\,(+),\,-x(+))$

case we do not allow a path of length q starting from ρ_0^+ and ending at ρ_q^+, i.e. $E_1 = 1$. □

Example 5.3. Let $G = \mathrm{BD}_{12}(7)$ with $q = 2$, $k = 3$. The G-graphs is shown in Fig. 4. The open choices for Γ_A and Γ_{C^-} are shown in Fig. 5. Using the quiver in Fig. 3 we can calculate the basis polynomials in every irreducible representation (e.g. Table I). The equations of the open cover as hypersurfaces in $\mathbf{C}^3$ are $U_A : c_0d_1 - (c_0d_1^2+1)G_1$, $U_{C^+} : b_2D_1 - (b_2-1)E_1$, $U_{C^-} : b_2D_1 - (b_2-1)G_1$, $U_{D^+} : e_1f_0 - (e_1^2f_0-1)D_1$ and $U_{D^-} : g_1h_0 - (g_1^2h_0-1)D_1$. The dual graph of the exceptional divisor in G-Hilb$(\mathbf{C}^2)$ is $\substack{-2\\-2}\,{-3}\,{-2}$.

References

[1] I. Assem, D. Simson and A. Skowroński, *Elements of the representation theory of associative algebras. Vol. 1*, London Mathematical Society Student Texts, 65, Cambridge Univ. Press, Cambridge, 2006.

[2] R. Bocklandt, T. Schedler and M. Wemyss, Superpotentials and higher order derivations, J. Pure Appl. Algebra **214** (2010), no. 9, 1501–1522.

[3] A. Craw, D. Maclagan and R. R. Thomas, Moduli of McKay quiver representations. I. The coherent component, Proc. Lond. Math. Soc. (3) **95** (2007), no. 1, 179–198.

[4] A. Ishii, On the McKay correspondence for a finite small subgroup of GL$(2,\mathbf{C})$, J. Reine Angew. Math. **549** (2002), 221–233.

[5] Y. Ito and I. Nakamura, Hilbert schemes and simple singularities, in *New trends in algebraic geometry* (*Warwick, 1996*), 151–233, London Math. Soc. Lecture Note Ser., 264, Cambridge Univ. Press, Cambridge.

[6] A. D. King, Moduli of representations of finite-dimensional algebras, Quart. J. Math. Oxford Ser. (2) **45** (1994), no. 180, 515–530.

[7] Becky Leng, The McKay correspondence and orbifold Riemann-Roch, Ph.D. Thesis, University of Warwick, 2002.

[8] J. McKay, Graphs, singularities, and finite groups, in *The Santa Cruz Conference on Finite Groups* (*Univ. California, Santa Cruz, Calif., 1979*), 183–186, Proc. Sympos. Pure Math., 37 Amer. Math. Soc., Providence, RI, 1980.

[9] I. Nakamura, Hilbert schemes of abelian group orbits, J. Algebraic Geom. **10** (2001), no. 4, 757–779.

[10] Alvaro Nolla de Celis, Dihedral groups and G-Hilbert schemes, Ph.D. Thesis, University of Warwick, 112 pp., 2008.

[11] Alvaro Nolla de Celis, G-graphs and special representations for binary dihedral groups in GL$(2,\mathbf{C})$. (to appear in Glasgow Mathematical Journal).

[12] M. Reid, La correspondance de McKay, Astérisque No. 276 (2002), 53–72.

[13] Michael Wemyss, Reconstruction algebras of type D (I). *arXiv:0905.1154v2*, 2009.

[14] Michael Wemyss, Reconstruction algebras of type D (II). *arXiv:0905.1155v1*, 2009.

[15] J. Wunram, Reflexive modules on quotient surface singularities, Math. Ann. **279** (1988), no. 4, 583–598.

[16] Y. Yoshino, *Cohen-Macaulay modules over Cohen-Macaulay rings*, London Mathematical Society Lecture Note Series, 146, Cambridge Univ. Press, Cambridge, 1990.