On the growth of hyperbolic 3-dimensional generalized simplex reflection groups

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Abstract: We prove that the growth rates of three-dimensional generalized simplex reflection groups, i.e. three-dimensional non-compact hyperbolic Coxeter groups with four generators are always Perron numbers.

Key words: Growth function; Coxeter group; Perron number.

1. Introduction. A convex polyhedron P of finite volume in the n-dimensional hyperbolic space $\mathbf{H}^n$ is called a *Coxeter polyhedron* if its dihedral angles are submultiples of π. Any Coxeter polyhedron is a fundamental domain of the discrete group Γ generated by the set S consisting of the reflections with respects to its facets. We call (Γ, S) an *n-dimensional hyperbolic Coxeter group*. In particular when P is a (generalized) simplex of $\mathbf{H}^n$, (Γ, S) is also called a (*generalized*) *simplex reflection group* [9]. In this situation we can define the *word length* $\ell_S(x)$ *of* $x \in \Gamma$ *with respect to* S by the smallest integer $n \geq 0$ for which there exist $s_1, s_2, \cdots, s_n \in S$ such that $x = s_1 s_2 \cdots s_n$. The *growth function* $f_S(t)$ of (Γ, S) is the formal power series $\sum_{k=0}^{\infty} a_k t^k$ where a_k is the number of elements $g \in \Gamma$ satisfying $\ell_S(g) = k$. It is known that the *growth rate* of (Γ, S), $\omega := \limsup_{k\to\infty} \sqrt[k]{a_k}$ is bigger than 1 [3] and less than or equal to the cardinality $|S|$ of S. By means of Cauchy-Hadamard formula, the *radius of convergence* R of $f_S(t)$ is the reciprocal of ω, i.e. $1/|S| \leq R < 1$. In practice the growth function $f_S(t)$ which is analytic on $|t| < R$ extends to a rational function $P(t)/Q(t)$ on $\mathbf{C}$ by analytic continuation where $P(t), Q(t) \in \mathbf{Z}[t]$ are relatively prime. There are formulas due to Solomon and Steinberg to calculate the rational function $P(t)/Q(t)$ from the Coxeter diagram of (Γ, S) [11,12]. See also [4].

Theorem 1 (Solomon's formula)**.** *The growth function $f_S(t)$ of an irreducible spherical Coxeter group (Γ, S) can be written as $f_S(t) = \prod_{i=1}^{k} [m_i + 1]$ where $[n] := 1 + t + \cdots + t^{n-1}$ and $\{m_1, m_2, \cdots, m_k\}$ is the set of exponents of (Γ, S).*

Theorem 2 (Steinberg's formula)**.** *Let (Γ, S) be a hyperbolic Coxeter group. Let us denote the Coxeter subgroup of (Γ, S) generated by the subset $T \subseteq S$ by (Γ_T, T), and denote its growth function by $f_T(t)$. Set $\mathcal{F} = \{T \subseteq S : \Gamma_T$ is finite$\}$. Then*

$$\frac{1}{f_S(t^{-1})} = \sum_{T \in \mathcal{F}} \frac{(-1)^{|T|}}{f_T(t)}.$$

In this case, $t = R$ is a pole of $f_S(t)$. Hence R is a real zero of the denominator $Q(t)$ closest to the origin $0 \in \mathbf{C}$ of all zeros of $Q(t)$. Solomon's formula implies that $P(0) = 1$. Hence $a_0 = 1$ means that $Q(0) = 1$. Therefore $\omega > 1$, the reciprocal of R, becomes a real algebraic integer whose conjugates have moduli less than or equal to the modulus of ω. If $t = R$ is the unique zero of $Q(t)$ with the smallest modulus, then $\omega > 1$ is a real algebraic integer whose conjugates have moduli less than the modulus of ω: such a real algebraic integer is called a *Perron number*.

For two and three-dimensional cocompact hyperbolic Coxeter groups, Cannon-Wagreich and Parry showed that the growth rates are Salem numbers [1,8], where a real algebraic integer $\tau > 1$ is called a *Salem number* if τ^{-1} is an algebraic conjugate of τ and all algebraic conjugates of τ other than τ and τ^{-1} lie on the unit circle. From the definition, a Salem number is a Perron number.

Kellerhals and Perren calculated the growth functions of four-dimensional cocompact hyperbolic Coxeter groups with at most 6 generators and showed that ω are not Salem numbers while they checked that ω are Perron numbers numerically [6].

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In the non-compact case, Floyd proved that the growth rates of two-dimensional non-compact hyperbolic Coxeter groups are *Pisot-Vijayaraghavan numbers*, where a real algebraic integer $\tau > 1$ is called a Pisot-Vijayaraghavan number if algebraic conjugates of τ other than τ lie in the unit disk [2]. A Pisot-Vijayaraghavan number is also a Perron number by definition.

From these results for low-dimensional cases, Kellerhals and Perren conjectured that the growth rates of hyperbolic Coxeter groups are always Perron numbers. In the present paper, we go to the next stage: three-dimensional non-compact hyperbolic Coxeter groups of finite covolume. We will show that the growth rate of a three-dimensional generalized simplex reflection group is a Perron number.

In this paper we consider hyperbolic Coxeter groups with 4 generators, and we can also prove the same result for hyperbolic Coxeter groups with 5 generators, even though the same idea doesn't work anymore; the details will be presented in our forthcoming paper [7].

2. Denominators of growth functions. There are exactly 23 three-dimensional generalized simplex reflection groups [5,9]. By means of Steinberg's formula we can calculate growth functions of them.

Proposition 1. *The denominator polynomials $Q(t)$ of the growth functions $f_S(t) = P(t)/Q(t)$ of 23 three-dimensional generalized simplex reflection groups (Γ, S) are as follows*:

- $(t-1)(3t^2+t-1)$
- $(t-1)(3t^3+t^2+t-1)$
- $(t-1)(2t^4+3t^3+t^2-1)$
- $(t-1)(t^5+t^4+t-1)$
- $(t-1)(2t^5+t^4+t^2+t-1)$
- $(t-1)(3t^5+t^4+t^3+t^2+t-1)$
- $(t-1)(t^7+t^6+t^5+t^4+t^3-1)$
- $(t-1)(t^7+t^6+t^5+t^4-1)$
- $(t-1)(t^7+t^6+2t^5+2t^4+t^3+t^2-1)$
- $(t-1)(t^7+t^6+2t^5+t^4+t^3+t-1)$
- $(t-1)(t^8+2t^7+2t^6+3t^5+t^4+t^3-1)$
- $(t-1)(t^9+t^7+t^6+t^4+t^2+t-1)$
- $(t-1)(t^{13}+t^{12}+2t^{11}+2t^{10}+2t^9+2t^8 +2t^7+2t^6+2t^5+t^4+t^3-1)$
- $(t-1)(t^2+t+1)(t^2+t-1)$
- $(t-1)(t^4+t^3+t^2+t+1)(t^2+t-1)$
- $(t-1)(t^3+t-1)$
- $(t-1)(t^4+t^3+t^2+t+1)(t^3+t-1)$
- $(t-1)(t^4+t^3+t^2+t-1)$
- $(t-1)(t^4+t^3+t^2+t+1)(t^4+t^3+t^2+t-1)$
- $(t-1)(t^5+t^4+t^2+t-1)$
- $(t-1)(t^5+t^3+t-1)$
- $(t-1)(t^6+t^5+t^4+t^3+t^2+t-1)$
- $(t-1)(t^{10}+t^9+t^8+t^7+t^6+t^5+t^4+t^3 +t^2+t-1)$.

We remark that the factor $(t-1)$ appears in every denominator of $f_S(t)$ because of the fact that $1/f_S(1) = \chi(\Gamma) = 0$ in the odd-dimensional case due to a result of Serre [10].

3. Main result.

Theorem 3. *The growth rate of a three-dimensional generalized simplex reflection group is a Perron number.*

In Table I below, we show the distributions of poles of $f_S(t)$ for a particular case of three-dimensional generalized simplex reflection groups.

By Proposition 1, the following lemma is sufficient to prove the theorem.

Lemma 1. *Consider the polynomial of degree $n \geq 2$*

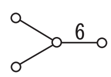
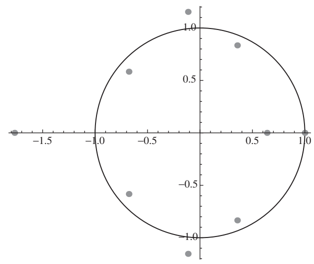
$$g(t) = \sum_{k=1}^{n} a_k t^k - 1,$$

where a_k is a non-negative integer. We also assume that the greatest common divisor of $\{k \in \mathbf{N} \mid a_k \neq 0\}$ is 1. Then there is a real number r_0, $0 < r_0 < 1$ which is the unique zero of $g(t)$ having the smallest absolute value of all zeros of $g(t)$.

Proof. Let us put $h(t) = \sum_{k=1}^{n} a_k t^k$. Note that $g(t) = 0$ if and only if $h(t) = 1$.

(Step 1) Observe $h(0) = 0$, $h(1) > 1$, and $h(t)$ is strictly monotone increasing where t is in the

Table I

Coxeter diagram	
$f_S(t)$	$\dfrac{(t+1)^3(t^2+1)(t^2-t+1)(t^2+t+1)}{(t-1)(t^8+2t^7+2t^6+3t^5+t^4+t^3-1)}$
poles of $f_S(t)$	

open interval $(0,1)$. From the intermediate value theorem, there exists the unique real number r_0 in $(0,1)$ such that $h(r_0)=1$.

(Step 2) Suppose there exists a complex number z whose absolute value is less than r_0 and satisfying the condition $h(z)=1$. Denote $z=re^{i\theta}$ where $0<r<r_0$ and $0\leqslant\theta<2\pi$. Then

$$1=|h(z)|=\left|\sum_{k=1}^{n}a_k(re^{i\theta})^k\right|\leq\sum_{k=1}^{n}|(a_kr^k)e^{ik\theta}|$$
$$=\sum_{k=1}^{n}a_kr^k=h(r)<h(r_0)=1,$$

which is a contradiction. Hence r_0 has the smallest absolute value of all zeros of $g(t)$.

(Step 3) Consider a complex number z whose absolute value is equal to r_0. Set $z=r_0e^{i\theta}$ and $0\leqslant\theta<2\pi$. Then $1=\sum_{k=1}^{n}a_kr_0^ke^{ik\theta}$ implies

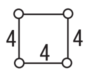
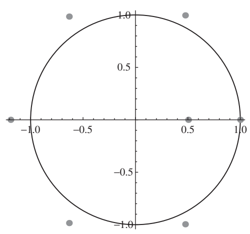
$$1=\sum_{k=1}^{n}a_kr_0^k\cos k\theta\leq\sum_{k=1}^{n}a_kr_0^k=1$$

Hence $\cos k\theta=1$ for any $k\in\mathbf{N}$ with $a_k\neq 0$. The assumption that the greatest common divisor of $\{k\in\mathbf{N}\mid a_k\neq 0\}$ is 1 means that $\theta=0$. Therefore $z=r_0$, and we conclude that r_0 is a unique zero of $g(t)$ having the smallest absolute value of all zeros of $g(t)$. □

4. Remark. By Proposition 1, the next lemma shows that some growth rates of three-dimensional generalized simplex reflection groups are not only Perron numbers but also Pisot-Vijayaraghavan numbers (see Table II below).

Lemma 2. *For $n\geq 2$, the polynomial $g(t)=\sum_{k=1}^{n}t^k-1$ has the unique zero in the unit disk $\{t\in\mathbf{C}\mid |t|<1\}$ and does not have zeros on the unit circle $|t|=1$.*

Table II

Coxeter diagram	
$f_S(t)$	$\dfrac{(t+1)^3(t^2+1)(t^2-t+1)}{(t-1)(t^6+t^5+t^4+t^3+t^2+t-1)}$
poles of $f_S(t)$	

Proof. Define $h_1(t)=t^{n+1}$, $h_2(t)=-2t+1$, and

$$h(t)=h_1(t)+h_2(t)=t^{n+1}-2t+1=(t-1)g(t).$$

Then for any $1/2<r<1$ sufficiently close to 1, $h(r)<0$. Any complex number t on the circle $\{t\in\mathbf{C}\mid |t|=r\}$ satisfies

$$|h_1(t)|=|t^{n+1}|=r^{n+1}<2r-1\leq|2t-1|=|h_2(t)|.$$

Because $h_2(t)$ has the unique zero $t=1/2$ in the disk $|t|<r$, it follows from Rouché's theorem that $h(t)$ also has the unique zero in the disk $|t|<r$. Since this holds for any $r<1$ sufficiently close to 1, it means that $h(t)$, hence $g(t)$ has the unique zero in the unit disk $|t|<1$. Finally we show that $g(t)$ does not have zeros on the unit circle $|t|=1$. Suppose there exists $\theta\in\mathbf{R}$ such that $g(e^{i\theta})=0$. Then $h(e^{i\theta})=0$ implies that $1=|e^{i(n+1)\theta}|=|2e^{i\theta}-1|$. Hence $e^{i\theta}=1$, which contradicts to $g(1)\neq 0$. Therefore $g(t)$ has the unique zero in the unit disk $\{t\in\mathbf{C}\mid |t|<1\}$ and does not have zeros on the unit circle $|t|=1$. □

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