Potential functions via toric degenerations

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Abstract: We construct an integrable system on an open subset of a Fano manifold equipped with a toric degeneration, and compute the potential function for its Lagrangian torus fiber if the central fiber is a toric Fano variety admitting a small resolution.

Key words: Toric degeneration; Lagrangian torus fibration; potential function.

An integrable system on a symplectic manifold (M, ω) of dimension $2N$ is a set of N functions which are functionally independent and mutually Poisson-commutative. An integrable system defines a Hamiltonian $\mathbf{R}^N$-action on M, and any regular compact connected orbit of the $\mathbf{R}^N$-action is a Lagrangian torus by the Arnold-Liouville theorem.

For a Lagrangian submanifold L in a symplectic manifold, the cohomology group $H^*(L; \Lambda_0)$ with coefficient in the Novikov ring

$$\Lambda_0 = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \;\middle|\; a_i \in \mathbf{Q},\ \lambda_i \in \mathbf{R}^{\geq 0},\ \lim_{i \to \infty} \lambda_i = \infty \right\}$$

has a structure of a weak A_∞-algebra [FOOO09]. A solution to the Maurer-Cartan equation

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b, \dots, b) \equiv 0 \mod \mathrm{PD}([L])$$

is called a weak bounding cochain, which can be used to define the deformed Floer cohomology. The potential function is a map $\mathfrak{PO} : \mathcal{M}(L) \to \Lambda_0$ from the moduli space $\mathcal{M}(L)$ of weak bounding cochains such that

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b, \dots, b) = \mathfrak{PO}(b) \cdot \mathrm{PD}([L]).$$

The moment map for the torus action on a toric Fano manifold with respect to a torus-invariant Kähler form provides an example of an integrable system. The potential function for its Lagrangian torus fiber is computed in [CO06, FOOO10].

We have introduced the notion of a *toric degeneration of an integrable system* in [NNU10, Definition 1.1] and shown that the Gelfand-Cetlin system on a flag manifold of type A admits a toric degeneration. In this paper, we construct an integrable system from a toric degeneration of a projective manifold. Let $f : \mathfrak{X} \to B$ be a flat family of projective varieties over a complex manifold B. Assume that B contains two points 0 and 1 such that $X_t = f^{-1}(t)$ is smooth for general $t \in B$ including $t = 1$ and the central fiber X_0 is a toric variety. Assume further that the singular locus of the total space $\mathfrak{X}$ is contained in the singular locus of X_0, and the regular locus of the total space has a Kähler form which restricts to a torus-invariant Kähler form on the regular locus X_0^{reg} of X_0. Choose a piecewise smooth path $\gamma : [0, 1] \to B$ such that $\gamma(0) = 0$, $\gamma(1) = 1$ and $X_{\gamma(t)}$ is smooth for $t \in (0, 1]$. Then the symplectic parallel transport along γ, defined by the horizontal distribution given as the orthogonal complement to the tangent space of the fiber of f with respect to the symplectic form (see e.g. [Sei08, Section (15a)]), gives a symplectomorphism $\widetilde{\gamma} : X_0^{\mathrm{reg}} \to X_1^{\mathrm{reg}}$ from X_0^{reg} to an open subset X_1^{reg} of X_1. By transporting the toric integrable system $\Phi_0 : X_0 \to \mathbf{R}^N$ to X_1 by $\widetilde{\gamma}$, one obtains an integrable system $\Phi = \Phi_0 \circ \widetilde{\gamma}^{-1} : X_1^{\mathrm{reg}} \to \mathbf{R}^N$ on X_1^{reg}. Let $\ell_i(u) = \langle v_i, u \rangle - \tau_i$ be the affine functions defining the faces of the moment polytope; $\Delta = \Phi_0(X_0) = \{u \in \mathbf{R}^N \mid \ell_i(u) \geq 0,\ i = 1, \dots, m\}$. Although Φ is defined only on an open subset of X_1, the proof of [NNU10, Theorem 10.1] goes through without any change and gives the following:

Theorem 1. *Assume that X_0 is a Fano variety admitting a small resolution. Then for any $u \in \mathrm{Int}\Delta$, one has an inclusion* $H^1(L(u); \Lambda_0) \subset$

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$\mathcal{M}(L(u))$ for the Lagrangian torus fiber $L(u) = \Phi^{-1}(u)$, and the potential function is given by

$$\mathfrak{PO}(x) = \sum_{i=1}^{m} e^{\langle v_i, x\rangle} T^{\ell_i(u)}$$

for $x \in H^1(L(u), \Lambda_0)$.

Here, a resolution of singularities is said to be small if the exceptional set does not contain a divisor. If the central fiber does not have a small resolution, then there can be additional contribution to the potential function. See [Aur09,FOOOb] for the discussion on the degeneration of $\mathbf{P}^1 \times \mathbf{P}^1$ to $\mathbf{P}(1,1,2)$ (or its non-small crepant resolution $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(2))$).

The potential function in Theorem 1 has a critical point in the interior of the moment polytope [NNU10, Proposition 12.3]. As a corollary, one obtains a non-displaceable Lagrangian torus just as in [FOOO10, Theorem 1.5]:

Corollary 2. *If a Fano manifold X admits a flat degeneration into a toric Fano variety with a small resolution, then there is a Lagrangian torus L in X satisfying*

$$\psi(L) \cap L \neq \emptyset$$

for any Hamiltonian diffeomorphism $\psi : X \to X$.

As an example, consider the complete intersection $X = Q_1 \cap Q_2$ of two quadrics in $\mathbf{P}^5$, which is isomorphic to the moduli space of stable rank two vector bundles with a fixed determinant of odd degree on a genus two curve [New68,NR69]. We equip X with the Kähler form $\omega = \lambda\,\omega_{\mathrm{FS}}|_X$ where $\lambda > 0$ and ω_{FS} is the Fubini-Study form on $\mathbf{P}^5$. Although X has several toric degenerations (i.e. a degeneration into a variety defined by binomial equations), one has to choose the one with a small resolution to apply Theorem 1. Our choice for the central fiber is the complete intersection $z_0 z_1 = z_2 z_3 = z_4 z_5$ with the torus action

$$[z_0 : z_1 : z_2 : z_3 : z_4 : z_5] \mapsto [\alpha z_0 : \beta z_1 : \gamma z_2 : \alpha\beta\gamma^{-1} z_3 : \alpha\beta z_4 : z_5]$$

for $(\alpha, \beta, \gamma) \in (\mathbf{C}^\times)^3$. The singular locus of this toric variety consists of six ordinary double points, and hence it admits a small resolution. By applying the results above, we obtain the following

Theorem 3. *The moduli space of stable rank two vector bundles with a fixed determinant of odd degree on a genus two curve admits a structure of a completely integrable system, whose moment polytope is the octahedron with vertices $(\lambda, 0, 0)$, $(0, \lambda, 0)$, $(0, 0, \lambda)$, $(\lambda, \lambda, -\lambda)$, $(\lambda, \lambda, 0)$, and $(0, 0, 0)$. The potential function for its Lagrangian torus fiber is given by*

$$\begin{aligned}\mathfrak{PO} = {} & e^{x_2+x_3} T^{u_2+u_3} + e^{-x_1} T^{-u_1+\lambda} + e^{-x_2} T^{-u_2+\lambda} \\ & + e^{x_1+x_3} T^{u_1+u_3} + e^{x_2} T^{u_2} + e^{-x_1-x_3} T^{-u_1-u_3+\lambda} \\ & + e^{-x_2-x_3} T^{-u_2-u_3+\lambda} + e^{x_1} T^{u_1}.\end{aligned}$$

This potential function can be written as

$$\begin{aligned}\mathfrak{PO} = {} & y_2 y_3 + \frac{Q}{y_1} + \frac{Q}{y_2} + y_1 y_3 \\ & + y_2 + \frac{Q}{y_1 y_3} + \frac{Q}{y_2 y_3} + y_1,\end{aligned}$$

by setting $Q = T^\lambda$ and $y_i = e^{x_i} T^{u_i}$, $i = 1, 2, 3$. This has two isolated critical points $(y_1, y_2, y_3) = (\sqrt{Q}, \sqrt{Q}, 1), (-\sqrt{Q}, -\sqrt{Q}, 1)$ with critical values $\pm 8\sqrt{Q}$ and non-isolated critical points consisting of three rational components $y_1 + y_2 = y_3 + 1 = 0$, $y_1 + y_2 = y_1^2 y_3 - Q = 0$, and $y_1 y_2 - Q = y_3 + 1 = 0$ with the critical value 0 (cf. [Prz09, Example 22]). The valuations of these critical points lie in the interior of the moment polytope, so that one obtains the following:

Corollary 4. *The moduli space of stable rank two vector bundles with a fixed determinant of odd degree on a genus two curve has a continuum of non-displaceable Lagrangian tori.*

The existence of a continuum of non-displaceable Lagrangian tori is previously known in toric examples using bulk deformations [FOOOa, Theorem 1.1].

The split-closed derived Fukaya category of the moduli space of stable rank two vector bundles with a fixed determinant of odd degree on a genus two curve contains an orthogonal summand equivalent to that of a genus two curve [Smi10, Theorem 1.1], and it is natural to expect that the Lagrangian tori corresponding to non-isolated critical points generate this summand, whereas the Lagrangian tori for two isolated critical points generate its orthogonal complement.

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