A property of the Fourier transform of probability measures on the real line related to the renewal theorem

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Abstract: We study the weak convergence of some measures related to the renewal theorem, extending a result by Feller and Orey.

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1. Introduction and the main result.

Let F be a probability measure on $\mathbf{R}$, F^{n*} be its n-fold convolution. We assume $m = \int_{-\infty}^{\infty} x\,F(dx) \in (0,\infty)$ since it is the most interesting case in the renewal theory. We denote the Fourier transform $\int_{-\infty}^{\infty} e^{izx}F(dx)$ by $\varphi(z)$.

If $A \subset \mathbf{R}$ is a Borel set and x is a real number, the sets $-A$, xA, and $x+A$ are defined in the obvious way by symmetry, expansion (or contraction), and translation. We say that F is periodic with the period $\omega > 0$ if ω is the greatest positive number such that F is supported on $\omega\mathbf{Z}$. If such ω does not exist, we set $\omega = 0$.

Let $\{X_n\}_{n=0,1,\dots}$ be a sequence of independent random variables with the common distribution F and set $S_0 = 0$, $S_n = \sum_{k=1}^n X_k$. Thus $\{S_n\}_{n=0,1,\dots}$ forms a transient random walk on $\mathbf{R}$ going to $+\infty$. We also set, for any interval I, $U(I) = \sum_{n=0}^{\infty} F^{n*}(I)$, which is the 0-resolvent measure for the random walk $\{S_n\}$.

As the renewal theory (see [5], [1], [4], [2]) reveals, there are following cases: If $\omega > 0$, then $\lim_{n\to\infty} U(\{n\omega\}) = \dfrac{\omega}{m}$; If $\omega = 0$, then $\lim_{x\to\infty} U(x+I) = \dfrac{|I|}{m}$ for any interval I where $|I|$ denotes the length of I. In any case, $\lim_{x\to-\infty} U(x+I) = 0$.

For this, Feller and Orey [6] give a rather short proof, which is based on the symmetrized measure V defined by $V(I) := \frac{1}{2}(U(I)+U(-I))$. Let us review very briefly their method in the case $\omega = 0$. They prove

$$(1) \qquad \lim_{x\to\infty} V(x+I) = \frac{|I|}{2m}$$

and make use of transience of $\{S_n\}$. The proof of (1) relies on the following weak convergence (2) of a family of finite measures. Let $m_s(dz) = \frac{1}{1+z^2}\Re(\frac{1}{1-s\varphi(z)})dz$ and $m(dz) = \frac{\pi}{m}\delta_0(dz) + \frac{1}{1+z^2}\Re(\frac{1}{1-\varphi(z)})dz$, a mixture of a point mass and an absolutely continuous one. It is shown in [6] that

$$(2) \qquad m_s(dz) \Longrightarrow m(dz) \quad \text{as } s \to 1-0$$

if $\omega = 0$, where $\Longrightarrow$ indicates weak convergence.

Remark 1.1. It holds $\Re(\frac{1}{1-s\varphi(z)}) \geq \frac{1}{2}$ and $\Re(\frac{1}{1-\varphi(z)}) \geq \frac{1}{2}$. Indeed, $w = \frac{1}{1-z}$ maps the unit disc $\{z \in \mathbf{C} \mid |z| \leq 1\}$ conformally to $\{\infty\} \cup \{w \in \mathbf{C} \mid \Re w \geq \frac{1}{2}\}$. An extreme example can be found in Example 2.1 in Section 2, although in the case $\omega > 0$. As we make $s \to 1-0$, the density $\frac{1}{1+z^2}\Re(\frac{1}{1-s\varphi(z)})$ of $m_s(dz)$ produces an acute thorn, which will form a point mass of $m(dz)$. Some examples of thorns are observed in Examples 2.1 and 2.2.

Remark 1.2. At every z such that $\varphi(z) = 1$, we can prove $\varphi'(z) = im$, whether $\omega = 0$ or $\omega > 0$. Hence $\frac{1}{1-\varphi(z)}$ has only isolated singularities, which forms a negligible set, so that the measure $\frac{1}{1+z^2}\Re(\frac{1}{1-\varphi(z)})dz$ is well-defined. The set of singularity is $\frac{2\pi}{\omega}\mathbf{Z}$ if $\omega > 0$ while $z = 0$ is the only singularity if $\omega = 0$.

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Remark 1.3. In many cases, $\Re(\frac{1}{1-\varphi(z)})$ behaves rather mildly near a singularity a: If $\int_{-\infty}^{\infty} |x|^{1+\delta} F(dx) < \infty$ for some $\delta \in (0,1)$, then $\Re(\frac{1}{1-\varphi(z)}) = O(|z-a|^{-1+\delta})$ as $z \to a$. This is an exercise involving the expansion $\varphi(z) = 1 + im(z-a) + O(|z-a|^{1+\delta})$.

In this note, we are motivated to understand (2) deeper and aim to establish the following result which includes also the case $\omega > 0$.

Theorem 1.1. *For any $\alpha > 0$ and $0 \le s < 1$, let $m_s^{(\alpha)}(dz) = \frac{1}{1+|z|^{\alpha+1}} \Re(\frac{1}{1-s\varphi(z)}) dz$.*

Then the family of finite measures $m_s^{(\alpha)}(dz)$ converges weakly, say, to $m^{(\alpha)}(dz)$:

$$m_s^{(\alpha)}(dz) \Longrightarrow m^{(\alpha)}(dz) \quad \text{as } s \to 1-0. \tag{3}$$

Moreover, if $\omega = 0$ then $m^{(\alpha)}(dz) = \frac{\pi}{m}\delta_0(dz) + \frac{1}{1+|z|^{\alpha+1}} \Re(\frac{1}{1-\varphi(z)}) dz$; if $\omega > 0$ then $m^{(\alpha)}(dz) = \sum_{n\in\mathbf{Z}} \frac{\pi}{m(1+(2\pi|n|/\omega)^{\alpha+1})} \delta_{2\pi n/\omega}(dz) + \frac{1}{1+|z|^{\alpha+1}} \Re(\frac{1}{1-\varphi(z)}) dz$.

The proof will be given in Section 3.

Theorem 1.1 gives an explanation for the roles played by the assumption $\omega = 0$ and the factor $1/(1+z^2)$ in (2). Moreover, if we make $\alpha \le 0$ in the expression of $m_s^{(\alpha)}(dz)$ and $m^{(\alpha)}(dz)$, we easily deduce that they are infinite measures from Remark 1.1. In this sense, the statement of Theorem 1.1 is exhaustive concerning the value of α that enables weak convergence.

2. Examples. In this section, we investigate several examples of F and φ. Let $\alpha > 0$.

Example 2.1. If $\omega > 0$, $\varphi(z)$ is a periodic function with the fundamental period $\frac{2\pi}{\omega}$. The simplest case among them is $F(dz) = \delta_m(dz)$: the unit mass at $m = \omega > 0$. In this case, $\varphi(z) = e^{imz}$ and $\Re(\frac{1}{1-\varphi(z)}) = \frac{1}{2}$. The limit measure is hence $m^{(\alpha)}(dz) = \sum_{n\in\mathbf{Z}} \frac{\pi}{m(1+(2\pi|n|/m)^{\alpha+1})} \delta_{2\pi n/m}(dz) + \frac{1}{2(1+|z|^{\alpha+1})} dz$. Next let us observe how $m_s^{(\alpha)}(dz)$ produces a series of acute thorns at each point in $\frac{2\pi}{m}\mathbf{Z}$. We have

$$\Re\left(\frac{1}{1-s\varphi(z)}\right) = \Re\left(\frac{1}{1-se^{imz}}\right) = \frac{1}{2} + \frac{(1-s^2)/2}{(1+s^2) - 2s\cos(mz)}.$$

Here the first term corresponds to the absolutely continuous part of $m^{(\alpha)}(dz)$. In a neighborhood of $z = 2\pi n/m$, where n is an integer, it holds

$$\cos(mz) = \cos(m(z - 2\pi n/m)) = 1 - (1+o(1))\frac{1}{2}m^2(z-2\pi n/m)^2$$

and hence

$$\frac{(1-s^2)/2}{(1+s^2) - 2s\cos(mz)} = (1+o(1))\frac{1-s}{(1-s)^2 + m^2(z-2\pi n/m)^2}$$

as $s \to 1-0$. The last term is very close to a scaled/translated version $\frac{1}{1-s} f(\frac{z-2\pi n/m}{1-s})$ of a function $f(x) = \frac{1}{1+m^2x^2}$, approximating a point mass $\nu\delta_{2\pi n/m}$ with $\nu = \int_{-\infty}^{\infty} f(x)dx = \pi/m$.

Example 2.2. If $\omega = 0$ and F is not singular with respect to the Lebesgue measure, (3) follows from (2) in a straightforward manner as follows. To begin with, we note that $\sup_{\varepsilon<|z|<\infty} |\varphi(z)| < 1$ for any $\varepsilon > 0$ and hence $\Re(\frac{1}{1-s\varphi(z)})$ converges to $\Re(\frac{1}{1-\varphi(z)})$ uniformly on $\{\varepsilon < |z| < \infty\}$. In view of (2), $1_{[-1,1]}(z)\Re(\frac{1}{1-s\varphi(z)})dz$ converges weakly to $\frac{\pi}{m}\delta_0(dz) + 1_{[-1,1]}(z)\Re(\frac{1}{1-\varphi(z)})dz$ as $s \to 1-0$, which convergence can be traced back to [3]. For $|z| > 1$, $\sup_{0<s<1}(\frac{1}{1-s\varphi(z)}) < \infty$. It is then immediate to deduce (3) since $\frac{1}{1+|z|^{\alpha+1}}$ is an integrable function. Among Example 2.2, the exponential distribution is the most remarkable case: $F(dx) = \frac{1}{m}e^{-x/m}dx$. In this case, $\varphi(z) = \frac{1}{1-imz}$ and $\Re(\frac{1}{1-\varphi(z)}) = 1$. The limit measure is hence $m^{(\alpha)}(dz) = \frac{\pi}{m}\delta_0(dz) + \frac{1}{1+|z|^{\alpha+1}} dz$. Next let us observe how $m_s^{(\alpha)}(dz)$ produces an acute thorn at $z = 0$. We have

$$\Re\left(\frac{1}{1-s\varphi(z)}\right) = \Re\left(\frac{1}{1-s/(1-imz)}\right) = 1 + s\frac{1-s}{(1-s)^2 + m^2z^2}.$$

Here the first term corresponds to the absolutely continuous part of $m^{(\alpha)}(dz)$ and the second term is very close to a scaled version $\frac{1}{1-s}f(\frac{z}{1-s})$ of a function $f(x) = \frac{1}{1+m^2x^2}$, approximating $\nu\delta_0$ with $\nu = \int_{-\infty}^{\infty} f(x)dx = \pi/m$.

Example 2.3. The case $\omega = 0$ and F is singular is the most troublesome one. To be specific, let $a > 0$, $b > 0$, and $0 < c < 1$ be such that b/a is an irrational number and set $F = c\delta_a + (1-c)\delta_b$. Its Fourier transform $\varphi(z) = c\exp(iaz) + (1-c)\exp(ibz)$ satisfies $\liminf_{z\to\pm\infty} |\varphi(z) - 1| \le$

$\liminf_{k\in\mathbf{Z},k\to\pm\infty}|\varphi(2\pi k/a)-1|=0$. Indeed, $\varphi(2\pi k/a)=c+(1-c)\exp(2\pi\frac{b}{a}ki)$ and the sequence $\{\exp(2\pi\frac{b}{a}ki);k\in\mathbf{Z}\}$ runs densely over the unit disc in $\mathbf{C}$. Hence it holds $\limsup_{z\to\pm\infty}\Re(\frac{1}{1-\varphi(z)})dz=\infty$ and, for any fixed $s\in[0,1)$, $\limsup_{z\to\pm\infty}\Re(\frac{1}{1-s\varphi(z)})=1/(1-s)$. So one can not expect a priori bound $C(1+|z|^{\alpha+1})^{-1}$ for the density of $m_s^{(\alpha)}$ on $\{|z|>1\}$ as in Example 2.2. Still Theorem 1.1 implies that $m_s^{(\alpha)}$ converges weakly.

3. Proof of Theorem 1. Since the random walk $\{S_n\}_{n=0,1,\dots}$ is transient, we have $U((-h,h))=V((-h,h))<\infty$ for any $h>0$.

Define a family of measures V_s for $0\le s<1$ by

$$V_s(I)=\frac{1}{2}\sum_{n=0}^{\infty}s^n(F^{n*}(I)+F^{n*}(-I)).$$

Each V_s is a finite measure on $\mathbf{R}$. As $s\to 1-0$, $V_s((-h,h))\nearrow V((-h,h))<\infty$. The following statement is given in [6] but we prove it here for the sake of reader's convenience. Let $\mathcal{F}g(z)=\int_{-\infty}^{\infty}e^{izx}g(x)dx$ and $\mathcal{F}^{-1}\gamma(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ixz}\gamma(z)dz\equiv\frac{1}{2\pi}\mathcal{F}\gamma(-x)$ for integrable functions $g(x)$ and $\gamma(z)$.

Lemma 3.1. *For any function $g(x)\in L^1(\mathbf{R})$ such that $\mathcal{F}g(z)\in L^1(\mathbf{R})$, we have, for any $y\in\mathbf{R}$,*

$$\int_{-\infty}^{\infty}g(y-x)V_s(dx)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-iyz}\mathcal{F}g(z)\Re\left(\frac{1}{1-s\varphi(z)}\right)dz. \tag{4}$$

Proof. The Fourier transform of V_s is given by

$$\int_{-\infty}^{\infty}e^{izx}V_s(dx)=\frac{1}{2}\sum_{n=0}^{\infty}s^n(\varphi(z)^n+\varphi(-z)^n)=\Re\left(\frac{1}{1-s\varphi(z)}\right).$$

The equation (4) follows from the Parseval identity or the Funibi theorem. □

In the next lemma we prove the existence of a function with a crucial property.

Lemma 3.2. *Let $0<\alpha<1$ and $\tau(z)=((1-|z|)\vee 0)^2$, $\delta_\alpha(z)=\exp(-|z|^\alpha)$, and $\psi_\alpha(z)=\tau(z)\delta_\alpha(z)$. We also set $t=\mathcal{F}^{-1}\tau$, $d_\alpha=\mathcal{F}^{-1}\delta_\alpha$, and $p_\alpha=\mathcal{F}^{-1}\psi_\alpha$.*

Then ψ_α is bounded, nonnegative, supported on a compact set; p_α is bounded, strictly positive, and $p_\alpha(x)\asymp\frac{1}{|x|^{\alpha+1}}\wedge 1$, where '$\asymp$' means that the ratio $r(x)$ between both sides satisfies $0<\inf_{x\in\mathbf{R}}r(x)\le\sup_{x\in\mathbf{R}}r(x)<\infty$. In particular, ψ_α and p_α are both integrable and continuous.

Moreover, the functions that appear here are even and real-valued.

Proof. It follows from the formula I.2.4 in [7] that $t(x)=\frac{4}{x^2}(1-\frac{\sin x}{x})\asymp\frac{1}{|x|^2}\wedge 1$.

It is known that $d_\alpha(x)$ is the density of a symmetric α-stable law. As such, $d_\alpha(x)$ is infinitely differentiable (see, e.g., [8, exercise 1.5 (p.49)]), strictly positive, and satisfies $d_\alpha(x)\asymp\frac{1}{|x|^{\alpha+1}}\wedge 1$.

Let '$*$' denote the convolution of two functions. Then $p_\alpha(x)=\mathcal{F}^{-1}(\tau\delta_\alpha)(x)=(t*d_\alpha)(x)$, from which follows $p_\alpha(x)\asymp\frac{1}{|x|^{\alpha+1}}\wedge 1$. The other statements can be deduced easily. □

Proof of Theorem 1. For $h\in(0,1)$, set

$$g_h(x):=h\,\psi_\alpha(x/h^{1/\alpha}).$$

Since ψ_α is an even function, $\frac{1}{2\pi}\mathcal{F}g_h(z)=\mathcal{F}^{-1}g_h(z)=h^{1+1/\alpha}p_\alpha(h^{1/\alpha}z)$. Thus it holds $\mathrm{supp}(g_h)=[-h^{1/\alpha},h^{1/\alpha}]$, $\|g_h\|_\infty=h$, and $\mathcal{F}g_h(z)\asymp\frac{1}{|z|^{\alpha+1}}\wedge h^{1+1/\alpha}$.

Choosing $g=g_h$ and $y=0$ in (4), we obtain

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{F}g_h(z)\Re\left(\frac{1}{1-s\varphi(z)}\right)dz&=\int_{-\infty}^{\infty}g_h(x)V_s(dx)\\&\le h\,V_s([-h^{1/\alpha},h^{1/\alpha}])\\&\le h\,V_s([-1,1])\le h\,V([-1,1]).\end{aligned} \tag{5}$$

On one hand, there exists a positive constant C_0 (depending on α) such that

$$\mathcal{F}g_h(z)|z|^{\alpha+1}>\frac{1}{C_0}$$

if $|z|>h^{-1/\alpha}$. We have from (5) that

$$\begin{aligned}&m_s^{(\alpha)}([-h^{-1/\alpha},h^{-1/\alpha}]^c)\\&\le\int_{|z|>h^{-1/\alpha}}\frac{1}{|z|^{\alpha+1}}\Re\left(\frac{1}{1-s\varphi(z)}\right)dz\\&\le\int_{|z|>h^{-1/\alpha}}C_0\mathcal{F}g_h(z)\Re\left(\frac{1}{1-s\varphi(z)}\right)dz\\&\le 2\pi C_0h\,V([-1,1])\end{aligned}$$

for any $h\in(0,1)$ and $s\in[0,1)$.

On the other hand, if we fix $h\in(0,1)$, then there exists a positive constant $C_1(h)$ depending on h (and α) such that

$$\mathcal{F}g_h(z)>\frac{1}{C_1(h)}$$

for any $z \in [-h^{-1/\alpha}, h^{-1/\alpha}]$. Hence

$$\begin{aligned} m_s^{(\alpha)}&([-h^{-1/\alpha}, h^{-1/\alpha}]) \\ &\leq \int_{|z|\leq h^{-1/\alpha}} \Re\left(\frac{1}{1-s\varphi(z)}\right) dz \\ &\leq \int_{|z|\leq h^{-1/\alpha}} C_1(h)\mathcal{F}g_h(z)\Re\left(\frac{1}{1-s\varphi(z)}\right) dz \\ &\leq 2\pi C_1(h) h\, V([-1,1]) \end{aligned}$$

for any $s \in [0, 1)$.

These bounds imply that $\{m_s^{(\alpha)}(dz); s \in [0,1)\}$ is a tight family of finite measures on $\mathbf{R}$ and there exists a finite measure $m^{(\alpha)}(dz)$ such that (3) holds.

If $\omega = 0$, the density $\frac{1}{1+|z|^{\alpha+1}}\Re(\frac{1}{1-s\varphi(z)})$ converges uniformly to $\frac{1}{1+|z|^{\alpha+1}}\Re(\frac{1}{1-\varphi(z)})$ as $s \to 1-0$ in every compact interval excluding the origin. Hence $m^{(\alpha)}(dz) = \nu\delta_0(dz) + \frac{1}{1+|z|^{\alpha+1}}\Re(\frac{1}{1-\varphi(z)})dz$ where $\nu \in [0,\infty)$ is the mass assigned to the origin by the limit measure. To be consistent with (2), we must have $\nu = \frac{\pi}{m}$.

If $\omega > 0$, then $\varphi(z) = 1$ if and only if $z \in \frac{2\pi}{\omega}\mathbf{Z}$. It follows that $\frac{1}{1+|z|^{\alpha+1}}\Re(\frac{1}{1-s\varphi(z)})$ converges, as $s \to 1-0$, to $\frac{1}{1+|z|^{\alpha+1}}\Re(\frac{1}{1-\varphi(z)})$ uniformly on any compact set K such that $K \cap \frac{2\pi}{\omega}\mathbf{Z} = \emptyset$. Hence the limit measure can have point masses only at points belonging to $\frac{2\pi}{\omega}\mathbf{Z}$. It is straightforward to verify

$$m^{(\alpha)}(\{2\pi n/\omega\}) = \frac{m^{(\alpha)}(\{0\})}{(1+(2\pi|n|/\omega)^{\alpha+1})}$$

by periodicity.

To prove $m^{(\alpha)}(\{0\}) = \frac{\pi}{m}$, we introduce $\tilde{F}_\varepsilon = F * N(0,\varepsilon)$, where '$*$' denotes the convolution of two measures and $N(0,\varepsilon)$ is the normal distribution with mean 0 and variance $\varepsilon \in (0,\infty)$. It is absolutely continuous and Theorem 1.1 (the non-periodic case) is applicable.

Since $\tilde{F}_\varepsilon$ is the probability distribution of the sum of X_1 and an independent centered normal random variable,

$$\int_{-\infty}^{\infty} x\, \tilde{F}_\varepsilon(dx) = m. \tag{6}$$

The Fourier transform $\tilde{\varphi}_\varepsilon(z)$ of $\tilde{F}_\varepsilon$ is given by $e^{-\varepsilon z^2/2}\varphi(z)$. Let

$$m_s^{(\alpha;\varepsilon)}(dz) = \frac{1}{1+|z|^{\alpha+1}}\Re\left(\frac{1}{1-se^{-\varepsilon z^2/2}\varphi(z)}\right)dz.$$

Then this family converges weakly to, say, $m^{(\alpha;\varepsilon)}(dz)$. In particular, $m^{(\alpha;\varepsilon)}(\{0\}) = \frac{\pi}{m}$ by (6).

We denote the Radon-Nikodym density $\frac{dm_s^{(\alpha)}}{dm_s^{(\alpha;\varepsilon)}}(z) = \frac{\Re(\frac{1}{1-s\varphi(z)})dz}{\Re(\frac{1}{1-se^{-\varepsilon z^2/2}\varphi(z)})dz}$ by $\xi(\varepsilon,s,z)$.

We define the error terms $R(z)$ and $I(z)$ in the expansion $\varphi(z) = 1 + imz + R(z) + iI(z)$ so that $|R(z)| + |I(z)| = o(z)$ as $z \to 0$ and $R(z)$ and $I(z)$ are real valued.

For all $\varepsilon \in (0, \frac{1}{3})$ that is sufficiently small, we can find a neighborhood $U_\varepsilon \subset (-\frac{1}{2}, \frac{1}{2})$ of $z = 0$ such that $1 - 2\varepsilon \leq se^{-\varepsilon z^2/2} \leq 1$, $|I(z)| \leq \varepsilon|z|$, and $|R(z)| \leq \varepsilon|z|$ for any $s \in [1-\varepsilon, 1)$ and $z \in U_\varepsilon$. Moreover, it follows that $R(z) \leq -\frac{1}{2}(m-\varepsilon)^2 z^2 < 0$ from $|\varphi(z)| \leq 1$. We set

$$\begin{aligned} C_1(\varepsilon) &:= \inf_{s\in[1-\varepsilon,1), z\in U_\varepsilon} \xi(\varepsilon,s,z), \\ C_2(\varepsilon) &:= \sup_{s\in[1-\varepsilon,1), z\in U_\varepsilon} \xi(\varepsilon,s,z). \end{aligned}$$

It is elementary but tedious to prove that

$$\lim_{\varepsilon\to+0} C_1(\varepsilon) = \lim_{\varepsilon\to+0} C_2(\varepsilon) = 1$$

using the above estimates. We omit its proof. By the definiton of $m^{(\alpha;\varepsilon)}$, we have

$$C_1(\varepsilon)m^{(\alpha;\varepsilon)}(\{0\}) \leq m^{(\alpha)}(\{0\}) \leq C_2(\varepsilon)m^{(\alpha;\varepsilon)}(\{0\}).$$

Since ε is arbitrary and $m^{(\alpha;\varepsilon)}(\{0\}) = \frac{\pi}{m}$, we have $m^{(\alpha)}(\{0\}) = \frac{\pi}{m}$. □

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