Selberg type zeta function for the Hilbert modular group of a real quadratic field

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Abstract: In this article we announce fundamental results of Selberg type zeta functions for the Hilbert modular group of a real quadratic field; the meromorphic extension over $\mathbf{C}$, its functional equation and some arithmetic applications.

Key words: Hilbert modular group; Selberg zeta function.

1. Introduction. We consider Selberg type zeta functions attached to the Hilbert modular group of a real quadratic field. First of all, we recall the original Selberg zeta function constructed by Selberg in 1956. Let Γ be a co-finite discrete subgroup of $\mathrm{PSL}(2,\mathbf{R})$ acting on the upper half plane $\mathbf{H}$. Take a hyperbolic element $\gamma \in \Gamma$, that is $|\mathrm{tr}(\gamma)| > 2$, then the centralizer of γ in Γ is infinite cyclic and γ is conjugate in $\mathrm{PSL}(2,\mathbf{R})$ to $\begin{pmatrix} N(\gamma)^{1/2} & 0 \\ 0 & N(\gamma)^{-1/2} \end{pmatrix}$ with $N(\gamma) > 1$. Put $\mathrm{Prim}(\Gamma)$ be the set of Γ-conjugacy classes of the primitive hyperbolic elements in Γ. For $\Re(s) > 1$, the Selberg zeta function for Γ is defined by the following Euler product:

$$Z_\Gamma(s) := \prod_{p \in \mathrm{Prim}(\Gamma)} \prod_{k=0}^{\infty} (1 - N(p)^{-(k+s)}).$$

Selberg defined this zeta function and proved (Cf. Selberg [8,9] and Hejhal [6]):

(a) $Z_\Gamma(s)$ defined for $\Re(s) > 1$ extends meromorphically over the whole complex plane.

(b) $Z_\Gamma(s)$ has "non-trivial" zeros at $s = \frac{1}{2} \pm i r_n$ of order equal to the multiplicity of the eigenvalue $1/4 + r_n^2$ of the Laplacian $\Delta_0 = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ acting on $L^2(\Gamma\backslash\mathbf{H})$.

(c) $Z_\Gamma(s)$ satisfies a functional equation between s and $1-s$.

The theory of Selberg zeta functions for locally symmetric spaces of rank one is evolved by Gangolli (compact case) and Gangolli-Warner [3] (noncompact case). For higher rank cases, Deitmar [1] defined and studied "generalized Selberg zeta functions" for compact higher rank locally symmetric spaces. Therefore, our concern is to define and study "Selberg type zeta functions" for *noncompact* higher rank locally symmetric spaces such as Hilbert modular surfaces.

Let us introduce Selberg type zeta functions for the Hilbert modular group of a real quadratic field. Let $K/\mathbf{Q}$ be a real quadratic field with class number one and $\mathcal{O}_K$ be the ring of integers of K. Put D be the discriminant of K and $\varepsilon > 1$ be the fundamental unit of K. We denote the generator of $\mathrm{Gal}(K/\mathbf{Q})$ by σ and put $a' := \sigma(a)$ and $N(a) := aa'$ for $a \in K$. We also put $\gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathcal{O}_K)$. Let $\Gamma_K := \{(\gamma,\gamma') \mid \gamma \in \mathrm{PSL}(2,\mathcal{O}_K)\}$ be the Hilbert modular group of K. It is known that Γ_K is a co-finite (*non-cocompact*) irreducible discrete subgroup of $\mathrm{PSL}(2,\mathbf{R}) \times \mathrm{PSL}(2,\mathbf{R})$ and Γ_K acts on the product $\mathbf{H}^2$ of two copies of the upper half plane $\mathbf{H}$ by component-wise linear fractional transformation. Γ_K have only one cusp (∞,∞), i.e., Γ_K-inequivalent parabolic fixed point. $X_K := \Gamma_K\backslash\mathbf{H}^2$ is called the Hilbert modular surface of K.

Let $(\gamma,\gamma') \in \Gamma_K$ be hyperbolic-elliptic, i.e., $|\mathrm{tr}(\gamma)| > 2$ and $|\mathrm{tr}(\gamma')| < 2$. Then the centralizer of hyperbolic-elliptic (γ,γ') in Γ_K is infinite cyclic.

Definition 1.1 (Selberg type zeta function for Γ_K with the weight $(0,m)$)**.** For an even integer $m \geq 2$ and $s \in \mathbf{C}$ with $\Re(s) > 1$, we define

$$Z_K(s;m) := \prod_{(p,p')} \prod_{k=0}^{\infty} (1 - e^{i(m-2)\omega} N(p)^{-(k+s)})^{-1}.$$

Here, (p,p') run through the set of primitive hyperbolic-elliptic Γ_K-conjugacy classes of Γ_K, and (p,p') is conjugate in $\mathrm{PSL}(2,\mathbf{R})^2$ to

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$$(p,p') \sim \left(\begin{pmatrix} N(p)^{1/2} & 0 \\ 0 & N(p)^{-1/2} \end{pmatrix}, \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix} \right).$$

Here, $N(p) > 1$, $\omega \in (0,\pi)$ and $\omega \notin \pi\mathbf{Q}$. The product is absolutely convergent for $\Re(s) > 1$.

2. Analytic properties of $Z_K(s;m)$. We state our main theorems on analytic properties of $Z_K(s;m)$.

Theorem 2.1. *For an even integer $m \geq 2$, $Z_K(s;m)$ a priori defined for $\Re(s) > 1$ has a meromorphic extension over the whole complex plane.*

Our Selberg zeta functions $Z_K(s;m)$ have also "non-trivial" zeros or poles and they have connections with the eigenvalues of two Laplacians. Let $\Delta_0^{(1)} := -y_1^2(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2})$ and $\Delta_m^{(2)} := -y_2^2(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y_2^2}) + im\, y_2 \frac{\partial}{\partial x_2}$ be the Laplacians of weight 0 and m for $(z_1,z_2) \in \mathbf{H}^2$. Two Laplacians $\Delta_0^{(1)}$ and $\Delta_m^{(2)}$ act on

$$L^2(m) := L^2_{\mathrm{dis}}(\Gamma_K\backslash\mathbf{H}^2;(0,m))$$

: the space of Hilbert Maass forms of weight $(0,m)$. For an even integer q, we consider the subspaces of $L^2(q)$ and $L^2(q-2)$ given by

$$V_q^{(2)} := \left\{ f \in L^2(q) \,\middle|\, \Delta_q^{(2)} f = \frac{q}{2}\left(1-\frac{q}{2}\right) f \right\},$$
$$W_{q-2}^{(2)} := \left\{ f \in L^2(q-2) \,\middle|\, \Delta_{q-2}^{(2)} f = \frac{q}{2}\left(1-\frac{q}{2}\right) f \right\}.$$

Theorem 2.2 (Zeros and poles of $Z_K(s;m)$)**.** *Let $m \geq 4$ be an even integer.*

- *$Z_K(s;m)$ has zeros at $s = \frac{1}{2} \pm i\rho_j(m)$ of order equal to the multiplicity of the eigenvalue $\frac{1}{4} + \rho_j(m)^2$ of $\Delta_0^{(1)}$ acting on $V_m^{(2)}$ and has simple zeros at $s = 1 - \frac{m}{2} + \frac{\pi i k}{\log\varepsilon}$ for $k \in \mathbf{Z}$.*
- *$Z_K(s;m)$ has poles at $s = \frac{1}{2} \pm i\rho_j(m-2)$ of order equal to the multiplicity of the eigenvalue $\frac{1}{4} + \rho_j(m-2)^2$ of $\Delta_0^{(1)}$ acting on $V_{m-2}^{(2)}$ and has simple poles at $s = 2 - \frac{m}{2} + \frac{\pi i k}{\log\varepsilon}$ for $k \in \mathbf{Z}$.*
- *$Z_K(s;m)$ has zeros or poles (according to their orders are positive or negative) at $s = -k$ ($k \in \mathbf{N} \cup \{0\}$) of order $(2k+1)E(X_K) + 2\sum_{j=1}^N [k/\nu_j] - \sum_{j=1}^N \beta_{k,j}(m)$.*
- *If $m = 4$, $Z_K(s,m)$ has additional simple zeros at $s = 0$ and $s = 1$.*

Here, $E(X_K)$ denotes the Euler characteristic of X_K, natural numbers $\nu_1,\nu_2,\cdots,\nu_N$ denote the orders of primitive elliptic conjugacy classes of Γ_K and $\beta_{j,k}(m)$ are explicitly given integers. When the location of two zeros or poles coincide, the orders of them are added.

On the contrary to the case of $m \geq 4$, $Z_K(s;2)$ has no "non-trivial" poles.

Theorem 2.3 (Zeros and poles of $Z_K(s;2)$)**.**

- *$Z_K(s;2)$ has a double pole at $s = 1$.*
- *$Z_K(s;2)$ has zeros at $s = \frac{1}{2} \pm i\rho_j(2)$ of order equal to the multiplicity of the eigenvalue $\frac{1}{4} + \rho_j(2)^2$ of $\Delta_0^{(1)}$ acting on $V_2^{(2)}$ and has zeros at $s = \frac{1}{2} \pm i\mu_j(-2)$ of order equal to the multiplicity of the eigenvalue $\frac{1}{4} + \mu_j(-2)^2$ of $\Delta_0^{(1)}$ acting on $W_{-2}^{(2)}$.*
- *$Z_K(s;2)$ has double zeros at $s = \pm\frac{k\pi i}{\log\varepsilon}$ ($k \in \mathbf{N}$).*
- *$Z_K(s;2)$ has a zero at $s = 0$ of order $E(X_K)$.*
- *$Z_K(s;2)$ has zeros or poles (according to their orders are positive or negative) at $s = -k$ ($k \in \mathbf{N}$) of order $(2k+1)E(X_K) + 2\sum_{j=1}^N [k/\nu_j] - 2kN$.*

When the location of two zeros or poles coincide, the orders of them are added.

Actually $Z_K(s;m)$ has infinite "non-trivial" zeros by the following "Weyl's law".

Theorem 2.4. *For an even integer $m \geq 2$, let*

$$N_m^+(T) := \#\{j \mid 1/4 + \rho_j(m)^2 \leq T\}$$

for $T > 0$. Then we have

$$N_m^+(T) \sim (m-1)\frac{\mathrm{vol}(\Gamma_K\backslash\mathbf{H}^2)}{16\pi^2}\, T \quad (T \to \infty).$$

Our $Z_K(s;m)$ also satisfy a symmetric functional equation.

Theorem 2.5 (Functional equation)**.** *The Selberg type zeta function $Z_K(s;m)$ satisfies the functional equation*

$$\hat{Z}_K(s;m) = \hat{Z}_K(1-s;m).$$

Here the completed zeta function $\hat{Z}_K(s,m)$ is given by

$$\hat{Z}_K(s;m) := Z_K(s;m)\, Z_{\mathrm{id}}(s)\, Z_{\mathrm{ell}}(s;m) \cdot Z_{\mathrm{par/sct}}(s;m)\, Z_{\mathrm{hyp2/sct}}(s;m)$$

with

$$Z_{\mathrm{id}}(s) := (\Gamma_2(s)\Gamma_2(s+1))^{2\zeta_K(-1)}$$

$$Z_{\mathrm{ell}}(s;m) := \prod_{j=1}^{N}\prod_{l=0}^{\nu_j-1} \Gamma(\tfrac{s+l}{\nu_j})^{\frac{\nu_j-1-\alpha_l(m,j)-\overline{\alpha_l}(m,j)}{\nu_j}}$$

$$Z_{\mathrm{par/sct}}(s;m) := \begin{cases} 1 & (m \geq 4) \\ \varepsilon^{-2s} & (m = 2) \end{cases}$$

$$Z_{\mathrm{hyp2/sct}}(s;m) := \begin{cases} \zeta_\varepsilon(s+\frac{m}{2}-1)\,\zeta_\varepsilon(s+\frac{m}{2}-2)^{-1} & (m \geq 4) \\ \zeta_\varepsilon(s)^2 & (m = 2) \end{cases}$$

where, $\Gamma_2(s)$ *is the double Gamma function* (*Cf.* [5, Definition 4.10, p. 751]), $\zeta_K(s)$ *is the Dedekind zeta functions of* K *and* $\alpha_l(m,j)$, $\overline{\alpha_l}(m,j) \in \{0,1,\cdots,\nu_j-1\}$ *are explicitly given integers and* $\zeta_\varepsilon(s) := (1-\varepsilon^{-2s})^{-1}$.

3. Ruelle type zeta functions. We consider the Ruelle type zeta function of Γ_K.

Definition 3.1 (Ruelle type zeta function for Γ_K)**.** For $\Re(s) > 1$, the Ruelle type zeta function for Γ_K is defined by the following absolutely convergent Euler product:

$$R_K(s) := \prod_{(p,p')} (1 - N(p)^{-s})^{-1}.$$

Here, (p,p') run through the set of primitive hyperbolic-elliptic Γ_K-conjugacy classes of Γ_K, and (p,p') is conjugate in $\mathrm{PSL}(2,\mathbf{R})^2$ to

$$(p,p') \sim \left(\begin{pmatrix} N(p)^{1/2} & 0 \\ 0 & N(p)^{-1/2} \end{pmatrix}, \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix} \right).$$

Here, $N(p) > 1$, $\omega \in (0,\pi)$ and $\omega \notin \pi\mathbf{Q}$.

By the relation

$$R_K(s) = \frac{Z_K(s;2)}{Z_K(s+1;2)}$$

and Theorem 2.3, we have

Theorem 3.2. *The Ruelle type zeta function* $R_K(s)$ *has a meromorphic continuation to the whole complex plane.* $R_K(s)$ *has a double pole at* $s=1$ *and nonzero for* $\Re(s) \geq 1$.

As a byproduct of Theorem 2.5, we obtain a simple functional equation for $R_K(s)$ and an explicit formula of the coefficient of the leading term of $R_K(s)$ at $s=0$.

Theorem 3.3 (Functional equation of $R_K(s)$)**.** *Let* D *be the discriminant of* K *and* $D \geq 13$. *Then, the function* $R_K(s)$ *satisfy the functional equation*

$$\begin{aligned} &R_K(s)\,R_K(-s) \\ &\quad = (-1)^{E(X_K)}\, 2^{2E(X_K)} \sin(\pi s)^{2E(X_K)-2a_2(\Gamma)-2a_3(\Gamma)} \\ &\qquad \cdot \sin\left(\frac{\pi s}{2}\right)^{2a_2(\Gamma)} \sin\left(\frac{\pi s}{3}\right)^{2a_3(\Gamma)} \left(\frac{\zeta_\varepsilon(s-1)\,\zeta_\varepsilon(s+1)}{\zeta_\varepsilon(s)^2}\right)^2. \end{aligned}$$

Let $R_K^*(0)$ *be the leading coefficient of the Laurent expansion of* $R_K(s)$ *at* $s=0$, *that is*

$$R_K^*(0) := \lim_{s\to 0} \frac{R_K(s)}{s^{E(X_K)+2}}.$$

Then we have

$$|R_K^*(0)| = \frac{(2\pi)^{E(X_K)}}{2^{a_2(\Gamma)}\,3^{a_3(\Gamma)}} \frac{(2\varepsilon\log\varepsilon)^2}{(\varepsilon^2-1)^2}.$$

Here, $E(X_K)$ *denotes the Euler characteristic of* X_K, ε *is the fundamental unit of* K, $\zeta_\varepsilon(s) = (1-\varepsilon^{-2s})^{-1}$ *and* $a_r(\Gamma)$ *is the number of elliptic fixed points in* X_K *for which corresponding points have isotropy groups of order* r.

4. Differences of the Selberg trace formulas and class numbers of binary quadratic forms over $\boldsymbol{\mathcal{O}_K}$**.** Analytic properties and functional equations of $Z_K(s;m)$ and $R_K(s)$ are obtained by using the "differences" of the Selberg trace formula for Hilbert modular surfaces. The key point is considering the differences between two Selberg trace formulas with different weights. For this we shall extend the Selberg trace formula for Hilbert modular group Γ_K with trivial weight (Cf. Efrat [2] and Zograf [10]) to that with nontrivial weights. Based on our Selberg trace formula for Γ_K with weight $(0,m)$, we can treat and obtain the differences and double differences of the Selberg trace formula. The details of proofs are given in [4].

As an application of "Double differences of the Selberg trace formula", we obtain a prime geodesic type theorem. Let $\mathrm{P}\Gamma_{\mathrm{HE}}$ be the set of primitive hyperbolic-elliptic Γ_K-conjugacy classes of Γ_K.

Theorem 4.1 (Prime geodesic type theorem)**.** *For* $X \geq 2$, *we have*

$$\begin{aligned} \sum_{\substack{(p,p')\in \mathrm{P}\Gamma_{\mathrm{HE}} \\ N(p)\leq X}} 1 &= 2\,\mathrm{Li}(X) - \sum_{1/2<s_j(2)<1} \mathrm{Li}(X^{s_j(2)}) \\ &\quad - \sum_{1/2<s_j(-2)<1} \mathrm{Li}(X^{s_j(-2)}) \\ &\quad + O(X^{3/4}/\log X). \end{aligned}$$

Here, $s_j(2)(1-s_j(2))$ *and* $s_j(-2)(1-s_j(-2))$ *are eigenvalues of the Laplacian* $\Delta_0^{(1)}$ *acting on* $V_2^{(2)}$ *and* $W_{-2}^{(2)}$ *respectively and* $\mathrm{Li}(x) := \int_2^x 1/\log t\,dt$.

Besides, we have a generalization of Sarnak's theorem [7] on class numbers of indefinite binary quadratic forms over $\mathbf{Z}$ to that for class numbers of indefinite binary quadratic forms over $\mathcal{O}_K$. Put $\mathcal{D}_{+-} := \{d \in \mathcal{O}_K \mid \exists b \in \mathcal{O}_K \text{ s.t. } d \equiv b^2 \pmod 4,$ d nota square in $\mathcal{O}_K$, $d>0$, $d'<0\}$. For each $d \in \mathcal{D}_{+-}$, let $h_K(d)$ denote the number of inequivalent primitive binary quadratic forms over $\mathcal{O}_K$ of discriminant d, and let $(x_d, y_d) \in \mathcal{O}_K \times \mathcal{O}_K$ be the fundamental solution of the Pellian equation

$x^2 - dy^2 = 4$. Put $\varepsilon_K(d) := (x_d + \sqrt{d}\, y_d)/2$. By Theorem 4.1, we obtain

Theorem 4.2. *For $x \geq 2$, we have*

$$\sum_{\substack{d \in \mathcal{D}_{+-} \\ \varepsilon_K(d) \leq x}} h_K(d) = 2\,\mathrm{Li}(x^2) - \sum_{1/2 < s_j(2) < 1} \mathrm{Li}(x^{2s_j(2)}) - \sum_{1/2 < s_j(-2) < 1} \mathrm{Li}(x^{2s_j(-2)}) + O(x^{3/2}/\log x) \quad (x \to \infty).$$

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