Realizing a complex of unstable modules

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Abstract: In a preceding article [NST10] the authors and Tran Ngoc Nam constructed a minimal injective resolution of the mod 2 cohomology of a Thom spectrum. A Segal conjecture type theorem for this spectrum was proved. In this paper one shows that the above mentioned resolutions can be realized topologically. In fact there exists a family of cofibrations inducing short exact sequences in mod 2 cohomology. The resolutions above are obtained by splicing together these short exact sequences. Thus the injective resolutions are realizable in the best possible sense. In fact our construction appears to be in some sense an injective closure of one of Takayasu. It strongly suggests that one can construct geometrically (not only homotopically) certain dual Brown-Gitler spectra.

Key words: Unstable module; Brown-Gitler spectrum; Adams spectral sequence.

1. Introduction. In this note H^*X, resp. H_*X, will denote the mod 2 singular cohomology, resp. homology, of a space or a spectra X. All spaces or spectra will be supposed to be 2-completed and to have finite mod 2 cohomology in each degree. Let $\mathcal{A}$ be the mod 2 Steenrod algebra and let $\mathcal{U}$ be the category of unstable $\mathcal{A}$-modules.

Let L_n be the Steinberg summand in the mod 2 cohomology $H^*B(\mathbf{Z}/2)^n$, as defined by Mitchell and Priddy [MP83]. Let also $J(\ell)$ be the ℓ-th Brown-Gitler module. These are both injective unstable $\mathcal{A}$-modules as well as their tensor product $L_k \otimes J(\ell)$ following Lannes and Zarati [LZ86].

The Brown-Gitler modules are the mod 2 cohomology of spectra, even of spaces if ℓ is odd. We will denote these spectra by $\mathbf{T}(\ell)$. If ℓ is odd this is a suspension spectrum.

Let $V_n = (\mathbf{Z}/2)^n$. Let us denote by L'_n the unstable module which is obtained as follows. Let e_n be the Steinberg idempotent in $\mathbf{F}_2[\mathrm{GL}_n(\mathbf{F}_2)]$. Let $\widetilde{reg}_n\colon V_n \to O(2^n - 1)$ be the real reduced regular representation of V_n.

Let $\widetilde{reg}_n^{\oplus k} = \widetilde{reg}_n + \cdots + \widetilde{reg}_n$, be the direct sum of k copies of this representation. Let $BV_n^{\widetilde{reg}_n^{\oplus k}}$ be the Thom space associated to $\widetilde{reg}_n^{\oplus k}$, *i.e.* the Thom space of the vector bundle $EV_n \times_{V_n} \mathbf{R}^{k(2^n-1)} \to BV_n$. The Steinberg idempotent determines a map of these Thom spaces after one suspension. Let us denote [Tak99] by $\mathbf{M}(n)_k$ the Thom spectrum $e_n \cdot BV_n^{\widetilde{reg}_n^{\oplus k}}$. In particular if $k = 1$ it is the spectrum $\mathbf{L}(n)$ of Mitchell and Priddy, and we will simply denote it $\mathbf{L}'(n)$ if $k = 2$. We will denote by L_n (as usual) and by L'_n their mod 2 cohomology.

Theorem 1.1 [NST09,NST10]. *For all $n \geq 1$ there exists an acyclic complex of unstable injective $\mathcal{A}$-modules*

$$0 \to L'_n \to L_n \to L_{n-1} \otimes J(1) \to L_{n-2} \otimes J(3) \to \cdots \to L(1) \otimes J(2^{n-1} - 1) \to J(2^n - 1) \to 0.$$

This complex is a minimal injective resolution of L'_n.

Here is the first result of this article.

Theorem 1.2. *The above complexes are realizable in the sense there exists a complex of spectra*

$$\mathbf{T}(2^n - 1) \to \mathbf{L}(1) \wedge \mathbf{T}(2^{n-1} - 1) \to \cdots \to \mathbf{L}(i) \wedge \mathbf{T}(2^{n-i} - 1) \to \cdots \to \mathbf{L}(n) \to \mathbf{L}'(n)$$

so that when applying mod 2 singular cohomolgy one gets the algebraic complexes. The composite of two successive maps is homotopically trivial, moreover all Toda brackets vanish.

This follows from the following stronger result:

Theorem 1.3. *There are cofibrations of spectra, $1 \leq k \leq n$,*

$$\mathcal{C}(k-1) \to \mathbf{L}(k) \wedge \mathbf{T}(2^{n-k} - 1) \to \mathcal{C}(k)$$

so that:

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(a) *The induced sequences in mod 2 cohomology are short exact*;
(b) $\mathcal{C}(n) = \mathbf{L}'(n)$;
(c) $\mathcal{C}(0) = \mathbf{T}(2^n - 1)$.

Because of [NST09,NST10], and further applications we are focused on the case of 1.2. However 1.2 is quite general. The proof of the general case proceeds exactly as in the particular one. Thus, we will only state

Theorem 1.4. *Let M be the mod 2 cohomology of a spectrum X. Let us assume that M has an injective resolution of finite type in the category of unstable modules $\mathcal{U}$. By finite type one means that any term in the injective resolution is a finite direct sum of indecomposable modules. Then this resolution can be realized topologically by spectra, in the sense described above, as soon as the algebraic map from M to its injective hull in $\mathcal{U}$ can be realized.*

One needs the condition that each term of the resolution is a finite direct sum of unstable injective modules in order to apply Theorem 6.4 of [LS89].

This needs some more explanation. Theorem 6.4 of [LS89] is in general not true when deleting the finitness hypothesis. But one knows from [LS89] that any direct sum of injective unstable modules is injective. One could take resolutions by products of Brown-Gitler modules. The same argument works to realize them without restrictions (the cohomology of an infinite wedge being the product). But proceeding like this one will get resolutions that will be far from being minimal (in general).

It is clear that 1.3 leads to the construction of spectral sequences that we will consider either in the last section of this paper.

The above theorems are in the category of spectra, however the spectra $\mathbf{L}(i) \wedge \mathbf{T}(2^{n-i} - 1)$ are suspension spectra if $n > i$. In Theorem 1.2 only $\mathbf{L}(n)$ and $\mathbf{L}'(n)$ are not. They are only direct summand of a suspension spectrum (after one suspension). It is thus reasonable to ask whether or not the complex of spectra can be realized as a complex of spaces (in an appropriate sense) after one suspension. It looks necessary to restict to space-like spectra. The technology to deal with this question is given by the work of P. Goerss, J. Lannes and F. Morel [GLM92]. However, even if some preliminary computations are positive, this is extremely technical, and we have not been able up to now to get significant results. Thus we only propose

Conjecture 1.5. *The cofibrations in* 1.2 *can be realized as cofibrations of spaces after one suspension.*

2. Spectra satisfying the Brown-Gitler property and related spectra. In this section we recollect some facts from [HK00]. Most (but not all) of the material in this paper comes from earlier work, nicely collected here. Moreover we will need an original result of this paper. We shall say that a spectrum is *space-like* if it is a wedge summand of a suspension spectrum. For such a spectrum S the evaluation map $\Sigma^\infty\Omega^\infty S \to S$ has a section and induces a monomorphism in cohomology.

Let now S be a spectrum whose mod 2 singular cohomology is an injective unstable $\mathcal{A}$-module. Following N. Kuhn we shall say that S has the Brown-Gitler property if the natural map

$$f \mapsto f^*, \quad [S, X] \to \mathrm{Hom}_{\mathcal{U}}(H^*X, H^*S)$$

from homotopy classes of maps from S to X to the set of maps of unstable modules $\mathrm{Hom}_{\mathcal{U}}(H^*X, H^*S)$ is onto for all space-like spectrum X.

Theorem 2.1 [HK00]**.** *Let S be a spectrum whose mod 2 cohomology is an injective unstable $\mathcal{A}$-module. Then S satisfies the Brown-Gitler property if and only if either of the following condition holds*:
(a) *The evaluation map induces a monomorphism $H^*S \to H^*\Sigma^\infty\Omega^\infty S$ in cohomology*;
(b) *The spectrum is space-like*;
(c) *There exists a space-like spectrum Z and a map $f : S \to Z$ which induces an epimorphism in cohomology.*

It is clear that a spectrum satisfying the Brown-Gitler property is uniquely defined by its cohomology.

Recall that an unstable module M is reduced, if either it does not contain a non-trivial suspension, or if the map from M into itself, $x \in M^n$, $\mathrm{Sq}_0 \colon x \mapsto \mathrm{Sq}^n x$ is injective.

Below are the main examples of interest for us of such spectra.

2.1. The Mitchell-Priddy spectra. The standard $\mathrm{GL}_n(\mathbf{F}_2)$ action on V_n induces one on BV_n, on the direct sum of all non-trivial line bundles over BV_n, and on the direct sum of k copies of this vector bundle. Thus it acts also on their Thom spaces $BV_n^{\widetilde{reg_n}^{\oplus k}}$. As a consequence the Steinberg idempotent e_n [Ste56] induces a stable map on $BV_n^{\widetilde{reg_n}^{\oplus k}}$, this map is defined after one suspension. Consider the telescope spectrum of this map [Tak99], this is a stable summand of $BV_n^{\widetilde{reg}_n^{\oplus k}}$, denoted by $e_n \cdot BV_n^{\widetilde{reg}_n^{\oplus k}}$. Following Takayasu we will shorten $e_n \cdot BV_n^{\widetilde{reg}_n^{\oplus k}}$ to $\mathbf{M}(n)_k$.

In particular $\mathbf{M}(n)_0 = \mathbf{M}(n)$, $\mathbf{M}(n)_1 = \mathbf{L}(n)$ are the spectra of Michell and Priddy and by definition $\mathbf{M}(n)_2 = \mathbf{L}'(n)$. The spectra $\mathbf{M}(n)$ and $\mathbf{L}(n)$ satisfy the Brown-Gitler property.

The cohomology of these spectra follows from the Thom isomorphism theorem

$$H^*\mathbf{M}(n)_k = \omega_n^k e_n \mathbf{F}_2[x_1, \cdots, x_n] = \omega_n^{k-1} L_n$$

where ω_n is the Euler class of the reduced regular representation $\widetilde{reg}_n$, or the top Dickson invariant of $\mathrm{GL}_n(\mathbf{F}_2)$: the product of all non-trivial linear forms in $\mathbf{F}_2[x_1, \cdots, x_n]$.

Theorem 2.2 (Takayasu [Tak99]). *For $k \geq 0$, there exist a cofibration sequence*

$$\Sigma^k \mathbf{M}(n-1)_{2k+1} \xrightarrow{i_{n,k}} \mathbf{M}(n)_k \xrightarrow{j_{n,k}} \mathbf{M}(n)_{k+1}.$$

This induces the following short exact sequence in cohomology

$$0 \to \omega_n^k L_n \to \omega_n^{k-1} L_n \to \Sigma^k \omega_{n-1}^{2k} L_{n-1} \to 0.$$

Splicing the cofibration sequences together yields a complex of spectra

$$\Sigma^{2^n-1}\mathbf{M}(0)_{2^n} \to \cdots \to \Sigma^{2^k-1}\mathbf{M}(n-k)_{2^k} \to \cdots \to \Sigma\mathbf{M}(n-1)_2 \to \mathbf{M}(n)_1 \to \mathbf{M}(n)_2.$$

2.2. The Brown-Gitler spectra. Let $J(\ell)$ be the ℓ-th Brown-Gitler module (cf. [Sch94, Chapter 2]). It is characterised by the fact that there is a natural equivalence $\mathrm{Hom}_{\mathcal{U}}(M, J(\ell)) \cong M^{\ell *}$. The unstable module $J(\ell)$ is the injective hull of $\Sigma^\ell \mathbf{F}_2$ in the category $\mathcal{U}$. It is known that there exist spectra $\mathbf{T}(\ell)$ with mod 2 singular cohomology $J(\ell)$ [GLM92]. This was originally proved by E. Brown et S. Gitler. If ℓ is odd $\mathbf{T}(\ell)$ is a suspension spectrum. More precisely we have

Theorem 2.3 [GLM92]. *Let $n \geq 2$. There exists a 1-connected pointed space $\mathbf{T}_1(n)$ (that we are going to call the n-th Brown-Gitler space), unique up to homotopy, such that the following holds*

(a) *There is an isomorphism of unstable modules $H^*\mathbf{T}_1(n) \cong \Sigma J(n)$.*
(b) *The induced map in cohomology by the evaluation $\Sigma\Omega\mathbf{T}_1(n) \to \mathbf{T}_1(n)$ is a monomorphism.*
(c) *$\mathbf{T}_1(n)$ is 2-complete.*

This space has moreover the following properties

(d) *It is retract of the suspension of a pointed space.*
(e) *For all pointed spaces Y the natural map*
$$[\mathbf{T}_1(n), \Sigma Y]_* \to \mathrm{Hom}_{\mathcal{U}}(\tilde{H}^*(\Sigma Y), \Sigma J(n))$$
is surjective.

Thus the suspension spectrum of $\mathbf{T}_1(n)$ is $\Sigma\mathbf{T}(n)$, and if n is odd $\mathbf{T}(n)$ is the suspension spectrum of the space $\mathbf{T}_1(n-1)$.

These were the first known spectra to have the two first conditions of Theorem 2.1.

2.3. The Lannes-Zarati spectra.

Theorem 2.4 [LZ86]. *The unstable module $L_n \otimes J(k)$ is an injective object in $\mathcal{U}$.*

This is just a particular case of the Lannes-Zarati theorem. This module is the reduced mod 2 cohomology of $\mathbf{L}(n) \wedge \mathbf{T}(k)$. If k is odd this is the suspension spectrum of $\mathbf{L}(n) \wedge \mathbf{T}_1(k-1)$. This is because $\mathbf{T}_1(k-1)$ has a co-H-space structure.

In general, if e_λ is an idempotent of the semigroup ring $\mathbf{F}_2[\mathrm{End}(V_n)]$, we denote by $L(\lambda)$ the stable summand of the suspention spectrum of BV_n^+ (the disjoint union of the classifying space BV_n with a base point). A Lannes-Zarati spectrum is a spectrum of the form $L(\lambda) \wedge J(k)$.

3. Proof of theorem 1.3. Let us recall some notations. The morphisms

$$f_{n-k,n}: L_{k+1} \otimes J(2^{n-k-1} - 1) \to L_k \otimes J(2^{n-k} - 1)$$

of the complex of Theorem 1.1 are defined in [NST10,NST09]. Let us denote by C_k the image of the map $f_{n-k,n}$, which is also the kernel of $f_{n-k+1,n}$. If the spectra $\mathcal{C}(k)$ exist one has $H^*\mathcal{C}(k) \cong C_k$.

We are now going to proceed by descending induction. The first step of the proof is done by observing that the cofibration $\mathcal{C}(n-1) \to \mathbf{L}(n) \to \mathcal{C}(n)$ is nothing else than Takayasu's cofibration

$$\Sigma\mathbf{M}(n-1)_3 \to \mathbf{M}(n)_1 \to \mathbf{M}(n)_2.$$

However the right map is just the zero section, and the left hand spectrum could be defined as above.

We will assume that we have got a spectrum $\mathcal{C}(k-1)$, $2 \leq k \leq n$, which is the fibre of a map from $\mathbf{L}(k) \wedge \mathbf{T}(2^{n-k} - 1)$ to $\mathcal{C}(k)$ and short exact sequences in cohomology

$$H^*(\mathcal{C}(k)) \to H^*(\mathbf{L}(k) \wedge \mathbf{T}(2^{n-k} - 1)) \to H^*(\mathcal{C}(k-1)).$$

We will make use of the following

Proposition 3.1 [HK00]. *Let X be a 0-connected spectrum such that H_*X is of finite type and let Z be a space. Suppose there exists a map $f: X \to$*

$\Sigma^\infty Z$ which is an epimorphism in cohomology. Then the evaluation map $\Sigma^\infty\Omega^\infty X \to X$ induces a monomorphism in cohomology.

One can apply this proposition to the spectrum $\mathcal{C}(k-1)$ and the suspension spectrum $\Sigma^\infty Z$ of which $\mathbf{L}(k) \wedge \mathbf{T}(2^{n-k}-1)$ is a wedge summand. Thus the evaluation map $\Sigma^\infty\Omega^\infty\mathcal{C}(k-1) \to \mathcal{C}(k-1)$ induces a monomorphism in cohomology

$$C_{k-1} \cong H^*\mathcal{C}(k-1) \to H^*\Omega^\infty\mathcal{C}(k-1).$$

As a consequence, as $L_k \otimes J(2^{n-k}-1)$ is injective in $\mathcal{U}$, the canonical map $C_{k-1} \to L_{k-1} \otimes J(2^{n-k+1}-1)$ extends to a map of unstable modules

$$H^*\Omega^\infty\mathcal{C}(k-1) \to L_{k-1} \otimes J(2^{n-k+1}-1).$$

It follows from 2.1 that this map is induced by a map of spectra

$$\gamma\colon \mathbf{L}(k-1) \wedge \mathbf{T}(2^{n-k+1}-1) \to \Sigma^\infty\Omega^\infty\mathcal{C}(k-1).$$

Composing γ with the evaluation $\epsilon\colon \Sigma^\infty\Omega^\infty\mathcal{C}(k-1) \to \mathcal{C}(k-1)$ one gets a map

$$\epsilon\gamma\colon \mathbf{L}(k-1) \wedge \mathbf{T}(2^{n-k+1}-1) \to \mathcal{C}(k-1)$$

inducing in cohomology the canonical map $C_{k-1} \to L_{k-1} \otimes J(2^{n-k+1}-1)$. One takes the fibre $\mathcal{C}(k-2)$ of the map $\epsilon\gamma$ to obtain a cofibration of spectra which induces a short exact sequence in cohomology.

The final step of the induction deserves care. One needs to show that the spectrum $\mathcal{C}(0)$ is indeed equivalent to $\mathbf{T}(2^n-1)$, *a priori* it has the right cohomology. By construction there is a map $\mathcal{C}(0) \to \mathbf{L}(1) \wedge \mathbf{T}(2^{n-1}-1)$ which induces an epimorphism in cohomology. As $\mathbf{L}(1) \wedge \mathbf{T}(2^{n-1}-1)$ is space-like, it follows from 2.1 that $\mathcal{C}(0)$ satisfies the Brown-Gitler property, thus $\mathcal{C}(0) \simeq \mathbf{T}(2^n-1)$.

Remark 3.2.

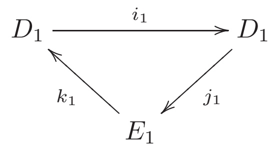
(a) One has a fibration of spaces: $\Omega^\infty\mathcal{C}(k-1) \to \Omega^\infty\mathbf{L}(k-1) \wedge \mathbf{T}(2^{n-k+1}-1) \to \Omega^\infty\mathcal{C}(k)$. This is even a fibration of H-spaces which allows us to compute $H^*\Omega^\infty\mathcal{C}(k-1)$. In fact the Eilenberg-Moore spectral sequence shows that: $H^*\Omega^\infty\mathcal{C}(k-1) \cong H^*(\Omega^\infty\mathbf{L}(k-1) \wedge \mathbf{T}(2^{n-k+1}-1)) \otimes_{H^*\Omega^\infty\mathcal{C}(k)} \mathbf{F}_2$. However we don't need this computation in the above proof, thus we don't describe (in terms of Dyer-Lashof operations) the result further.

(b) The final step of the proof suggests it is possible to describe the space $\mathbf{T}_1(2^n-2)$ as a sub-complex of $(B\mathbf{Z}/2)^{\wedge n}$. We hope to be able to say more about that later.

4. A dual Adams spectral sequence. In this section we construct a spectral sequence which arises from the topological realisation of an unstable injective resolution.

Let $X = X_0$ be a spectrum with $\mathcal{U}$-injective cohomology. Suppose that H^*X admits an injective resolution $0 \to H^*X \to I^0 \to I^1 \to \cdots$, where, for each $j \geq 0$, the unstable module I^j is the mod 2 cohomology of a finite wedge of Lannes-Zarati spectra. If the inclusion of unstable modules $H^*X \hookrightarrow I^0$ is realised by a map of spectra $S_0 \to X_0$, then the above injective resolution is realised in the sense that, for each $j \geq 0$, there exists a cofibration of spectra $X_{j+1} \to S_j \to X_j$ which induces a short exact sequence in mod 2 cohomology. The spectrum S_j, whose cohomology is the unstable module I^j, is a finite wedge of Lannes-Zarati spectra.

Now let Y be a spectrum. Applying the functor $[-, Y]$, the homotopy classes of maps to Y, to the cofibrations $X_{j+1} \to S_j \to X_j$, $j \geq 0$, gives rise to an exact couple



where $E_1^{s,t} = [S_s, \Sigma^t Y]$ and $D_1^{s,t} = [X_s, \Sigma^t Y]$. The map i_1 is of bidegree $(-1,1)$, j_1 of bidegree $(0,0)$ and k_1 of bidegree $(1,0)$. These data determine a spectral sequence

$$E_1^{s,t} = [S_s, \Sigma^t Y] \Longrightarrow [X, \Sigma^{s+t} Y],$$

with the differential in the r-th page is of bidegree $(r, 1-r)$. The spectral sequence converges if, for example, the injective resolution above is of finite length.

We now consider the first page of this spectral sequence. It is enough to consider the computation of $[S_*, \Sigma^* Y]$ where S_* is a Lannes-Zarati spectrum. Suppose that $S_* = L(\lambda) \otimes T(k)$ with $L(\lambda)$ a stable summand of (the suspension spectrum of) some BV_n^+ associated to an idempotent e_λ of the semi-group ring $\mathbf{F}_2[\mathrm{End}(V_n)]$.

We have

$$[L(\lambda) \otimes T(k), \Sigma^t Y] = e_\lambda[BV_n^+ \wedge T(k), \Sigma^t Y].$$

Here, given spectra X and Y, $[X, Y]$ denotes the homotopy classes of maps from X to Y. If X and Y are pointed spaces, we denote by $\mathrm{Map}_*(X, Y)$ the space of pointed maps from X to Y and by $[X, Y]_*$ the homotopy classes of pointed maps from X to Y.

We consider the following cases.

1) *k is odd.* The spectrum $T(k)$ is then the suspension spectrum of the space $T_1(k-1)$. It follows

$$[BV_n^+ \wedge T(k), \Sigma^t Y] \cong [BV_n^+ \wedge T_1(k-1), \Omega^\infty \Sigma^t Y]_* \cong [T_1(k-1), \mathrm{Map}_*(BV_n^+, \Omega^\infty \Sigma^t Y)]_*.$$

Following [GLM92], we have

$$[T_1(k-1), \mathrm{Map}_*(BV_n^+, \Omega^\infty \Sigma^t Y)]_* \cong \mathrm{Hom}_{\mathcal{H}_*^{\mathbf{F}_2}}(W^{\mathbf{F}_2}(k-1), H_*\Omega_0 \mathrm{Map}_*(BV_n^+, \Omega^\infty \Sigma^t Y)).$$

Here $\mathcal{H}_*^{\mathbf{F}_2}$ denotes the category of graded cocommutative connected Hopf algebras over $\mathbf{F}_2$ and $W^{\mathbf{F}_2}(\ell)$ denotes the projective cover in $\mathcal{H}_*^{\mathbf{F}_2}$ of the tensor algebra generated by a primitive element of degree ℓ.

It is easy to verify that we have $\Omega_0 \mathrm{Map}_*(BV_n^+, \Omega^\infty \Sigma^t Y) \simeq \mathrm{Map}(BV_n, \Omega_0^\infty \Sigma^{t-1} Y)$ with $\mathrm{Map}(X, Y)$ denoting the space of maps from the unpointed space X to Y. From the above isomorphisms we get

$$[L(\lambda) \otimes T(k), \Sigma^t Y] \cong e_\lambda \mathrm{Hom}_{\mathcal{H}_*^{\mathbf{F}_2}}(W^{\mathbf{F}_2}(k-1), H_* \mathrm{Map}(BV_n, \Omega_0^\infty \Sigma^{t-1} Y)).$$

2) *k is even.* The spectrum $\Sigma T(k)$ is then the suspension spectrum of the space $T_1(k)$. We use the same arguments as above to obtain the isomorphism

$$[L(\lambda) \otimes T(k), \Sigma^t Y] \cong e_\lambda \mathrm{Hom}_{\mathcal{H}_*^{\mathbf{F}_2}}(W^{\mathbf{F}_2}(k), H_* \mathrm{Map}(BV_n, \Omega_0^\infty \Sigma^t Y)).$$

To finish let us point out that this spectral sequence gives back very quickly, using the Dyer-Lashof algebra, the computations of $[\mathbf{L}'(n), \mathbf{S}^k]$ done in [NST10] in dimension $k \geq n + 2^{n-1}$, in lower dimensions differentials appear.

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