Multi-specialization and multi-asymptotic expansions

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Abstract: We extend the notion of specialization functor to the case of several closed submanifolds satisfying some suitable conditions. Applying this functor to the sheaf of Whitney holomorphic functions we construct different kinds of sheaves of multi-asymptotically developable functions, whose definitions are natural extensions of the definition of strongly asymptotically developable functions introduced by Majima.

Key words: Normal deformation; asymptotic expansions; sheaves on subanalytic sites.

1. Multi-normal deformation. Let X be a real analytic manifold with $\dim X = n$, and let $\chi = \{M_1, \dots, M_\ell\}$ be a family of closed real analytic submanifolds in X ($\ell \geq 1$). We set, for $N \in \chi$ and $p \in N$,

$$\mathrm{NR}_p(N) := \{M_j \in \chi;\, p \in M_j,\, N \not\subseteq M_j \text{ and } M_j \not\subseteq N\}.$$

Let us consider the following conditions for χ.

H1 Each $M_j \in \chi$ is connected and the submanifolds are mutually distinct, i.e. $M_j \neq M_{j'}$ for $j \neq j'$.

H2 For any $N \in \chi$ and $p \in N$ with $\mathrm{NR}_p(N) \neq \emptyset$, we have

$$\left(\bigcap_{M_j \in \mathrm{NR}_p(N)} T_p M_j \right) + T_p N = T_p X. \tag{1.1}$$

Note that, if χ satisfies the condition H2, the configuration of two submanifolds must be either (a) or (b) below.

(a) Both submanifolds intersect transversely.

(b) One of them contains the other.

To better understand the situation let us give some examples. Let $X = \mathbf{R}^2$ with coordinates (x_1, x_2).

(i) Let $\chi = \{M_1, M_2\}$ with $M_i = \{x_i = 0\}$, $i = 1, 2$. Then clearly χ satisfies H1 and $T_0M_1 + T_0M_2 = T_0X$. Hence χ satisfies H2.

(ii) Let $\chi = \{M_1, M_2\}$ with $M_1 = \{x_1 = x_2 = 0\}$, $M_2 = \{x_2 = 0\}$. Then clearly χ satisfies H1 and $NR_0(M_i) = \emptyset$ for $i = 1, 2$. Hence χ satisfies H2.

(iii) Let $\chi = \{M_1, M_2, M_3\}$ with $M_1 = \{x_1 = x_2 = 0\}$, $M_2 = \{x_1 = 0\}$, $M_3 = \{x_2 = 0\}$. Then clearly χ satisfies H1. We have $NR_0(M_1) = \emptyset$ and $T_0M_2 + T_0M_3 = T_0X$. Hence χ satisfies H2.

(iv) Let $\chi = \{M_1, M_2, M_3\}$ with $M_1 = \{x_1 = x_2\}$, $M_2 = \{x_1 = 0\}$, $M_3 = \{x_2 = 0\}$. Then clearly χ satisfies H1. We have $\bigcap_{i \neq j} T_0M_j = \{0\}$ for $i = 1, 2, 3$. Then χ does not satisfy H2.

(v) Let $\chi = \{M_1, M_2, M_3\}$ with $M_1 = \{x_1 = x_2^2\}$, $M_2 = \{x_1 = 0\}$, $M_3 = \{x_2 = 0\}$. Then clearly χ satisfies H1. We have $\bigcap_{1 \neq j} T_0M_j = \{0\}$. Then χ does not satisfy H2 (even if $\bigcap_{3 \neq j} T_0M_j + T_0M_3 = T_0M_2 + T_0M_3 = T_0X$).

We set, for $N \in \chi$, $\iota_\chi(N) = X$ if there exists no $M_j \in \chi$ with $N \subsetneq M_j$ and $\iota_\chi(N) = \bigcap_{N \subsetneq M_j} M_j$ otherwise. The set $\iota_\chi(N)$ is often denoted by $\iota(N)$ for simplicity. We assume for simplicity that $M_j \neq \iota(M_j)$ for any $j \in \{1, 2, \dots, \ell\}$.

We define recursively the multi-normal deformation of X. When $\sharp\chi = 1$ it is the classical normal deformation of [5]: it is given by an analytic manifold $\widetilde{X}_{M_1}$, an application $(p_{M_1}, t) : \widetilde{X}_{M_1} \to X \times \mathbf{R}$, and an action of $\mathbf{R} \setminus \{0\}$ on $\widetilde{X}_{M_1}$ $(\widetilde{x}, r) \mapsto \widetilde{x} \cdot r$ such that $p_{M_1}^{-1}(X \setminus M_1) \simeq (X \setminus M_1) \times (\mathbf{R} \setminus \{0\})$, $t^{-1}(c) \simeq X$ for each $c \neq 0$ and $t^{-1}(0) \simeq T_{M_1}X$.

Set $\widetilde{\Omega}_{M_1} = \{(x, t_1);\, t_1 \neq 0\}$ and define $\widetilde{M}_2 := \overline{(p_{M_1}|_{\widetilde{\Omega}_{M_1}})^{-1}(M_2)}$. Then $\widetilde{M}_2$ is a closed smooth submanifold of $\widetilde{X}_{M_1}$. Now we can define the normal deformation along M_1, M_2 as $\widetilde{X}_{M_1, M_2} := (\widetilde{X}_{M_1})^{\sim}_{\widetilde{M}_2}$. Then we can define recursively the normal deformation along χ as

$$\widetilde{X} = \widetilde{X}_{M_1, \dots, M_\ell} := (\widetilde{X}_{M_1, \dots, M_{\ell-1}})^{\sim}_{\widetilde{M}_\ell}.$$

Set $S = \{t_1, \dots, t_\ell = 0\}$, $M = \bigcap_{i=1}^{\ell} M_i$ and $\Omega = \{t_1, \dots, t_\ell > 0\}$. Then we have the commutative diagram

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$$(1.2)\qquad \begin{tikzcd} S \arrow[r, "s"] \arrow[d, "\tau"] & \widetilde{X} \arrow[d, "p"] & \Omega \arrow[l, "i_\Omega"'] \arrow[dl, "\widetilde{p}"] \\ M \arrow[r, "i_M"] & X. \end{tikzcd}$$

To better understand the situation, let us choose local coordinates $(x_1,\dots,x_n)$ and let $I_1,\dots I_\ell \subseteq \{1,\dots,n\}$ be subsets such that $M_i = \{x_k = 0\,;\, k \in I_i\}$. Set $J_i = \{k \in \{1,\dots,\ell\}\,;\, i \in I_k\}$, $t_{J_i} = \prod_{k\in J_i} t_k$, where $t_1,\dots,t_\ell \in \mathbf{R}$ and $t_{J_i} = 1$ if $J_i = \emptyset$. Then $p : \widetilde{X} \to X$ is defined by

$$(x_1,\dots,x_n,t_1,\dots,t_\ell) \mapsto (t_{J_1}x_1,\dots,t_{J_n}x_n).$$

The zero section $S = \{t_1 = \ldots = t_\ell = 0\} \subset \widetilde{X}$ is isomorphic to

$$\underset{X,1\le j\le\ell}{\times} T_{M_j}\iota(M_j) := T_{M_1}\iota(M_1)\underset{X}{\times}\cdots\underset{X}{\times}T_{M_\ell}\iota(M_\ell).$$

Let $q \in \bigcap_{1\le j\le\ell} M_j$ and $p_j = (q;\, \xi_j)$ be a point in $T_{M_j}\iota(M_j)$ $(j = 1,2,\dots,\ell)$. We set $p = p_1 \underset{X}{\times}\dots\underset{X}{\times} p_\ell \in \underset{X,1\le j\le\ell}{\times} T_{M_j}\iota(M_j)$, and $\tilde{p}_j = (q;\, \tilde{\xi}_j) \in T_{M_j}X$ denotes the image of the point p_j by the canonical embedding $T_{M_j}\iota(M_j) \hookrightarrow T_{M_j}X$. We denote by $\mathrm{Cone}_{\chi,j}(p)$ $(j = 1,2,\dots,\ell)$ the set of open conic cones in $(T_{M_j}X)_q \simeq \mathbf{R}^{n-\dim M_j}$ that contain the point $\tilde{\xi}_j \in (T_{M_j}X)_q \simeq \mathbf{R}^{n-\dim M_j}$.

Definition 1.1. We say that an open set $G \subset (TX)_q$ is a multi-cone along χ with direction to $p \in \left(\underset{X,1\le j\le\ell}{\times} T_{M_j}\iota(M_j)\right)_q$ if G is written in the form

$$G = \bigcap_{1\le j\le\ell} \pi_{j,q}^{-1}(G_j) \quad G_j \in \mathrm{Cone}_{\chi,j}(p)$$

where $\pi_{j,q} : (TX)_q \to (T_{M_j}X)_q$ is the canonical projection. We denote by $\mathrm{Cone}_\chi(p)$ the set of multi-cones along χ with direction to p.

Definition 1.2. Let Z be a subset of X. The multi-normal cone to Z along χ is the set $C_\chi(Z) = \overline{\widetilde{p}^{-1}(Z)} \cap S$.

For any $q \in X$, there exists an isomorphism $\psi : X \simeq (TX)_q$ near q and $\psi(q) = (q;\, 0)$ that satisfies $\psi(M_j) = (TM_j)_q$ for any $j = 1,\dots,\ell$. Let Z be a subset of X. We have the following equivalence: $p \notin C_\chi(Z)$ if and only if there exist an open subset $\psi(q) \in U \subset (TX)_q$ and a multi-cone $G \in \mathrm{Cone}_\chi(p)$ such that $\psi(Z) \cap G \cap U = \emptyset$ holds.

This definition is also compatible with the restriction to a subfamily of χ. Namely, let $k \le \ell$ and $\{j_1,\dots,j_k\}$ be a subset of $\{1,2,\dots,\ell\}$. Set $\chi_k = \{M_{j_1},\dots,M_{j_k}\}$ and $\underset{M,\chi_k}{\times} T_{M_{j_i}}\iota_\chi(M_{j_i}) := T_{M_{j_1}}\iota_\chi(M_{j_1}) \underset{X}{\times} \cdots \underset{X}{\times} T_{M_{j_k}}\iota_\chi(M_{j_k}) \underset{X}{\times} M$. Let Z be a subset of X. Then we have

$$C_\chi(Z) \cap \underset{M,\chi_k}{\times} T_{M_{j_i}}\iota_\chi(M_{j_i}) = C_{\chi_k}(Z) \cap \underset{M,\chi_k}{\times} T_{M_{j_i}}\iota_\chi(M_{j_i}).$$

In the following we will denote with the same symbol $C_{\chi_k}(Z)$ the normal cone with respect to χ_k and its inverse image via the map $\widetilde{X} \to \widetilde{X}_{M_{j_1},\dots,M_{j_k}}$.

2. Multi-specialization. Let k be a field and denote by $\mathrm{Mod}(k_{X_{sa}})$ (resp. $D^b(k_{X_{sa}})$) the category (resp. bounded derived category) of sheaves on the subanalytic site X_{sa}. For the theory of sheaves on subanalytic sites we refer to [6,8]. For the classical construction of the specialization we refer to [5].

Definition 2.1. The multi-specialization along χ is the functor

$$\nu_\chi^{sa} : D^b(k_{X_{sa}}) \to D^b(k_{S_{sa}}),$$
$$F \mapsto s^{-1}\mathrm{R}\Gamma_\Omega p^{-1}F.$$

We can give a description of the sections of the multi-specialization of $F \in D^b(k_{X_{sa}})$: let V be a conic subanalytic open subset of S. Then:

$$H^j(V; \nu_M^{sa}F) \simeq \varinjlim_U H^j(U;F),$$

where U ranges through the family of open subanalytic subsets of X such that $C_\chi(X \setminus U) \cap V = \emptyset$. Let $p = (q,\xi) \in \underset{X,1\le j\le\ell}{\times} T_{M_j}\iota(M_j)$, let $B_\epsilon \subset (TX)_q$ be an open ball of radius $\epsilon > 0$ with its center at the origin and set

$$\mathrm{Cone}_\chi(p,\,\epsilon) := \{G \cap B_\epsilon;\, G \in \mathrm{Cone}_\chi(p)\}.$$

Applying the functor $\rho^{-1} : D^b(k_{S_{sa}}) \to D^b(k_S)$ (see [8] for details) we can calculate the fibers at $p \in \underset{X,1\le j\le\ell}{\times} T_{M_j}\iota(M_j)$ which are given by

$$(\rho^{-1}H^j\nu_\chi^{sa}F)_p \simeq \varinjlim_W H^j(W;F),$$

where W ranges through the family $\mathrm{Cone}_\chi(p,\,\epsilon)$ for $\epsilon > 0$.

Let $f : X \to Y$ be a morphism of real analytic manifolds, $\chi^M = \{M_1,\dots,M_\ell\}$, $\chi^N = \{N_1,\dots,N_\ell\}$ two families of closed analytic submanifolds of X and Y respectively satisfying the conditions H1 and H2. Suppose that $f(M_i) \subseteq N_i$, $i = 1,\dots,\ell$. Then f naturally extends to a morphism $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ of multi-normal deformations. We call $T_\chi f$ the induced map on the zero section. Let $J \subseteq \{1,\dots,\ell\}$ and set $\chi_J = \{M_{j_1},\dots,M_{j_k}\}$, $j_k \in J$. The map $T_{\chi_J}f$ denotes the restriction of $\widetilde{f}$ to $\{t_{j_k} = 0,\, j_k \in J\}$.

Proposition 2.2. *Let* $F \in D^b(k_{X_{sa}})$, $G \in D^b(k_{Y_{sa}})$.

(i) *There exists a commutative diagram of canonical morphisms*

$$\begin{array}{ccc} R(T_\chi f)_{!!}\nu^{sa}_{\chi^M} F & \longrightarrow & \nu^{sa}_{\chi^N} Rf_{!!}F \\ \downarrow & & \downarrow \\ R(T_\chi f)_* \nu^{sa}_{\chi^M} F & \longleftarrow & \nu^{sa}_{\chi^N} Rf_* F. \end{array}$$

Moreover if $f : \mathrm{supp}F \to Y$ *and* $T_{\chi_J} f : C_{\chi_J}(\mathrm{supp}F) \to \{t_{j_1} = \cdots = t_{j_k} = 0\}$ *for each* $J = \{j_1, \ldots, j_k\}$ *are proper, and if* $\mathrm{supp}F \cap f^{-1}(N_j) \subseteq M_j$, $j \in \{1, \ldots, \ell\}$, *then the above morphisms are isomorphisms.*

(ii) *There exists a commutative diagram of canonical morphisms*

$$\begin{array}{ccc} \omega_{S_X/S_Y} \otimes (T_\chi f)^{-1}\nu^{sa}_{\chi^N} G & \longrightarrow & \nu^{sa}_{\chi^M}(\omega_{X/Y} \otimes f^{-1}G) \\ \downarrow & & \downarrow \\ T_\chi f^! \nu^{sa}_{\chi^N} G & \longleftarrow & \nu^{sa}_{\chi^M} f^! G. \end{array}$$

The above morphisms are isomorphisms on the open sets where $T_{\chi_J} f$ *is smooth for each* $J \subseteq \{1, \ldots, \ell\}$.

3. Multi-specialization and Majima's expansions. Let $X = \mathbf{C}^n$ with coordinates $(z_1, z_2 \ldots, z_n)$ and let $Z_j = \{z_j = 0\}$, $j = 1, \ldots, m$, $m \leq n$. We set $Z = \bigcup Z_j = \{z_1 z_2 \cdots z_m = 0\}$.

As usual, we denote by $\mathcal{A}_X$, $\mathcal{A}_X^{<0}$, $\mathcal{A}_X^{CF}$ the sheaves of strongly asymptotically developable holomorphic functions, flat asymptotic functions and consistent families of coefficients respectively of [7]. Let $\mathcal{O}_X^{\mathrm{w}}$, $\mathcal{O}_{X|X\setminus Z}^{\mathrm{w}}$, $\mathcal{O}_{X|Z}^{\mathrm{w}}$ denote the sheaves on the subanalytic site X_{sa} of Whitney holomorphic functions, flat Whitney holomorphic functions and Whitney holomorphic functions on Z respectively of [6]. Using the fact that (see [2,3]) strongly asymptotically developable holomorphic functions on a polysector S are Whitney functions on each polysector properly contained in S and the stalk formula we have the isomorphisms (outside the zero section)

$$\mathcal{A}_X \xrightarrow{\sim} \rho^{-1}\nu^{sa}_\chi \mathcal{O}_X^{\mathrm{w}},$$
$$\mathcal{A}_X^{<0} \xrightarrow{\sim} \rho^{-1}\nu^{sa}_\chi \mathcal{O}_{X|X\setminus Z}^{\mathrm{w}},$$
$$\mathcal{A}_X^{CF} \xrightarrow{\sim} \rho^{-1}\nu^{sa}_\chi \mathcal{O}_{X|Z}^{\mathrm{w}}.$$

4. Sheaves of multi-asymptotically developable functions and their relations. If we apply multi-specialization functor with respect to an arbitrary family χ of closed complex submanifolds satisfying the conditions H1 and H2 to Whitney holomorphic functions, we can obtain not only the sheaf of strongly asymptotically developable functions but also several kinds of sheaves of multi-asymptotically developable functions.

Theorem 4.1. *Let X be a complex manifold, and let $\chi = \{Z_1, \ldots, Z_l\}$ be a family of closed complex submanifolds satisfying the conditions $H1$ and $H2$. Then, outside the zero section, the complexes* $\rho^{-1}\nu^{sa}_\chi \mathcal{O}_X^{\mathrm{w}}$, $\rho^{-1}\nu^{sa}_\chi \mathcal{O}_{X|X\setminus Z}^{\mathrm{w}}$ *and* $\rho^{-1}\nu^{sa}_\chi \mathcal{O}_{X|Z}^{\mathrm{w}}$ *are concentrated in degree zero, and we have the Borel-Ritt exact sequence for multi-asymptotic expansions*

$$0 \to \mathcal{A}_X^{<0} \to \mathcal{A}_X \to \mathcal{A}_X^{CF} \to 0.$$

These sheaves $\mathcal{A}_X$, $\mathcal{A}_X^{<0}$ and $\mathcal{A}_X^{CF}$ can be also defined in a similar way to that of strongly asymptotically developable functions in [7]. We explain these multi-asymptotic expansions with examples, for the precise definitions we refer to [4]. Let us see two typical examples of multi-normal deformations in $\mathbf{C}^2$ with coordinates (z_1, z_2).

(Majima). Let $\chi = \{Z_1, Z_2\}$ with $Z_1 = \{z_1 = 0\}$ and $Z_2 = \{z_2 = 0\}$. We have $I_1 = \{1\}$, $I_2 = \{2\}$, $J_1 = \{1\}$, $J_2 = \{2\}$. The map $p : \widetilde{X} \to X$ is defined by $(z_1, z_2, t_1, t_2) \mapsto (t_1 z_1, t_2 z_2)$. We have $\iota(Z_1) = \iota(Z_2) = X$ and then the zero section of $\widetilde{X}$ is isomorphic to $T_{Z_1}X \underset{X}{\times} T_{Z_2}X$. Let $p = (0, 0;\, 1, 1)$, so $q = \pi(p) = 0$. Then a cofinal set of $\mathrm{Cone}_\chi(p, \epsilon)$ is nothing but the set of polysectors $S_1 \times S_2$ (S_1, S_2 sectors in $\mathbf{C} \simeq T_{Z_i}X$, $i = 1, 2$) along $Z_1 \cup Z_2$ with their direction to $(1, 1)$.

For $N = (n_1, n_2) \in \mathbf{N}^2$, an asymptotic expansion $\mathrm{App}^{<N}(F; z)$ is

$$\sum_{k<n_1} f_{\{1\},k}(z_2)\frac{z_1^k}{k!} + \sum_{k<n_2} f_{\{2\},k}(z_1)\frac{z_2^k}{k!} - \sum_{\substack{\alpha_1<n_1\\ \alpha_2<n_2}} f_{\{1,2\},\alpha}\frac{z_1^{\alpha_1} z_2^{\alpha_2}}{\alpha_1!\alpha_2!}$$

where $f_{\{1\},k}$ (resp. $f_{\{2\},k}$ and $f_{\{1,2,\},\alpha}$) is a holomorphic function on S_1 (resp. holomorphic on S_2, resp. constant). A holomorphic function f is strongly asymptotically developable to $F = (f_{\{1\},k_1}, f_{\{2\},k_2}, f_{\{1,2\},(k_1,k_2)})_{k_1,k_2}$ in the sense of [7] if, for any polysector S' properly contained in S and for any $N = (n_1, n_2) \in \mathbf{N}^2$, there exists a positive constant $C_{S',N}$ such that

$$\left|f(z) - \mathrm{App}^{<N}(F;\, z)\right| \leq C_{S',N}|z_1|^{n_1}|z_2|^{n_2}$$

with $z \in S'$. Since f is strongly asymptotically developable if and only if its restriction to any polysector S' properly contained in S has bounded derivatives, we have $(\mathcal{A}_X)_p \simeq (\rho^{-1}\nu_\chi^{sa}\mathcal{O}_X^{\mathrm{w}})_p$. This follows from the fact that f is Whitney on a sector if and only if it has bounded derivatives.

(Takeuchi). Let $\chi = \{M_1, M_2\}$ with $M_1 = \{0\}$, $M_2 = \{z_2 = 0\}$. We have $I_1 = \{1, 2\}$, $I_2 = \{2\}$, $J_1 = \{1\}$, $J_2 = \{1, 2\}$. The map $p : \widetilde{X} \to X$ is defined by $(z_1, z_2, t_1, t_2) \mapsto (t_1 z_1, t_1 t_2 z_2)$. We have $\iota(M_1) = M_2$, $\iota(M_2) = X$ and then the zero section of $\widetilde{X}$ is isomorphic to $T_{M_1}M_2 \underset{X}{\times} T_{M_2}X$. For $p = (0, 0;\, 1, 1)$ and $q = \pi(p) = 0$, a cofinal set of $\mathrm{Cone}_\chi(p, \epsilon)$ is given by the family of the sets

$$\{(z_1, z_2) \in S_1 \times S_2;\, |z_2| < \epsilon|z_1|,\}_{S_1\times S_2 \ni (1,1), \epsilon>0},$$

where S_1, S_2 are sectors in $\mathbf{C} \simeq T_{M_i}\iota(M_i)$, $i = 1, 2$.

For $N = (n_1, n_2) \in \mathbf{N}^2$, an asymptotic expansion $\mathrm{App}^{<N}(F; z)$ is

$$\sum_{\alpha_1+\alpha_2<n_1} f_{\{1\},\alpha}\frac{z_1^{\alpha_1}z_2^{\alpha_2}}{\alpha_1!\alpha_2!} + \sum_{k<n_2} f_{\{2\},k}(z_1)\frac{z_2^k}{k!} - \sum_{\substack{\alpha_1+\alpha_2<n_1\\ \alpha_2<n_2}} f_{\{1,2\},\alpha}\frac{z_1^{\alpha_1}z_2^{\alpha_2}}{\alpha_1!\alpha_2!}$$

where $f_{\{2\},k}$ (resp. $f_{\{1\},\alpha}$ and $f_{\{1,2,\},\alpha}$) is a holomorphic function on S_1 (resp. constant). A holomorphic function f is strongly asymptotically developable to $F = (f_{\{1\},(k_1,k_2)}, f_{\{2\},k_2}, f_{\{1,2\},(k_1,k_2)})_{k_1,k_2}$ if, for any proper convex multi-cone S' properly contained in S and for any $N = (n_1, n_2) \in \mathbf{N}^2$, there exists a positive constant $C_{S',N}$ such that

$$\left|f(z) - \mathrm{App}^{<N}(F;\, z)\right| \leq C_{S',N}(|z_1| + |z_2|)^{n_1-n_2}|z_2|^{n_2}$$

with $z \in S'$.

We finally explain a relation between the two sheaves appearing in the above examples. Set $Y = \mathbf{C}^2$, let $X = \{(z_1, z_2, \xi_1, \xi_2) \in \mathbf{C}^2 \times \mathbf{P}^1_{\mathbf{C}}, \xi_2 z_1 = \xi_1 z_2\}$ and let $\pi : X \to \mathbf{C}^2$, $(z_1, z_2, \xi_1, \xi_2) \mapsto (z_1, z_2)$ be the desingularization map. Let $M_1 = \{0\}$, $M_2 = \{z_2 = 0\}$, $Z_1 = \pi^{-1}(0)$, $Z_2 = \{\xi_2 = 0\}$. Locally on X, for example on $U_1 := \{\xi_1 \neq 0\}$, set $\lambda = \dfrac{\xi_2}{\xi_1}$. Then $z_2 = \dfrac{\xi_2}{\xi_1}z_1$ and we have an homeomorphism $\psi : \mathbf{C}^2 \xrightarrow{\sim} U_1$, $(\lambda, z_1) \mapsto (z_1, \lambda z_1, 1, \lambda)$. We have $\psi^{-1}(Z_1) = \{z_1 = 0\}$ and $\psi^{-1}(Z_2) = \{\lambda = 0\}$. We still denote them by Z_1, Z_2. The map $f := \pi|_{U_1} \circ \psi$ is given by $(\lambda, z_1) \mapsto (\lambda z_1, z_1)$. Let us consider $\widetilde{f}$. On the zero section, let (w_1, w_2) be the coordinates of $\mathbf{C}^2$, which is homeomorphic to the zero section of $\widetilde{Y}$. We have $J_1^Y = J_1^X = \{1\}$, $J_2^Y = \{1, 2\} = J_1^X \sqcup J_2^X$ and it is easy to check that the restriction of $\widetilde{f}$ to the zero section is given by

$$w_1 = \frac{\partial f}{\partial z_1}z_1 = z_1, \quad w_2 = \frac{\partial^2 f}{\partial\lambda\partial z_1}\lambda z_1 = \lambda z_1.$$

This map is not a bundle map, but preserves the natural actions of $(\mathbf{R}^+)^2$ induced by the multi-normal deformations. Moreover f satisfies the hypothesis of Proposition 2.2 (i). Hence desingularization maps give a link between two different kinds (Majima and Takeuchi) of multi-specializations.

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