Zeta functions of certain noncommutative algebras

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Abstract: For a fixed prime $l \in \mathbf{Z}$, we consider zeta functions for certain types of (not necessarily commutative) algebras over the completion $\mathbf{Q}_l$ of $\mathbf{Q}$ and show that they satisfy several properties analogous to those of the usual Hasse-Weil zeta function of an algebraic variety over a finite field.

Key words: Zeta functions; l-adic cohomology.

1. Introduction. The starting point of noncommutative geometry is the replacement of topological spaces by (not necessarily commutative) C^*-algebras (see [1]). It follows that, given a smooth scheme X over $Spec(\mathbf{Z})$, we can associate to X a manifold $X(\mathbf{C})$ over $\mathbf{C}$ and hence the commutative C^*-algebra $C^*(X(\mathbf{C}))$ of complex valued continuous functions on $X(\mathbf{C})$. In this paper, we consider certain not necessarily commutative algebras over a completion $\mathbf{Q}_l$ of $\mathbf{Q}$ ($l \in \mathbf{Z}$ being a given prime) that enjoy several properties associated to schemes over finite fields. We refer to these objects as "Q_l^*-algebras".

The zeta function of an algebraic variety over a finite field has been extended naturally to several more general settings (see, for instance, Deitmar-Koyama-Kurokawa [2], Deitmar [3], Kurokawa [6,8] or Kurokawa-Wakayama [7]). For Q_l^*-algebras with certain additional data (see Definition 2.3), we introduce a zeta function that extends the usual Hasse-Weil zeta function on an algebraic variety over a finite field. Further, we develop appropriate functional equations for these zeta functions and also verify that they are rational functions over $\mathbf{Q}_l$. We also extend classical results such as the Lefschetz fixed point formula to this context.

2. Q_l^*-algebras. Throughout this paper, let $p \in \mathbf{Z}$ denote a fixed prime and let $l \neq p$ be a prime different from p. We note that the involution on a usual C^*-algebra may be seen as an action of the group $Gal(\mathbf{C}/\mathbf{R})$. This suggests that a "$Q_l^*$-algebra" should carry an action of the Galois group $Gal(\overline{\mathbf{F}}_p/\mathbf{F}_p)$, where $\overline{\mathbf{F}}_p$ denotes the algebraic closure of $\mathbf{F}_p$. Then, we define:

Definition 2.1. Let $l \in \mathbf{Z}$ be a fixed prime in $\mathbf{Z}$, different from p. A Q_l^*-algebra consists of a (not necessarily unital) graded $\mathbf{Q}_l$-algebra $H = \oplus_{i=0}^{\infty} H^i$ satisfying the following two properties:

(a) Each H^i, $i \geq 0$ is a finite dimensional $\mathbf{Q}_l$-vector space.

(b) Each H^i, $i \geq 0$ carries a $\mathbf{Q}_l$-linear action of the group $Gal(\overline{\mathbf{F}}_p/\mathbf{F}_p)$ which is compatible with the graded algebra structure on H, i.e., for any $\sigma \in Gal(\overline{\mathbf{F}}_p/\mathbf{F}_p)$, $\forall x \in H^i$, $y \in H^j$, $i, j \geq 0$, we have $\sigma(x) \cdot \sigma(y) = \sigma(x \cdot y)$.

The category of Q_l^*-algebras will be denoted by $Alg_{Q_l^*}$. Let $Sm/\mathbf{F}_p$ denote the category of smooth projective schemes over $\mathbf{F}_p$.

Proposition 2.2. *The category $Alg_{Q_l^*}$ of Q_l^*-algebras is a monoidal category. Further, there exists a monoidal functor*

$$Q_l^* : Sm/\mathbf{F}_p \longrightarrow Alg_{Q_l^*}$$

that associates to each object X of $Sm/\mathbf{F}_p$ a graded commutative Q_l^*-algebra.

Proof. Let $H = \oplus_{i=0}^{\infty} H^i$ and $H' = \oplus_{i=0}^{\infty} H'^i$ be two given Q_l^*-algebras. Then, $H \otimes_{\mathbf{Q}_l} H'$ is clearly a graded $\mathbf{Q}_l$-algebra such that each

$$(H \otimes_{\mathbf{Q}_l} H')^i := \oplus_{j+j'=i} H^j \otimes_{\mathbf{Q}_l} H'^{j'}$$

is a finite dimensional $\mathbf{Q}_l$-vector space. The group $Gal(\overline{\mathbf{F}}_p/\mathbf{F}_p)$ also acts on each $(H \otimes_{\mathbf{Q}_l} H')^i$ via the diagonal action compatible with the product structure on $H \otimes_{\mathbf{Q}_l} H'$. Hence, $(H \otimes_{\mathbf{Q}_l} H')$ is also a Q_l^*-algebra.

Further, given any smooth projective scheme X over $Spec(\mathbf{F}_p)$, we let $\overline{X}$ denote the fibre product $X \times_{Spec(\mathbf{F}_p)} Spec(\overline{\mathbf{F}}_p)$. Then, we define

$$Q_l^*(X)^i := H^i(\overline{X}, \mathbf{Q}_l)$$

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Then, $Q_l^*(X) := \oplus_{i=0}^{\infty} Q_l^*(X)^i$ becomes a graded commutative algebra under the cup product on l-adic cohomologies and carries a natural action of $Gal(\overline{\mathbf{F}}_p/\mathbf{F}_p)$ induced by the natural action of $Gal(\overline{\mathbf{F}}_p/\mathbf{F}_p)$ on $\overline{X}$. Moreover, for any smooth projective schemes X, Y over $\mathbf{F}_p$, we have

$$\begin{aligned} Q_l^*(X \times Y)^i &= H^i(\overline{X} \times \overline{Y}, \mathbf{Q}_l) \\ &\cong \oplus_{j+j'=i} H^j(\overline{X}, \mathbf{Q}_l) \otimes_{\mathbf{Q}_l} H^{j'}(\overline{Y}, \mathbf{Q}_l) \\ &= \oplus_{j+j'=i} Q_l^*(X)^j \otimes_{\mathbf{Q}_l} Q_l^*(Y)^{j'} \end{aligned}$$

by Künneth theorem for l-adic cohomologies. It follows that Q_l^* is a symmetric monoidal functor. □

We will now exhibit several natural examples of $\mathbf{Q}_l^*$-algebras.

Examples: (1) Proposition 2.2 shows that to each smooth projective scheme X over $\mathbf{F}_p$, we can associate a natural graded commutative Q_l^*-algebra, which we have denoted by $Q_l^*(X)$.

(2) Let X be a smooth projective scheme over $\mathbf{F}_p$ and define $H = \oplus_{i=0}^{\infty} H^i$ by setting $H^i := H^i(\overline{X}, \mathbf{Q}_l)$ as in the proof of Proposition 2.2. Let $T : H = \oplus_{i=0}^{\infty} H^i \longrightarrow H = \oplus_{i=0}^{\infty} H^i$ be a $\mathbf{Q}_l$-linear operator of degree 0 that commutes with the action of $Gal(\overline{\mathbf{F}}_p/\mathbf{F}_p)$ (for instance, we could take T to be any linear combination of elements of $Gal(\overline{\mathbf{F}}_p/\mathbf{F}_p)$, since $Gal(\overline{\mathbf{F}}_p/\mathbf{F}_p) \cong \hat{\mathbf{Z}}$ is abelian). Then, we can define a multiplicative structure on H by setting

$$x \cdot^T y := x \cup T(y)$$

where $x \in H^i = H^i(\overline{X}, \mathbf{Q}_l)$, $y \in H^j = H^j(\overline{X}, \mathbf{Q}_l)$ for all $i, j \in \mathbf{Z}$ and $\cup$ denotes the usual cup product map on l-adic cohomologies. Then, H carries the structure of a graded algebra and $\sigma(x) \cdot^T \sigma(y) = \sigma(x \cdot^T y)$. We will denote this Q_l^*-algebra by $Q_l^*(X)_T$.

(3) More generally, suppose that A is any finite dimensional algebra over $\mathbf{Q}_l$ with an action of $Gal(\overline{\mathbf{F}}_p/\mathbf{F}_p)$. Then, we consider the universal algebra $\Omega(A)$ of A, defined as follows (see, for instance, [5]): let $\tilde{A}$ denote the algebra obtained by adjoining a unit to A (even if A is already unital) and set

$$\Omega^i(A) := \tilde{A} \otimes A^{\otimes i}.$$

The action of $Gal(\overline{\mathbf{F}}_p/\mathbf{F}_p)$ on A can be extended to $\Omega^i(A)$ by setting $\sigma((a_0 + \lambda \cdot 1) \otimes a_1 \otimes \ldots \otimes a_i) = (\sigma(a_0) + \lambda \cdot 1) \otimes \sigma(a_1) \otimes \ldots \otimes \sigma(a_i))$ for all $a_0, \ldots a_i \in A$. Then, it is clear that $\Omega(A) = \oplus_{i=0}^{\infty} \Omega^i(A)$ is a Q_l^*-algebra in the sense of Definition 2.1.

Definition 2.3. Let $n \geq 0$ be a given integer. By a cycle of dimension n, we will mean a pair $(H, \int)$ consisting of a Q_l^*-algebra $H = \oplus_{i=0}^{\infty} H^i$ such that $H^i = 0$ for all $i > n$ and a linear functional $\int : H^n \longrightarrow \mathbf{Q}_l$.

A cycle $(H, \int)$ of dimension n will be said to be smooth if: (a) the composition

$$H^i \otimes_{\mathbf{Q}_l} H^{n-i} \longrightarrow H^n \xrightarrow{\int} \mathbf{Q}_l$$

is a perfect pairing of $\mathbf{Q}_l$-vector spaces for all $0 \leq i \leq n$ and (b) the Kernel of $\int$ is invariant under the action of $Gal(\overline{\mathbf{F}}_p/\mathbf{F}_p)$, i.e., for any $\sigma \in Gal(\overline{\mathbf{F}}_p/\mathbf{F}_p)$, we have $\sigma(Ker(\int)) \subseteq Ker(\int)$.

We conclude this section by giving natural examples of smooth cycles $(H, \int)$:

(1) For any smooth and projective scheme X over $\mathbf{F}_p$ of dimension d and for any $\mathbf{Q}_l$-linear automorphism T on $\oplus_{i=0}^{2d} H^i(\overline{X}, \mathbf{Q}_l)$ of degree 0 that commutes with the action of $Gal(\overline{\mathbf{F}}_p/\mathbf{F}_p)$, Poincare duality

$$H^i(\overline{X}, \mathbf{Q}_l) \otimes_{\mathbf{Q}_l} H^{2d-i}(\overline{X}, \mathbf{Q}_l) \xrightarrow{1 \otimes T} H^{2d}(\overline{X}, \mathbf{Q}_l) \xrightarrow{\cong} \mathbf{Q}_l$$

enables us to define a smooth cycle $(Q_l^*(X)_T, \int_X)$ of dimension $2d$. For instance, we could choose T to be an element of $Gal(\overline{\mathbf{F}}_p/\mathbf{F}_p)$ itself.

(2) Let K be a field extension of $\mathbf{Q}_l$ and let $f : K \longrightarrow \mathbf{Q}_l$ denote a nonzero $\mathbf{Q}_l$-linear functional on K. Let V be an n-dimensional K-vector space with a K-linear action of $Gal(\overline{\mathbf{F}}_p/\mathbf{F}_p)$ and let $E = \{e_1, e_2, \ldots, e_n\}$ be a basis for V. We choose an isomorphism $i_E : \Lambda^n V \xrightarrow{\cong} K$ by taking $e_1 \wedge \ldots \wedge e_n$ to $1 \in K$. Let $k \geq 0$ and choose some $v \in \Lambda^k V$, $v \neq 0$. Then v may be expressed as a finite sum $v = \sum a_{i_1,\ldots,i_k} e_{i_1} \wedge \ldots \wedge e_{i_k}$ where each $a_{i_1,\ldots,i_k} \in K$ and $(i_1, \ldots, i_k)$ varies over all tuples $1 \leq i_1 < i_2 < \ldots < i_k \leq n$. Let $c \in K$ be such that $f(c) \neq 0$ and choose a tuple $1 \leq i'_1 < i'_2 < \ldots < i'_k \leq n$ such that $a_{i'_1 \ldots i'_k} \neq 0$. Then there exists $\{j_1, \ldots, j_{n-k}\}$ such that $\{i'_1, \ldots, i'_k\} \cup \{j_1, \ldots, j_{n-k}\} = \{1, 2, \ldots, n\}$. It follows that the composition

$$\Lambda^k V \otimes_{\mathbf{Q}_l} \Lambda^{n-k} V \longrightarrow \Lambda^k V \otimes_K \Lambda^{n-k} V \longrightarrow \Lambda^n V \xrightarrow[i_E]{\cong} K \xrightarrow{f} \mathbf{Q}_l$$

carries $v \otimes c \cdot a^{-1}_{i'_1 \ldots i'_k} e_{j_1} \wedge \ldots \wedge e_{j_{n-k}}$ to $\pm f(c) \neq 0$. Hence, for each $0 \leq k \leq n$, the composition above determines a perfect pairing of $\mathbf{Q}_l$-vector spaces. Further, if we assume that for each $\sigma \in Gal(\overline{\mathbf{F}}_p/\mathbf{F}_p)$, the determinant $det(\sigma) \in \mathbf{Q}_l$ (where σ is considered as a K-linear automorphism on V), it follows that the data $(\oplus_{i=0}^{\infty}\Lambda^i V, f\circ i_E)$ determines a smooth cycle of dimension n.

3. Zeta functions of cycles. In this section, we will associate a zeta function to each n-dimensional smooth cycle $(H,\int)$ and show that it satisfies several properties analogous to the (Hasse-Weil) zeta functions of varieties over $\mathbf{F}_p$.

Definition 3.1. Let $(H,\int)$ be an n-dimensional cycle and let F denote the Frobenius element of $Gal(\overline{\mathbf{F}}_p/\mathbf{F}_p)$. For any $k\geq 0$, we set

$$N_k(H,\textstyle\int) = \sum_{i=0}^{n}(-1)^i Tr(F^k : H^i \longrightarrow H^i)$$

Let z denote an indeterminate. Then, the zeta function $\zeta_{(H,\int)}(z)$ is defined as the formal series:

$$\zeta_{(H,\int)}(z) = exp\left(\sum_{k=1}^{\infty} N_k(H,\textstyle\int)\frac{z^k}{k}\right).$$

Proposition 3.2. *Let X be a smooth, projective scheme over $\mathbf{F}_p$ of dimension d. Then, we have $\zeta_X(z) = \zeta_{(Q_l^*(X),\int_X)}$, where $\zeta_X(z)$ denotes the Hasse-Weil zeta function associated to X.*

Proof. For any $i\geq 0$, by definition, the Q_l^*-algebra $Q_l^*(X)$ is given by $Q_l^*(X)^i := H^i(\overline{X},\mathbf{Q}_l)$ and $\int_X : H^{2d}(\overline{X},\mathbf{Q}_l)\longrightarrow \mathbf{Q}_l$ is defined by the isomorphism $H^{2d}(\overline{X},\mathbf{Q}_l)\cong\mathbf{Q}_l$. Then, the result follows directly from the well known Lefschetz fixed point formula. □

Let $(H,\int)$ and $(H',\int')$ be cycles of dimensions n and n' respectively. Then, we can define a "product cycle" $(H\otimes H',\int\otimes\int')$ of dimension $n+n'$ by setting

$$\left(\int\otimes\int'\right)(\omega\otimes\omega') = \left(\int\omega\right)\cdot\left(\int'\omega'\right)$$

for all $\omega\in H^n$, $\omega'\in H'^{n'}$.

Additionally, if $(H,\int)$ is smooth, we have $\sigma(Ker(\int))\subseteq Ker(\int)$ for each $\sigma\in Gal(\overline{\mathbf{F}}_p/\mathbf{F}_p)$ and $\int$ is a $\mathbf{Q}_l$-linear functional on H^n. Hence, for each $\sigma\in Gal(\overline{\mathbf{F}}_p/\mathbf{F}_p)$, there exists a scalar $\lambda_\sigma(H,\int)\in\mathbf{Q}_l$ such that we have

$$\textstyle\int\sigma(\omega) = \lambda_\sigma(H,\int)\cdot\int\omega \qquad \forall\omega\in H^n$$

Proposition 3.3. *(a) Let $(H,\int)$ and $(H',\int')$ be cycles of dimensions n and n' respectively. Then, for any $k\geq 0$, we have $N_k((H\otimes H',\int\otimes\int')) = N_k((H,\int))\cdot N_k((H',\int'))$.*
(b) If $(H,\int)$ and $(H',\int')$ are smooth cycles of dimensions n and n' respectively, so is the product cycle $(H\otimes H',\int\otimes\int')$.

Proof. (a) We choose any $k\geq 0$. Then, by definition

$$\begin{aligned}
&N_k((H\otimes H',\textstyle\int'')\\
&= \sum_{i=0}^{n+n'}(-1)^i Tr(F^k : (H\otimes H')^i \longrightarrow (H\otimes H')^i)\\
&= \sum_{i=0}^{n+n'}(-1)^i \sum_{j+j'=i} Tr(F^k|H^j)\cdot Tr(F^k|H'^{j'})\\
&= \sum_{i=0}^{n+n'}\sum_{j+j'=i}(-1)^j Tr(F^k|H^j)\cdot(-1)^{j'}Tr(F^k|H'^{j'})\\
&= \left(\sum_{l=0}^{n}(-1)^l Tr(F^k|H^l)\right)\cdot\left(\sum_{l'=0}^{n'}(-1)^{l'}Tr(F^k|H'^{l'})\right)\\
&= N_k(H,\textstyle\int)\cdot N_k(H',\int')
\end{aligned}$$

(b) For any $0\leq i\leq n+n'$, we know that $(H\otimes H')^i := \oplus_{j+j'=i}H^j\otimes H'^{j'}$. Then, it is clear that the linear functional $\int\otimes\int' : (H\otimes H')^{n+n'}\longrightarrow\mathbf{Q}_l$ defined by

$$\left(\int\otimes\int'\right)(\omega\otimes\omega') = \left(\int\omega\right)\cdot\left(\int'\omega'\right)$$

for all $\omega\in H^n$, $\omega'\in H'^{n'}$ composed with the product on $H\otimes H'$ defines a perfect pairing of $(H\otimes H')^i$ with $(H\otimes H')^{n+n'-i}$ for each $0\leq i\leq n+n'$. Choose any $\sigma\in Gal(\overline{\mathbf{F}}_p/\mathbf{F}_p)$. Since $(H,\int)$ and $(H',\int')$ are smooth, we have $\sigma(Ker(\int))\subseteq Ker(\int)$ and $\sigma(Ker(\int'))\subseteq Ker(\int')$. Suppose that we have a finite sum $\sum_{i=1}^{N}\omega_i\otimes\omega_i'$, $\omega_i\in H^n$, $\omega'\in H'^{n'}$ such that

$$\textstyle(\int\otimes\int')(\sum_{i=1}^{N}\omega_i\otimes\omega_i') = \sum_{i=1}^{N}(\int\omega_i)\cdot(\int'\omega_i') = 0$$

Then, it follows that

$$\begin{aligned}
&\textstyle(\int\otimes\int')(\sum_{i=1}^{N}\sigma(\omega_i)\otimes\sigma(\omega_i'))\\
&\textstyle= \sum_{i=1}^{N}(\int\sigma(\omega_i))\cdot(\int'\sigma(\omega_i'))\\
&\textstyle= \sum_{i=1}^{N}(\lambda_\sigma(H,\int)\lambda_\sigma(H',\int'))(\int\omega_i)\cdot(\int'\omega_i')\\
&\textstyle= (\lambda_\sigma(H,\int)\lambda_\sigma(H',\int'))\sum_{i=1}^{N}(\int\omega_i)\cdot(\int'\omega_i') = 0
\end{aligned}$$

from which it follows that $\sigma(Ker(\int\otimes\int'))\subseteq Ker(\int\otimes\int')$. Hence, $(H\otimes H',\int\otimes\int')$ is a smooth cycle of dimension $n+n'$. □

Our next objective is to prove a version of Lefschetz fixed point theorem for smooth cycles $(H,\int)$ of some given dimension n. We note that if $\varphi : X\longrightarrow X$ is a morphism of smooth schemes over $\mathbf{F}_p$, φ induces a morphism $\varphi^* : H^*(\overline{X},\mathbf{Q}_l)\longrightarrow H^*(\overline{X},\mathbf{Q}_l)$ of degree 0. If X has dimension d, the morphism φ^* can be described completely in terms of the class of Γ_φ in $H^{2d}(\overline{X}\times\overline{X})$, $\Gamma_\varphi\subseteq\overline{X}\times\overline{X}$ being the graph of φ. We will now associate to each morphism $\varphi : H^* \longrightarrow H^*$ of degree 0 on a smooth cycle $(H, \int)$ of dimension n a class $cl(\varphi) \in (H \otimes H)^n$.

Proposition 3.4. *Let $(H, \int)$ be a smooth cycle of dimension n. Let $\varphi : H^* \longrightarrow H^*$ be a linear operator of degree 0. Then, φ induces a natural class $cl(\varphi) \in (H \otimes H)^n$.*

Proof. Suppose that V is a finite dimensional $\mathbf{Q}_l$-vector space and let $\psi : V \longrightarrow V$ be a linear operator on V. Let $\mathfrak{B} = \{v_1, \dots, v_k\}$ be a given basis of V and let $\mathfrak{B}^* = \{v_1^*, \dots, v_k^*\}$ be the dual basis of $\mathfrak{B}$. Let V^* denote the linear dual of V. Then, it is easy to check that the sum $\sum_{i=1}^k \psi(v_i) \otimes v_i^* \in V \otimes V^*$ is independent of the choice of the basis $\mathcal{B}$. We set $cl_V(\psi) = \sum_{i=1}^k \psi(v_i) \otimes v_i^*$.

Given the smooth cycle $(H, \int)$ and a linear operator $\varphi : H^* \longrightarrow H^*$ of degree 0, we let $\varphi_i : H^i \longrightarrow H^i$, $i \geq 0$ denote the restriction of φ to each H^i. For each i, we define $cl_i(\varphi) = cl_{H^i}(\varphi_i) \in H^i \otimes H^{i*}$, where H^{i*} denotes the linear dual of H^i. Since $(H, \int)$ is a smooth cycle of dimension n, we may take $H^{i*} = H^{n-i}$. Then, we have $cl_i(\varphi) \in H^i \otimes H^{n-i}$. Finally, we set

$$cl(\varphi) = \sum_{i=0}^{n} cl_i(\varphi) \in \sum_{i=0}^{n} H^i \otimes H^{n-i} = (H \otimes H)^n.$$

□

In the notation of the proof of Proposition 3.4, for any linear operator $\psi : V \longrightarrow V$ on a finite dimensional vector space V of dimension k, we can also consider the transpose $cl_V^t(\psi)$ of $cl_V(\psi)$, defined as $cl_V^t(\psi) = \sum_{i=1}^k v_i^* \otimes \psi(v_i) \in V^* \otimes V$. Then, given a linear operator $\varphi : H^* \longrightarrow H^*$ of degree 0 on a smooth cycle $(H, \int)$, we can define

$$cl^t(\varphi) = \sum_{i=0}^{n} (-1)^i cl_{H^i}^t(\varphi_i) \in H^{n-i} \otimes H^i = (H \otimes H)^n$$

(since each $H^{n-i} = H^{i*}$) and refer to $cl^t(\varphi)$ as the graded transpose of $cl(\varphi)$. We can now prove a version of Lefschetz fixed point theorem.

Proposition 3.5. *Let $(H, \int)$ be a smooth n-dimensional cycle. Let F denote the Frobenius operator in the group $Gal(\overline{\mathbf{F}}_p/\mathbf{F}_p)$ and let I denote the identity map. Then, for any $k \geq 0$, we have:*

$$(\int \otimes \int)(cl^t(F^k) \cdot cl(I)) = N_k(H, \int) = \sum_{r=0}^{n} (-1)^r Tr(F^k|H^r)$$

Proof. For each $0 \leq r \leq n$, let $d_r = dim_{\mathbf{Q}_l}(H^r)$. We let $\mathfrak{E}_r = \{e_i^r\}_{1 \leq i \leq d_r}$ be a basis of H^r and let $\mathfrak{F}_r = \{f_i^{n-r}\}_{1 \leq i \leq d_r}$ denote a dual basis of $\mathfrak{E}_r$. Hence $\mathfrak{F}_r$ may be taken as a basis for H^{n-r}. Then, by definition:

$$cl^t(F^k) = \sum_{r=0}^{n} (-1)^r \sum_{i=0}^{d_r} f_i^{n-r} \otimes F^k(e_i^r)$$

and

$$cl(I) = \sum_{r=0}^{n} \sum_{i=0}^{d_r} e_i^r \otimes f_i^{n-r}.$$

Then, the product

$$\begin{aligned}
&(\int \otimes \int)(cl^t(F^k) \cdot cl(I)) = (\int \otimes \int) \\
&\left(\left(\sum_{r=0}^{n} (-1)^r \sum_{i=0}^{d_r} f_i^{n-r} \otimes F^k(e_i^r) \right) \cdot \left(\sum_{s=0}^{n} \sum_{i=0}^{d_s} e_i^s \otimes f_i^{n-s} \right) \right) \\
&\quad = \sum_{r=0}^{n} (-1)^r \sum_{i=0}^{d_r} \int (f_i^{n-r} \cdot e_i^r) \cdot \int (F^k(e_i^r) \cdot f_i^{n-r}) \\
&\quad = \sum_{r=0}^{n} (-1)^r Tr(F^k|H^r) = N_k(H, \int).
\end{aligned}$$

□

Proposition 3.6. *Let $(H, \int)$ be a cycle of dimension n. Then, the zeta function $\zeta_{(H, \int)}(z)$ of $(H, \int)$ is a rational function of z with $\mathbf{Q}_l$ coefficients.*

Proof. By definition, we know that

$$\begin{aligned}
\zeta_{(H,\int)}(z) &= exp\left(\sum_{k=1}^{\infty} \sum_{r=0}^{n} (-1)^r Tr(F^k|H^r) \tfrac{z^k}{k} \right) \\
&= \prod_{r=0}^{n} exp\left(\sum_{k=1}^{\infty} Tr(F^k|H^r) \tfrac{z^k}{k} \right)^{(-1)^r}.
\end{aligned}$$

Since the Frobenius F is a linear operator on each finite dimensional vector space H^r, we have

$$exp\left(\sum_{k=1}^{\infty} Tr(F^k|H^r) \frac{z^k}{k} \right) = det(1 - Fz|H^r)^{-1}.$$

For each r, the determinant $det(1 - Ft|H^r)$ is a polynomial in $\mathbf{Q}_l[t]$. Hence, the result follows. □

Given a smooth cycle $(H, \int)$ of dimension n, for any $0 \leq r \leq n$, we will always let $d_r = dim_{\mathbf{Q}_l}(H^r)$. Then, we denote by B the "Euler characteristic" $B := \sum_{r=0}^{n} (-1)^r d_r$ of the smooth cycle $(H, \int)$.

Further, we will always let $P_r(z) := det(1 - Fz|H^r)$. Then, if we set:

$$Q_r(z) := \frac{P_r(z)}{(-1)^{d_r} z^{d_r}} = det\left(F - \frac{1}{z} |H^r \right)$$

it makes sense to write $Q_r(\infty) := det(F|H^r)$. We also set

$$\tilde{\zeta}_{(H,\int)}(z) = \left(\prod_{r=0}^{n} Q_r(z)^{(-1)^r} \right)^{-1} = (-1)^B z^B \zeta_{(H,\int)}(z).$$

Accordingly, it makes sense to write:

$$\tilde{\zeta}_{(H,\int)}(\infty) := \left(\prod_{r=0}^{n} Q_r(\infty)^{(-1)^r}\right)^{-1} = \left(\prod_{r=0}^{n} det(F|H^r)^{(-1)^r}\right)^{-1}.$$

Proposition 3.7. *Let $(H, \int)$ be a smooth cycle of dimension n. Let $F \in Gal(\overline{\mathbf{F}}_p/\mathbf{F}_p)$ be the Frobenius and let $\lambda = \lambda_F(H, \int)$. Then*:
(*a*) *If n is even, we have the functional equation*:

$$\left(\zeta_{(H,\int)}\left(\frac{1}{\lambda z}\right)\right)^2 = \lambda^B z^{2B} \zeta_{(H,\int)}(z)^2.$$

(*b*) *If n is odd, we have the functional equation*:

$$\tilde{\zeta}_{(H,\int)}(z)\tilde{\zeta}_{(H,\int)}\left(\frac{1}{\lambda z}\right) = (-1)^B z^{-B} \tilde{\zeta}_{(H,\int)}(\infty).$$

Proof. For any $0 \le r \le n$, we have perfect pairings of $\mathbf{Q}_l$-vector spaces and a commutative diagram:

$$\begin{array}{ccccc} H^r \otimes_{\mathbf{Q}_l} H^{n-r} & \longrightarrow & H^n & \xrightarrow{\int} & \mathbf{Q}_l \\ {\scriptstyle F\otimes F}\downarrow & & {\scriptstyle F}\downarrow & & {\scriptstyle \lambda}\downarrow \\ H^r \otimes_{\mathbf{Q}_l} H^{n-r} & \longrightarrow & H^n & \xrightarrow{\int} & \mathbf{Q}_l\,. \end{array}$$

Since $\lambda \int(x \cdot y) = \int(F(x \cdot y)) = \int(F(x) \cdot F(y))$ for any $x \in H^r$, $y \in H^{n-r}$, it follows from [4, Appendix C, Lemma 4.3] that

$$\begin{aligned} P_{n-r}(z) &= det(1 - Fz|H^{n-r}) \\ &= \tfrac{(-1)^{d_r}\lambda^{d_r} z^{d_r}}{det(F|H^r)} det(1 - \tfrac{F}{\lambda z}|H^r) \\ &= \tfrac{(-1)^{d_r}\lambda^{d_r} z^{d_r}}{det(F|H^r)} P_r(\tfrac{1}{\lambda z}) \end{aligned}$$

and

$$det(F|H^{n-r}) = \frac{\lambda^{d_r}}{det(F|H^r)}.$$

(a) When n is even, we have:

$$\begin{aligned} \left(\zeta_{(H,\int)}(\tfrac{1}{\lambda z})\right)^2 &= \left(\prod_{r=0}^{n} P_r(\tfrac{1}{\lambda z})^{(-1)^r}\right)^{-2} \\ &= \left(\prod_{r=0}^{n} P_{n-r}(z)^{(-1)^{n-r}}\right)^{-2} \cdot \left(\prod_{r=0}^{n}\left(\tfrac{det(F|H^r)^2}{\lambda^{2d_r} z^{2d_r}}\right)^{(-1)^r}\right)^{-1} \\ &= (\zeta_{(H,\int)}(z))^2 \cdot \left(\lambda^{-B} z^{-2B}\right)^{-1} = \lambda^B z^{2B} \zeta_{(H,\int)}(z)^2. \end{aligned}$$

(b) Since $d_r = d_{n-r}$, it is clear that, for odd n:

$$Q_{n-r}(z) = \frac{(-1)^{d_r} z^{-d_r}}{det(F|H^r)} Q_r\left(\frac{1}{\lambda z}\right).$$

Hence:

$$\begin{aligned} \tilde{\zeta}_{(H,\int)}(\tfrac{1}{\lambda z}) &= \left(\prod_{r=0}^{n} Q_r(\tfrac{1}{\lambda z})^{(-1)^r}\right)^{-1} \\ &= \left(\prod_{r=0}^{n} Q_{n-r}(z)^{(-1)^{n-r}}\right) \cdot \left(\prod_{r=0}^{n}\left(\tfrac{det(F|H^r)}{(-1)^{d_r} z^{-d_r}}\right)^{(-1)^r}\right)^{-1} \\ &= (-1)^B z^{-B} (\tilde{\zeta}_{(H,\int)}(z))^{-1} \cdot \tilde{\zeta}_{(H,\int)}(\infty). \end{aligned}$$

□

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