On a minimal counterexample to the Alperin-McKay conjecture

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Abstract: We show for a minimal counterexample (G, B) to the Alperin-McKay conjecture, the Fitting subgroup of G is central and G has a unique G-conjugacy class of components.

Key word: Alperin-McKay conjecture.

1. Introduction. Let G be a finite group and p a prime. Let $(\mathcal{K}, R, k)$ be a p-modular system ([NT, p.230]); that is, R is a complete discrete valuation ring with quotient field $\mathcal{K}$ of characteristic 0 and k is the residue field of R of characteristic p. We assume that $\mathcal{K}$ contains a primitive $|G|$-th root of unity and that k is algebraically closed. Let B be a block of G. This means that B is a block ideal of RG. Let D be a defect group of B. Let $\tilde{B}$ be the Brauer correspondent of B with respect to D in $N_G(D)$. Let $k_0(B)$ be the number of irreducible characters of height 0 in B. The Alperin-McKay conjecture [Al] (AM-conjecture, for short) states that $k_0(B) = k_0(\tilde{B})$. In a previous paper [Mu], we have studied reduction of this conjecture. In this note, using a result in [Mu], we give further restrictions on a minimal counterexample (G, B) to the AM-conjecture. Here we say (G, B) is a minimal counterexample to the AM-conjecture, if B is a counterexample to the AM-conjecture and G is chosen so that $|G : Z(G)|$ is as small as possible. We prove the following

Theorem. *For a minimal counterexample (G, B) to the AM-conjecture, the following holds.*

(i) *$O_p(G)$ and $O_{p'}(G)$ are both central in G. In particular, the Fitting subgroup of G is a central subgroup of G.*

(ii) *G has a unique G-conjugacy class of components.*

It is known that the AM-conjecture is a consequence of Dade's projective conjecture [Da]. Eaton-Robinson [ER, Theorem 1 and Remarks] and Robinson [Ro, Theorem 1] have obtained restrictions similar to Theorem on a minimal counterexample to Dade's projective conjecture.

For the McKay conjecture, the prototype of the AM-conjecture, Isaacs, Malle and Navarro [IMN] have obtained a reduction theorem. Using this paper as a starting point, Späth [Sp] has recently obtained a reduction theorem for the AM-conjecture. Her approach is different from ours.

2. Proof of Theorem. For a block B of a group G, let e_B be the block idempotent of B. Let $F^*(G)$ (resp. $F(G)$) be the generalized Fitting (resp. Fitting) subgroup of G. We have proved the following, see [Mu, Proposition 9].

Lemma. *Let (G, B) be a minimal counterexample to the AM-conjecture. Let D be a defect group of B. Then the following holds.*

(i) *For any non-central normal subgroup K of G, $G = N_G(D)K$.*

(ii) *For any normal subgroup K of G, B covers a G-invariant block of K.*

(iii) *$G = N_G(D)F^*(G)$.*

For the other notation and terminology, see the books [NT, Th].

Proof of Theorem. (i) Assume $O_p(G)$ is not central in G. By Lemma (i), $G = N_G(D)O_p(G) = N_G(D)$, a contradiction. Thus $O_p(G)$ is central in G.

Assume $K := O_{p'}(G)$ is not central in G. By Lemma (i), $G = N_G(D)K$. Put $H = C_G(D)DK$. Then $H \triangleleft G$. By Lemma (ii) B covers a G-invariant block b of H. Then D is a defect group of b. Let Q be a subgroup of D. Then $N_H(Q) = C_G(D)N_{DK}(Q)$ and $C_H(Q) = C_G(D)C_{DK}(Q)$. So $N_H(Q)/C_H(Q) \simeq N_{DK}(Q)/C_{DK}(Q)$, which is a p-group, since DK is p-nilpotent. Thus b is nilpotent. Further, since $C_G(D) \leq H$ for the defect group D of b, B is a unique block of G covering b. Let $\tilde{B}$ (resp. $\tilde{b}$) be the Brauer correspondent of B (resp. b) with respect to D in $N_G(D)$ (resp. $N_H(D)$). By the Harris-Knörr theorem [HK, Theorem], $\tilde{B}$ is a unique block of

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$N_G(D)$ covering $\tilde{b}$. Let β be a block of $C_H(D)$ covered by $\tilde{b}$. Then $\tilde{B}$ is a unique block of $N_G(D)$ covering β. Let B_1 be the Fong-Reynolds correspondent of $\tilde{B}$ over β in $N_G(D)_\beta$, where $N_G(D)_\beta$ is the inertial group of β in $N_G(D)$. Then B_1 is a unique block of $N_G(D)_\beta$ covering β by the Fong-Reynolds theorem. Let D_δ be the defect pointed group of the pointed group $H_{\{e_b\}}$ on RH associated with the subpair $(D, \overline{e_\beta})$, where $\overline{e_\beta}$ denotes the natural image of e_β in $kC_H(D)$. Then $N_G(D_\delta) = N_G(D)_\beta$, cf. [Th, Proposition 40.13(b)]. Therefore, by [KP, 1.20.3], B and B_1 are isomorphic to full matrix algebras over the same R-algebra. Then $k_0(B) = k_0(B_1)$ by the Morita equivalence between B and B_1. By the Fong-Reynolds theorem, $k_0(B_1) = k_0(\tilde{B})$. Hence $k_0(B) = k_0(\tilde{B})$, a contradiction. Thus $O_{p'}(G)$ is central in G.

(ii) Let n be the number of G-conjugacy classes of components of G. If $n = 0$, then, by (i) and Lemma (iii), $G = N_G(D)F^*(G) = N_G(D)F(G) = N_G(D)$, a contradiction. If $n \geq 2$, let N_1 be the product of all components in a G-conjugacy class and let N_2 be the product of the remaining components. Since N_1 and N_2 are non-central normal subgroups of G, DN_1 and DN_2 are normal subgroups of G by Lemma (i). Clearly $(DN_1 \cap DN_2)/(N_1 \cap N_2)$ is a p-group. Since $N_1 \cap N_2 \leq Z(N_1N_2) \leq F(G)$, $N_1 \cap N_2$ is central in G by (i). Hence $DN_1 \cap DN_2$ is a direct product of a p-group and a p'-group. Then $D \leq O_p(DN_1 \cap DN_2)$. Since $O_p(DN_1 \cap DN_2) \lhd G$, we have $O_p(DN_1 \cap DN_2) \leq D$. So $D = O_p(DN_1 \cap DN_2) \lhd G$, a contradiction. Thus $n = 1$, as required. □

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