Analyticity of the closures of some Hodge theoretic subspaces

By Kazuya KATO,*) Chikara NAKAYAMA**) and Sampei USUI***)

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Abstract: In this paper, we prove a general theorem concerning the analyticity of the closure of a subspace defined by a family of variations of mixed Hodge structures, which includes the analyticity of the zero loci of degenerating normal functions. For the proof, we use a moduli of the valuative version of log mixed Hodge structures.

Key words: Hodge theory; log geometry; intermediate Jacobian; Néron model; admissible normal function; zero locus.

Introduction. The analyticity of the zero loci of degenerating normal functions is a subject of current interest in various contexts. For example, it is closely related to the existence of the Bloch-Beilinson filtration on Chow groups (cf. [C10, L01]).

In this paper, we give a new proof of this analyticity by proving a general theorem concerning the analyticity of the closure of a subspace defined by a family of variations of mixed Hodge structures.

First, we recall the definition of the classifying space D of mixed Hodge structures with polarized graded quotients, introduced in [U84], which is a natural generalization of Griffiths classifying space of polarized Hodge structures ([G68]).

Fix a 4-ple $\Lambda := (H_0, W, (\langle\ ,\ \rangle_k)_k, (h^{p,q})_{p,q})$, where H_0 is a finitely generated free $\mathbf{Z}$-module, W is an increasing filtration on $H_{0,\mathbf{Q}} := \mathbf{Q} \otimes_{\mathbf{Z}} H_0$, $\langle\ ,\ \rangle_k$ is a non-degenerate $\mathbf{Q}$-bilinear form $\mathrm{gr}^W_k \times \mathrm{gr}^W_k \to \mathbf{Q}$ given for each $k \in \mathbf{Z}$ which is symmetric if k is even and anti-symmetric if k is odd, and $h^{p,q}$ is a non-negative integer given for $p, q \in \mathbf{Z}$ such that $h^{p,q} = h^{q,p}$ for all p, q, and that $\mathrm{rank}_{\mathbf{Z}}(H_0) = \sum_{p,q} h^{p,q}$, $\dim_{\mathbf{Q}}(\mathrm{gr}^W_k) = \sum_{p+q=k} h^{p,q}$ for all k.

For $A = \mathbf{Z}, \mathbf{Q}, \mathbf{R}$, or $\mathbf{C}$, let G_A be the group of all A-automorphisms of $H_{0,A}$ which are compatible with W and $\langle\ ,\ \rangle_k$ for any k. For $A = \mathbf{Q}, \mathbf{R}$, or $\mathbf{C}$, let $\mathfrak{g}_A$ be the Lie algebra of G_A, which is identified with a subalgebra of $\mathrm{End}_A(H_{0,A})$ in the natural way (cf. [KNU10a] 1.2).

Let D be the set of all decreasing filtration F on $H_{0,\mathbf{C}}$ for which (H_0, W, F) is a mixed Hodge structure such that the $(p, k-p)$ Hodge number of $F(\mathrm{gr}^W_k)$ coincides with $h^{p,k-p}$ for any $p, k \in \mathbf{Z}$ and such that $F(\mathrm{gr}^W_k)$ is polarized by $\langle\ ,\ \rangle_k$ for all k. Let $\check{D} \supset D$ be the space defined in [KNU10a] 1.3.

In this paper, we construct valuatively toroidal partial compactifications of D. As an application, we prove the following theorem, which was announced in [KNU10a] Remark 2.

Theorem 0.1. *Let S be a complex analytic space, let Y be a closed analytic subspace of S, and let $S^* = S - Y$. Assume that S^* is non-singular and is dense in S. Let Γ be a neat subgroup of $G_{\mathbf{Z}}$. For $1 \le j \le n$, let $f_j : S^* \to \Gamma\backslash D$ be the period map associated to a variation of mixed Hodge structure with polarized graded quotients which is admissible with respect to S ([K86]). Let $Z^* = \{s \in S^* \mid f_1(s) = \cdots = f_n(s)\}$, and let $\overline{Z^*}$ be the closure of Z^* in S. Then $\overline{Z^*}$ is an analytic subset of S.*

We can deduce from Theorem 0.1 the following theorem of Brosnan–Pearlstein, whose case of weight -1 is proved independently by Schnell.

Theorem 0.2 (Brosnan–Pearlstein [BP.p], Schnell [Scn.p]). *Let S^* be a smooth complex algebraic variety. Let H be a variation of PHS of negative weight on S^*. Let $\nu : S^* \to J(H)$ be an admissible normal function ([Sa96]), where $J(H)$ is the intermediate Jacobian. Then the zero locus $Z(\nu) \subset S^*$ of ν is algebraic.*

Remarks. (1) The case of 0.1 where S is smooth and Y is a smooth divisor is also proved in §4 in [KNU10a]. The case of 0.2 where S^* is compactifiable by a smooth divisor was also proved by Brosnan–Pearlstein ([BP09a, BP09b]) and, independently, by Saito ([Sa.p]).

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*) Department of Mathematics, University of Chicago, 5734 S. University Avenue, Chicago, Illinois 60637, USA.

**) Graduate School of Science and Engineering, Tokyo Institute of Technology, Meguro-ku, Tokyo 152-8551, Japan.

***) Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan.

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(2) After submitting this paper, we found Brosnan–Pearlstein–Schnell further developed their methods. In fact, [BPS10] Proposition 6 contains more cases of our Theorem 0.1, and they communicated to the authors that their methods can be enhanced to reprove 0.1. Also, it is an interesting problem to study the relation of our 0.1 and Cattani–Deligne–Kaplan type results as in [BPS10] Theorem 1 on the locus on a Hodge class.

To deduce 0.2, after taking a smooth compactification of S^*, we apply the case of 0.1 where $\mathrm{gr}_k^W = 0$ unless $k = 0, 1$ and $\mathrm{gr}_0^W = \mathbf{Q}$. See Remark 1 of §4 in [KNU10a].

After a preparation in §1, we state some theorems on moduli spaces in §2–§3. Next we apply them to 0.1 in §4–§5. Finally, we prove the theorems in §2–§3 in the last section, which are essentially reduced to our previous works.

1. Variants of toric varieties. We recall some facts about toric varieties.

1.1. Fix a subgroup Γ of $G_{\mathbf{Z}}$. Let σ be a sharp rational nilpotent cone ([KNU10a] 2.1.1) in $\mathfrak{g}_{\mathbf{R}}$. Let $\Gamma(\sigma) := \Gamma \cap \exp(\sigma)$ and assume that σ is generated by $\log(\Gamma(\sigma))$ as a cone. Let $P(\sigma) := \mathrm{Hom}(\Gamma(\sigma), \mathbf{N})$, the dual monoid. Define the toric variety and the torus associated to σ by

$$\begin{aligned} &\mathrm{toric}_\sigma := \mathrm{Hom}(P(\sigma), \mathbf{C}^{\mathrm{mult}}) \\ &\supset \mathrm{torus}_\sigma := \mathrm{Hom}(P(\sigma), \mathbf{C}^\times) = \Gamma(\sigma)^{\mathrm{gp}} \otimes_{\mathbf{Z}} \mathbf{C}^\times, \end{aligned}$$

where $\mathbf{C}^{\mathrm{mult}}$ denotes $\mathbf{C}$ regarded as a multiplicative monoid.

1.2. Let $\mathrm{toric}_{\sigma,\mathrm{val}} := (\mathrm{toric}_\sigma)_{\mathrm{val}}$ be the projective limit of the log modifications of toric_σ, that is, the projective limit of toric varieties over $\mathbf{C}$ associated to rational finite subdivisions of the cone σ.

1.3. $|\mathrm{toric}|_\sigma$ and $|\mathrm{torus}|_\sigma$ are defined as $\mathrm{Hom}(P(\sigma), \mathbf{R}_{\geq 0}^{\mathrm{mult}})$ and $\mathrm{Hom}(P(\sigma), \mathbf{R}_{>0}^{\mathrm{mult}}) = \Gamma(\sigma)^{\mathrm{gp}} \otimes_{\mathbf{Z}} \mathbf{R}_{>0}^{\mathrm{mult}}$, respectively ([KNU10a] 2.2.3).

$|\mathrm{toric}|_{\sigma,\mathrm{val}}$ is defined as the closure of the inverse image of $|\mathrm{torus}|_\sigma$ in $\mathrm{toric}_{\sigma,\mathrm{val}}$.

$$\begin{array}{ccccc} \mathrm{torus}_\sigma & \hookrightarrow & \mathrm{toric}_{\sigma,\mathrm{val}} & \hookleftarrow & |\mathrm{toric}|_{\sigma,\mathrm{val}} \\ \parallel & & \downarrow & & \downarrow \\ \mathrm{torus}_\sigma & \hookrightarrow & \mathrm{toric}_\sigma & \hookleftarrow & |\mathrm{toric}|_\sigma \end{array}$$

2. D_{val} and $D^\sharp_{\mathrm{val}}$. In this section, we give a general theory on the valuatively toroidal partial compactifications of D. Here we state and prove the part which suffices for the proof of Theorem 0.1.

2.1. We recall the definition of the sets D_{val} and $D^\sharp_{\mathrm{val}}$ in [KNU.p] 3.2.1.

Let D_{val} (resp. $D^\sharp_{\mathrm{val}}$) be the set of all triples (A, V, Z), where A is a $\mathbf{Q}$-subspace of $\mathfrak{g}_{\mathbf{Q}}$ consisting of mutually commutative nilpotent elements, V is a sharp valuative submonoid of $A^* := \mathrm{Hom}_{\mathbf{Q}}(A, \mathbf{Q})$ ("valuative" means $V \cup V^{-1} = A^*$), and Z is a subset of $\check{D}$ such that $Z = \exp(A_{\mathbf{C}})F$ (resp. $Z = \exp(iA_{\mathbf{R}})F$) for any $F \in Z$ and that there exists a finitely generated rational subcone τ of $A_{\mathbf{R}}$ satisfying the conditions that Z is a τ-nilpotent orbit (resp. i-orbit) ([KNU10a] 2.1.3) and $(A \cap \tau)^\vee \subset V$ in A^*.

We have a canonical surjective map $D^\sharp_{\mathrm{val}} \to D_{\mathrm{val}}$, $(A, V, Z) \mapsto (A, V, \exp(A_{\mathbf{C}})Z)$.

2.2. We endow $D^\sharp_{\mathrm{val}}$ with the strongest topology for which the map $D^\sharp_{\sigma,\mathrm{val}} \to D^\sharp_{\mathrm{val}}$ is continuous for any sharp rational nilpotent cone σ. Here the topology of $D^\sharp_{\sigma,\mathrm{val}}$ is defined in [KNU.p] 3.2.7, that is, the quotient topology from $E^\sharp_{\sigma,\mathrm{val}} = |\mathrm{toric}|_{\sigma,\mathrm{val}} \times_{|\mathrm{toric}|_\sigma} E^\sharp_\sigma$.

We will prove the next theorem in §6.

Theorem 2.3. (i) *$D^\sharp_{\mathrm{val}}$ is Hausdorff.*

(ii) *Let Γ be a subgroup of $G_{\mathbf{Z}}$. Then the action of Γ on $D^\sharp_{\mathrm{val}}$ is proper. The quotient $\Gamma\backslash D^\sharp_{\mathrm{val}}$ is Hausdorff. If Γ is neat, the projection $D^\sharp_{\mathrm{val}} \to \Gamma\backslash D^\sharp_{\mathrm{val}}$ is a local homeomorphism.*

2.4. Let Γ be a subgroup of $G_{\mathbf{Z}}$. Let $\mathcal{C}_\Gamma$ be the set of all sharp rational nilpotent cones in $\mathfrak{g}_{\mathbf{R}}$ generated by the logarithms of a finite number of elements of Γ. Let $D_{\mathrm{val},(\Gamma)} \subset D_{\mathrm{val}}$ be the union of $D_{\sigma,\mathrm{val}}$ for $\sigma \in \mathcal{C}_\Gamma$.

We endow $\Gamma\backslash D_{\mathrm{val},(\Gamma)}$ with the topology, the sheaf of (complex) analytic functions and the log structure as follows:

A subset U of $\Gamma\backslash D_{\mathrm{val},(\Gamma)}$ is open if and only if, for any $\sigma \in \mathcal{C}_\Gamma$, the inverse image of U in $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_{\sigma,\mathrm{val}}$ is open. For an open set U of $\Gamma\backslash D_{\mathrm{val},(\Gamma)}$, a complex valued function f on U is analytic if and only if, for any $\sigma \in \mathcal{C}_\Gamma$, the pull-back of f on the inverse image of U in $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_{\sigma,\mathrm{val}}$ is analytic. The log structure of $\Gamma\backslash D_{\mathrm{val},(\Gamma)}$ is the following subsheaf of the sheaf of analytic functions on $\Gamma\backslash D_{\mathrm{val},(\Gamma)}$. For an open set U of $\Gamma\backslash D_{\mathrm{val},(\Gamma)}$ and for an analytic function f on U, f belongs to the log structure of $\Gamma\backslash D_{\mathrm{val},(\Gamma)}$ if and only if, for any $\sigma \in \mathcal{C}_\Gamma$, the pull-back of f on the inverse image of U in $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_{\sigma,\mathrm{val}}$ belongs to the log structure of $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_{\sigma,\mathrm{val}}$.

In the above, the structure of a log local ringed space $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_{\sigma,\mathrm{val}}$ is defined in [KNU.p] 3.2.7.

In a forthcoming paper, we will prove nice properties of $\Gamma\backslash D_{\mathrm{val},(\Gamma)}$ which are analogous to Theorem 2.3.

2.5. To state the next theorem, let Γ be a neat subgroup of $G_{\mathbf{Z}}$. We denote by $\mathrm{LMH}_{(\Lambda,\Gamma),\mathrm{val}}$ the sheaf on the category $\mathcal{B}(\log)$ ([KU09] 3.2.4) with the usual topology associated to the presheaf sending $S \in \mathcal{B}(\log)$ to the inductive limit of the sets of all isomorphism classes of log mixed Hodge structures on S' with polarized graded quotients of the given Hodge type Λ and with Γ-level structure ([KNU.p] 2.6.2), where S' runs over the sets of log modifications of S ([KU09] 3.6).

Let S be an object of $\mathcal{B}(\log)$. A morphism $f : S_{\mathrm{val}} \to \Gamma\backslash D_{\mathrm{val},(\Gamma)}$ of log local ringed spaces over $\mathbf{C}$ (see [KU09] 3.6.23 for S_{val}) is called a *good morphism* if the following condition (1) is satisfied.

(1) For any point p of S_{val}, there are an open neighborhood U of p in S_{val}, $\sigma \in \mathcal{C}_\Gamma$, and a morphism $U \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D_{\sigma,\mathrm{val}}$ which induces the restriction $f|_U$ such that the composite $U \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D_{\sigma,\mathrm{val}} \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ factors through an open subspace V of a log modification of an open subspace of S.

$$\begin{array}{ccccc} S_{\mathrm{val}} & \supset & U & \to & V \\ f\downarrow & & \downarrow & & \downarrow \\ \Gamma\backslash D_{\mathrm{val},(\Gamma)} & \leftarrow & \Gamma(\sigma)^{\mathrm{gp}}\backslash D_{\sigma,\mathrm{val}} & \to & \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma. \end{array}$$

The proofs of the following 2.6 and 2.7 will be given in §6.

Theorem 2.6. *Let Γ and S be as above. Then there is a canonical functorial bijection between $\mathrm{LMH}_{(\Lambda,\Gamma),\mathrm{val}}(S)$ and the set of good morphisms $S_{\mathrm{val}} \to \Gamma\backslash D_{\mathrm{val},(\Gamma)}$.*

We expect that all morphisms $f\colon S_{\mathrm{val}} \to \Gamma\backslash D_{\mathrm{val},(\Gamma)}$ are good (hence $\Gamma\backslash D_{\mathrm{val},(\Gamma)}$ is in fact a fine moduli), though we have not yet proved it.

Notice that in analogous theorems on fine moduli of log (mixed) Hodge structures in [KU09] and [KNU.p], we always fix a fan or a weak fan Σ ([KNU10b]) to restrict local monodromies. Thus, in those works, the theorems always lead to another problem of different nature to make such a Σ. Here in 2.6, we do not need any Σ in virtue of the valuative formulation. This is a new feature already in the pure case.

In the proof of Theorem 0.1 below, we only use the following complement 2.7 of Theorem 2.6 (which we will use in 5.1), but we expect that 2.6 should relate more closely to 0.1 and that 2.6 will play more important roles in some future applications.

Proposition 2.7. *Let $H \in \mathrm{LMH}_{(\Lambda,\Gamma),\mathrm{val}}(S)$. Then there is a natural continuous map $S^{\log}_{\mathrm{val}} \to \Gamma\backslash D^\sharp_{\mathrm{val}}$ which is compatible with the good morphism $S_{\mathrm{val}} \to \Gamma\backslash D_{\mathrm{val},(\Gamma)}$ corresponding to H by* 2.6.

3. Space $D^\sharp_{(\sigma),\mathrm{val}}$.

3.1. To prove the results in §2 later in §6, we introduce the following set $D^\sharp_{(\sigma),\mathrm{val}}$, where σ is a sharp rational nilpotent cone.

Let $D^\sharp_{(\sigma),\mathrm{val}} = \bigcup_\tau D^\sharp_{\tau,\mathrm{val}} \subset D^\sharp_{\mathrm{val}}$, where τ ranges over all rational subcones of σ.

3.2. We endow $D^\sharp_{(\sigma),\mathrm{val}}$ with the strongest topology for which the map $D^\sharp_{\tau,\mathrm{val}} \to D^\sharp_{(\sigma),\mathrm{val}}$ is continuous for any rational subcone τ of σ.

3.3. For a sharp rational nilpotent cone σ, we define the topological space $E^\sharp_{(\sigma),\mathrm{val}}$ as follows:

Let $\check{E}^\sharp_{\sigma,\mathrm{val}} = |\mathrm{toric}|_{\sigma,\mathrm{val}} \times \check{D}$. Recall that we defined $E^\sharp_{\sigma,\mathrm{val}} := |\mathrm{toric}|_{\sigma,\mathrm{val}} \times_{|\mathrm{toric}|_\sigma} E^\sharp_\sigma \subset \check{E}^\sharp_{\sigma,\mathrm{val}}$ in [KNU.p] 3.2.5.

We define the set $E^\sharp_{(\sigma),\mathrm{val}}$ such that $E^\sharp_{\sigma,\mathrm{val}} \subset E^\sharp_{(\sigma),\mathrm{val}} \subset \check{E}^\sharp_{\sigma,\mathrm{val}}$ as follows. Let τ be a rational subcone of σ. Let $U(\tau)$ be the space $|\mathrm{torus}|_\sigma \times^{|\mathrm{torus}|_\tau} E^\sharp_{\tau,\mathrm{val}} := (|\mathrm{torus}|_\sigma \times E^\sharp_{\tau,\mathrm{val}})/|\mathrm{torus}|_\tau$, where the action of $t \in |\mathrm{torus}|_\tau$ on the product is $(a, b) \mapsto (ta, t^{-1}b)$. Then we have the map $U(\tau) \to \check{E}^\sharp_{\sigma,\mathrm{val}}$ induced by the map $|\mathrm{torus}|_\sigma \times^{|\mathrm{torus}|_\tau} |\mathrm{toric}|_{\tau,\mathrm{val}} \to |\mathrm{toric}|_{\sigma,\mathrm{val}}$. We define $E^\sharp_{(\sigma),\mathrm{val}}$ as the union of the images of all such $U(\tau) \to \check{E}^\sharp_{\sigma,\mathrm{val}}$, where τ runs over the set of the rational subcones of σ.

We endow $E^\sharp_{(\sigma),\mathrm{val}}$ with the strongest topology for which the map $U(\tau) \to E^\sharp_{(\sigma),\mathrm{val}}$ is continuous for any rational subcone τ of σ.

We have a canonical projection $E^\sharp_{(\sigma),\mathrm{val}} \to D^\sharp_{(\sigma),\mathrm{val}}$.

Let $\sigma_{\mathbf{R}} \subset \mathfrak{g}_{\mathbf{R}}$ be the $\mathbf{R}$-linear span of σ. The continuous action of $i\sigma_{\mathbf{R}}$ on $E^\sharp_{\sigma,\mathrm{val}}$ in [KNU.p] 4.2.1 is naturally extended to a continuous action of $i\sigma_{\mathbf{R}}$ on $E^\sharp_{(\sigma),\mathrm{val}}$.

Theorem 3.4. (i) *The action of $i\sigma_{\mathbf{R}}$ on $E^\sharp_{(\sigma),\mathrm{val}}$ is free and proper.*

(ii) *$E^\sharp_{(\sigma),\mathrm{val}}$ is an $i\sigma_{\mathbf{R}}$-torsor over $D^\sharp_{(\sigma),\mathrm{val}}$ in the category of topological spaces.*

Theorem 3.5. *The inclusion $D^\sharp_{(\sigma),\mathrm{val}} \to D^\sharp_{\mathrm{val}}$ is an open map.*

The proofs of these theorems will be given in §6.

4. Preparation for proof of Theorem 0.1.

4.1. Let the situation be as in the hypothesis of Theorem 0.1. By resolution of singularity, we may and do assume that S is non-singular and Y is a divisor with normal crossings. We endow S with the log structure associated to Y.

To prove Theorem 0.1, it is sufficient to show the following assertion (A).

(A) Let $s \in S$. Then there is an open neighborhood U of s in S such that $\overline{Z^*} \cap U$ is an analytic subset of U.

We replace S by a small open neighborhood of s in S. Replacing S further by a finite ramified covering, which is proper, we may and do assume that the local monodromy groups along Y are unipotent.

4.2. Let $H^{(j)}$ be the variation of mixed Hodge structure on S^* corresponding to f_j. Then, $H^{(j)}$ extends uniquely to a log mixed Hodge structure on S, which we still denote by $H^{(j)}$.

We explain this extension. Let $\tau : S^{\log} \to S$ be as in [KU09] 0.2.9. First, the lattice $H^{(j)}_{\mathbf{Z}}$ extends as a locally constant sheaf over $S^{\log}$, which we still denote by the same symbol and which is isomorphic modulo Γ to the constant sheaf H_0. Further, by the admissibility of f_j and by the nilpotent orbit theorem of Schmid ([Scm73]) reformulated as in [KU09] 2.5.13–2.5.14 applied to each graded quotient gr^W_w, $H^{(j)}$ extends uniquely to a log mixed Hodge structure on S. Here we notice that, since the Griffiths transversality for $H^{(j)}$ on S^* is satisfied by definition, the Griffiths transversality on S is also satisfied. Also, we remark that the canonical extension of Deligne ([D70]) is nothing but the underlying $\mathcal{O}_S$-module $\tau_*(\mathcal{O}_S^{\log} \otimes H^{(j)}_{\mathbf{Z}})$ of the extended $H^{(j)}$.

For $s \in S$, we have the action $\rho_j : \pi_1(s^{\log}) \to \mathrm{Aut}(H_0)$ which is determined modulo Γ.

4.3. Let v be a point of S_{val} lying over s, and let V_v be the valuative submonoid of $(M_S^{\mathrm{gp}}/\mathcal{O}_S^\times)_s$ corresponding to v containing $(M_S/\mathcal{O}_S^\times)_s$. Let $\pi(v)$ be the $\mathbf{Z}$-dual of $(M_S^{\mathrm{gp}}/\mathcal{O}_S^\times)_s/V_v^\times$. If we log blow up S sufficiently around s and obtain S' then, for the image s' of v in S', there is a canonical isomorphism $\pi_1((s')^{\log}) \simeq \pi(v)$ which is compatible with the isomorphism between $\pi_1(s^{\log})$ and the $\mathbf{Z}$-dual of $(M_S^{\mathrm{gp}}/\mathcal{O}_S^\times)_s$.

Note that both $\pi_1(s^{\log})$ and $\pi(v)$ are free abelian groups of finite rank.

5. Proof of Theorem 0.1. Here we prove 0.1 using the results in §2–§3, which will be proved in §6.

5.1. Let the situation be as in 4.1. We will prove (A) in 4.1 by induction on the log rank of s.

By 4.2 and 2.7, we see that the map $f_j : S^* \to \Gamma\backslash D$ extends uniquely to a continuous map $S^{\log}_{\mathrm{val}} \to \Gamma\backslash D^\sharp_{\mathrm{val}}$ which we still denote by f_j. Let $Z = \{p \in S^{\log}_{\mathrm{val}} \mid f_1(p) = \cdots = f_n(p)\}$. Let $p \in Z$ and assume $p \in s^{\log}_{\mathrm{val}} \subset S^{\log}_{\mathrm{val}}$ (that is, p lies over $s \in S$). Since $D^\sharp_{\mathrm{val}} \to \Gamma\backslash D^\sharp_{\mathrm{val}}$ is a local homeomorphism (2.3 (ii)), there is an open neighborhood U_p of p in $S^{\log}_{\mathrm{val}}$ such that, for $1 \le j \le n$, $f_j : U_p \to \Gamma\backslash D^\sharp_{\mathrm{val}}$ lifts to a continuous map $\tilde{f}_j : U_p \to D^\sharp_{\mathrm{val}}$ and such that $Z_p := \{p' \in U_p \mid \tilde{f}_1(p') = \cdots = \tilde{f}_n(p')\} = U_p \cap Z$. We may assume that there is an open set U'_p of $S^{\log}$ such that the image of U_p in $S^{\log}$ is contained in U'_p and the $H^{(j)}_{\mathbf{Z}}$ are constant sheaves on U'_p.

Let $Z_p(s) = Z_p \cap s^{\log}_{\mathrm{val}}$. Since $s^{\log}_{\mathrm{val}}$ is compact, there is a finite subset B of $s^{\log}_{\mathrm{val}}$ such that $\bigcup_{p \in B} Z_p(s) = Z \cap s^{\log}_{\mathrm{val}}$.

We have either the following Case 1 in 5.2 or Case 2 in 5.3. For $p' \in S^{\log}_{\mathrm{val}}$ lying over s, let $\pi(p') := \pi(v)$ (4.3), where v is the image of p' in S_{val}.

5.2. Case 1. For some $p \in B$, the images of $\mathbf{Q} \otimes \pi(p') \to \mathbf{Q} \otimes \pi_1(s^{\log})$ for $p' \in Z_p(s)$ generate $\mathbf{Q} \otimes \pi_1(s^{\log})$.

In this case, all ρ_j $(1 \le j \le n)$ coincide. This is because, for any $p' \in Z_p(s)$, $\tilde{f}_1(p') = \cdots = \tilde{f}_n(p')$ by definition, hence in particular the restrictions of ρ_j to $\pi(p')$ for $1 \le j \le n$ coincide.

In such a case, we can apply [KNU.p] 7.5.1 (ii) to f_j simultaneously. More precisely, let Γ' be the image of $\pi_1(s^{\log})$ in $\mathrm{Aut}(H_0)$ under the common representations ρ_j $(1 \le j \le n)$. Then, for every j, the period map $f_j : S^* \to \Gamma\backslash D$ lifts to $S^* \to \Gamma'\backslash D$.

Let C be the union over $1 \le j \le n$ of the sets of local monodromy cones of $H^{(j)}$ in $\mathfrak{g}_{\mathbf{Q}}$ (cf. [KU09] 2.5.11). Using the argument in the proof of [KU09] 4.3.8, we can construct a fan Σ in $\mathfrak{g}_{\mathbf{Q}}$ which satisfies the following conditions (1)–(3) (cf. [KU09] 4.3.6).

(1) $\bigcup_{\sigma \in \Sigma} \sigma = \bigcup_{\sigma \in C} \sigma$.

(2) Σ is compatible with Γ'.

(3) For any $\sigma \in C$, $\sigma = \bigcup_{j=1}^m \tau_j$ for some $m \ge 1$ and for some $\tau_j \in \Sigma$.

In fact, we subdivide each $\sigma \in C$ into sharp cones and get a set B of sharp cones. For each $\sigma \in B$, take a finite proper fan Σ_σ of $\mathfrak{g}_{\mathbf{Q}}$ a member of which is σ. Let Σ' be the coarsest proper fan of $\mathfrak{g}_{\mathbf{Q}}$ which is a subdivision of Σ_σ for any $\sigma \in B$. Then the unique subfan Σ of Σ' supported by the union of all $\sigma \in B$ is the desired one.

Then, similarly to [KU09] 4.3.6, Σ is strongly compatible with Γ' and the liftings $S^* \to \Gamma'\backslash D$ of the period maps f_j extend to $S(\Sigma) \to \Gamma'\backslash D_\Sigma$, where $S(\Sigma)$ is the log modification of S corresponding to Σ

([KU09] 3.6). From this and the properness of $S(\Sigma) \to S$, we have (A).

5.3. Case 2. For any $p \in B$, the images of $\mathbf{Q} \otimes \pi(p') \to \mathbf{Q} \otimes \pi_1(s^{\log})$ for $p' \in Z_p(s)$ do not generate $\mathbf{Q} \otimes \pi_1(s^{\log})$.

In this case, for each $p \in B$, take a non-trivial $\mathbf{Q}$-linear map $h_p : \mathbf{Q} \otimes \pi_1(s^{\log}) \to \mathbf{Q}$ which kills the images $\mathbf{Q} \otimes \pi(p')$ for all $p' \in Z_p(s)$. For each $p \in B$, take the division of $\pi_1^+(s^{\log})$ by $A_p^0 := \{x \in \pi_1^+(s^{\log}) \mid h_p(x) = 0\}$, $A_p^+ := \{x \in \pi_1^+(s^{\log}) \mid h_p(x) \geq 0\}$, $A_p^- := \{x \in \pi_1^+(s^{\log}) \mid h_p(x) \leq 0\}$. Then we have clearly

(P) For $p' \in Z_p(s)$, the image of $\mathbf{Q} \otimes \pi(p')$ in $\mathbf{Q} \otimes \pi_1(s^{\log})$ is contained in $\mathbf{Q} \otimes A_p^0$.

Define the subdivision of $\pi_1^+(s^{\log})$ consisting of all cones of the form $R \cap \bigcap_{p \in B} A_p^{c(p)}$, where R is a face of $\pi_1^+(s^{\log})$ and $c(p) \in \{0, +, -\}$. Let S' be the log modification of an open neighborhood of s in S corresponding to this subdivision. Let S'' be a log modification of S' which is smooth.

By (P), any point of the closure of Z^* in S'' lying over s is of log rank strictly smaller than the log rank of s.

By the induction on the log rank, we can complete the proof of (A) and hence the proof of Theorem 0.1.

6. Proofs of theorems in §2 and §3. The proofs of the theorems in §2–§3 go on in a similar way as those of the theorems in [KNU.p].

6.1. Proof of Theorem 3.4. The freeness in (i) is reduced to [KNU.p] 4.2.2 (ii). The properness in (i) is an analogue of ibid. 4.2.5, and is similarly reduced to the variant of ibid. 4.2.3 (ii) with the subscripts "(σ)" instead of "σ". The proof of this variant is the same. Note that, by the definition of the topology of $D^\sharp_{\mathrm{val}}$ in 2.2, the map $\psi : D^\sharp_{\mathrm{val}} \to D^I_{\mathrm{SL}(2)}$ in [KNU.p] 3.3 is continuous. Note also that the topology of $E^\sharp_{(\sigma),\mathrm{val}}$ given in 3.3 is stronger or equal to the strong topology as a subset of $\check{E}^\sharp_{\sigma,\mathrm{val}}$.

The assertion (ii) is an analogue of [KNU.p] 4.2.8. By ibid. 4.2.7 (a) and by the assertion (i), it is enough to check the condition (1) in ibid. 4.2.7 (with $H = i\sigma_{\mathbf{R}}$ and $X = E^\sharp_{(\sigma),\mathrm{val}}$), which is done similarly as follows. Let $x = (q, F) \in E^\sharp_{(\sigma),\mathrm{val}}$. Let $A \subset \check{E}_\sigma$ be the analytic space containing the image of x constructed in the same way as in the pure case [KU09] 7.3.5. Let S_1 be the pull-back of A in $E^\sharp_{(\sigma),\mathrm{val}}$, let U be a sufficiently small neighborhood of 0 in $\sigma_{\mathbf{R}}$, and let $S = \{(q', \exp(a)F') \mid (q', F') \in S_1, a \in U\}$. Then, S satisfies the condition (1) in [KNU.p] 4.2.7.

6.2. Proof of Theorem 3.5 This is an analogue of [KNU.p] 4.3.1. In the same way as there, we reduce 3.5 to the following (1).

(1) If σ is a sharp rational nilpotent cone and τ is a rational subcone, then the inclusion map $D^\sharp_{(\tau),\mathrm{val}} \to D^\sharp_{(\sigma),\mathrm{val}}$ is an open map.

The proof of (1) is also similar. We use the fact that the topology of $D^\sharp_{(\sigma),\mathrm{val}}$ is the quotient topology from $E^\sharp_{(\sigma),\mathrm{val}}$.

6.3. Proof of Theorem 2.3. The assertion (i) is an analogue of [KNU.p] 4.3.3. By 3.5, it is sufficient to prove the variant of (1) in the proof of ibid. 4.3.3 with the subscripts "(σ)" by "σ". Similarly to ibid. 4.3.3, by 3.4 (ii) (which is proved in 6.1), this variant of ibid. 4.3.3 (1) is reduced to the variant of ibid. 4.2.3 (i) used in 6.1.

The assertion (ii) is an analogue of [KNU.p] 4.3.6. As we have seen in 6.1, the map $\psi : D^\sharp_{\mathrm{val}} \to D^I_{\mathrm{SL}(2)}$ is continuous. By [KNU11] 3.5.17, the action of Γ on $D^I_{\mathrm{SL}(2)}$ is proper. By [KU09] 7.2.6 (ii), the properness of the action of Γ on $D^\sharp_{\mathrm{val}}$ follows from these two facts together with 2.3 (i). Then, $\Gamma \backslash D^\sharp_{\mathrm{val}}$ is Hausdorff by ibid. 7.2.4. By ibid. 7.2.5, the last assertion follows from the above result and the freeness of the action of Γ on $D^\sharp_{\mathrm{val}}$ which is easily derived from [KNU.p] 4.3.5 (i).

The next is a complement to [KNU.p].

Proposition 6.4. *Let* Γ *be a subgroup of* $G_{\mathbf{Z}}$ *and let* Σ *be a fan in* $\mathfrak{g}_{\mathbf{Q}}$ ([KNU10a] 2.1.2) *which is strongly compatible with* Γ ([KNU10a] 2.1.5). *Then the following* (i) *and* (ii) *hold.*

(i) *There is a canonical isomorphism* $(\Gamma \backslash D_\Sigma)_{\mathrm{val}} = \Gamma \backslash D_{\Sigma,\mathrm{val}}$ *of log local ringed spaces over* $\mathbf{C}$.

(ii) *There is a canonical homeomorphism* $(\Gamma \backslash D_\Sigma)^{\log}_{\mathrm{val}} = \Gamma \backslash D^\sharp_{\Sigma,\mathrm{val}}$ *of topological spaces.*

For the definitions of the right hand sides, see [KNU.p] 3.2.7.

Proof. Let $\sigma \in \Sigma$, and we may replace Γ by $\Gamma(\sigma)^{\mathrm{gp}}$ and Σ by σ.

We prove (i). Let $\sigma_{\mathbf{C}} \subset \mathfrak{g}_{\mathbf{C}}$ be the $\mathbf{C}$-linear span of σ. By [KNU.p] 2.5.3, $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ is the quotient $\sigma_{\mathbf{C}} \backslash E_\sigma$. By this, we see that $(\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma)_{\mathrm{val}}$ is the quotient $\sigma_{\mathbf{C}} \backslash E_{\sigma,\mathrm{val}}$. On the other hand, it is seen similarly as ibid. 2.5.3 that $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_{\sigma,\mathrm{val}}$ is also the quotient $\sigma_{\mathbf{C}} \backslash E_{\sigma,\mathrm{val}}$ (cf. ibid. 4.4.6). Hence we have (i).

The assertion (ii) is similarly proved by using [KNU.p] 4.2.8 instead of ibid. 2.5.3. □

6.5. Proof of Theorem 2.6. This is an analogue of [KNU.p] 2.6.6 (whose pure Hodge theoretic version is [KU09] Theorem B), and is reduced to [KNU.p] 2.6.6 as follows:

Assume that we are given an element H of $\mathrm{LMH}_{(\Lambda,\Gamma),\mathrm{val}}(S)$. We provide the corresponding good morphism $S_{\mathrm{val}} \to \Gamma\backslash D_{\mathrm{val},(\Gamma)}$. Assume first that H comes from an LMH over S. Then, by the mixed Hodge theoretic version of [KU09] 4.3.8 (which is used in the proof of [KNU.p] 7.5.1) and by [KNU.p] 2.6.6, locally on S, there are a log modification S', a subgroup Γ' of Γ, and a fan Σ in $\mathfrak{g}_{\mathbf{Q}}$ which is strongly compatible with Γ' such that H comes from a morphism $S' \to \Gamma'\backslash D_\Sigma$. Taking $(-)_{\mathrm{val}}$, we have $(S')_{\mathrm{val}} \to (\Gamma'\backslash D_\Sigma)_{\mathrm{val}} = \Gamma'\backslash D_{\Sigma,\mathrm{val}} \to \Gamma\backslash D_{\mathrm{val},(\Gamma)}$ by 6.4 (i). In the general case, the above construction glues to give the desired morphism.

Conversely, let $S_{\mathrm{val}} \to \Gamma\backslash D_{\mathrm{val},(\Gamma)}$ be a good morphism. Then, again by [KNU.p] 2.6.6, locally on S, we have LMHs of type Λ with Γ-level structures on various open subspaces of various log modifications. Since $S_{\mathrm{val}} \to S$ is proper ([KU09] 3.6.24), locally on S, these LMHs glue into an LMH on one log modification of S. Thus we get an element of $\mathrm{LMH}_{(\Lambda,\Gamma),\mathrm{val}}(S)$.

6.6. Proof of Proposition 2.7. The proof goes exactly in parallel to the former half of 6.5, by using 6.4 (ii) instead of 6.4 (i). Let $S' \to \Gamma'\backslash D_\Sigma$ be the morphism in 6.5 from which H comes locally. Taking $(-)^{\log}_{\mathrm{val}}$ this time, we have $(S')^{\log}_{\mathrm{val}} \to (\Gamma'\backslash D_\Sigma)^{\log}_{\mathrm{val}} = \Gamma'\backslash D^\sharp_{\Sigma,\mathrm{val}} \to \Gamma\backslash D^\sharp_{\mathrm{val}}$ by 6.4 (ii). The rest is the same.

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