# On the Hasegawa–Wakatani equations with vanishing resistivity

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**Abstract:** In this paper, we are concerned with the drift wave turbulence in a strong magnetic field. We prove the existence and uniqueness of a strong global in time solution to the initial boundary value problem for the model equations of drift wave turbulence similar to Hasegawa–Mima equation. Then we prove that the solution of Hasegawa–Wakatani equations established in [5] converges to that of Hasegawa–Mima like equation established at the first stage as the resistivity tends to zero on some time interval.

**Key words:** Hasegawa–Wakatani equations; Hasegawa–Mima equation; drift wave turbulence in Tokamak; Sobolev spaces.

**1. Introduction.** Tokamak is the most advanced magnetic confinement device, in which an axisymmetric plasma is confined by a strong magnetic field. It has been well known that the spatial gradients in plasma lead to the drift waves and the drift wave turbulence is a natural cause of anomalous transport from which the dramatic reduction in confinement results. Thereby the analysis of such drift wave turbulences is important from various point of view.

In order to describe the resistive drift wave turbulence in Tokamak, Hasegawa and Wakatani [3,4] proposed in 1983 the following equations for the perturbations of plasma density $n$ and the electrostatic potential $\phi$:

$$(1.1)\quad \begin{cases} \left(\dfrac{\partial}{\partial t} - (\nabla\phi\times\mathbf{e})\cdot\nabla\right)\Delta\phi \\ \qquad = -\dfrac{c_1}{\bar{n}}\dfrac{\partial^2}{\partial x_3^2}(\phi - n) + c_2\Delta^2\phi, \\ \left(\dfrac{\partial}{\partial t} - (\nabla\phi\times\mathbf{e})\cdot\nabla\right)(n + \log\bar{n}) \\ \qquad = -\dfrac{c_1}{\bar{n}}\dfrac{\partial^2}{\partial x_3^2}(\phi - n) \end{cases}$$

(Hasegawa–Wakatani equations) from the two fluids model in a homogeneous strong magnetic field $\mathbf{B} = B_0\mathbf{e}$ and an inhomogeneous plasma equilibrium density $\bar{n} = \bar{n}(|x'|)$ $(x = (x_1, x_2, x_3) = (x', x_3))$. Here $B_0$ is the strength of a magnetic field assumed to be a constant, $\mathbf{e} = (0,0,1)$, $c_1 = T_e/(e^2\eta\omega_{ci})$, $c_2 = \mu/(\rho_s^2\omega_{ci})$, $T_e$ is the electron temperature, $e$ is the elementary charge, $\mu$ is the kinematic ion-viscosity coefficient, $\eta$ is the resistivity, $m_i$ is the ion mass, $\omega_{ci} = eB_0/m_i$ is the cyclotoron frequency and $\rho_s = \sqrt{T_e}/(\omega_{ci}\sqrt{m_i})$ is the ion Larmor radius. For simplicity we assume that $c_1$ and $c_2$ are positive constants.

In advance of Hasegawa–Wakatani equations Hasegawa and Mima [1, 2] in 1977 proposed the equation

$$(1.2)\quad \left(\frac{\partial}{\partial t} - (\nabla\phi\times\mathbf{e})\cdot\nabla\right)(\Delta\phi - \phi - \log\bar{n}) = 0$$

(Hasegawa–Mima equation) from the one fluid model under the same magnetic field and plasma equilibrium state.

Concerning the mathematical results for (1.1) and (1.2) we refer to [5] and references therein.

By differencing the first and the second equations of (1.1) and by denoting $\varepsilon = 1/c_1$, (1.1) is equivalent to

$$(1.3)\quad \begin{cases} \left(\dfrac{\partial}{\partial t} - (\nabla\phi\times\mathbf{e})\cdot\nabla\right)(\Delta\phi - n - \log\bar{n}) \\ \qquad = c_2\Delta^2\phi, \\ \varepsilon\left(\dfrac{\partial}{\partial t} - (\nabla\phi\times\mathbf{e})\cdot\nabla\right)(n + \log\bar{n}) \\ \qquad = -\dfrac{1}{\bar{n}}\dfrac{\partial^2}{\partial x_3^2}(\phi - n). \end{cases}$$

For given an initial electrostatic potential $\phi_0^\varepsilon$, an initial plasma density $n_0^\varepsilon$ and the background density $\bar{n} = \bar{n}(|x'|)$, let $(\phi^\varepsilon, n^\varepsilon) = (\phi^\varepsilon, n^\varepsilon)(x,t)$ be a

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solution of the initial boundary value problem (1.3) with $\varepsilon > 0$ for $x \in \Omega$, $t > 0$ and the initial-boundary conditions

$$(1.4)\quad \begin{cases} \phi^\varepsilon(x,0) = \phi_0^\varepsilon(x),\ n^\varepsilon(x,0) = n_0^\varepsilon(x) \\ \qquad\qquad\qquad\qquad (x \in \Omega), \\ \phi^\varepsilon(x,t) = \Delta\phi^\varepsilon(x,t) = n^\varepsilon(x,t) = 0 \\ \qquad\qquad\qquad\qquad (x \in \Gamma,\ t > 0), \\ \phi^\varepsilon,\ n^\varepsilon, \quad \text{periodic in the } x_3\text{-direction.} \end{cases}$$

Here $\Omega = \omega \times (-L, L)$ is a 3-dimensional torus, $\omega = \{x' = (x_1, x_2) \in \mathbf{R}^2 \mid |x'| < R\}$, $\partial\omega = \{x' = (x_1, x_2) \in \mathbf{R}^2 \mid |x'| = R\}$, $\Gamma = \partial\omega \times [-L, L]$, $R$ and $L$ are positive numbers.

It is more convenient to change $n^\varepsilon(x,t)$ and $n_0^\varepsilon(x)$ by $n^\varepsilon(x,t) + \log\bar{n}(|x'|) - \log\bar{n}(R)$ and $n_0^\varepsilon(x) + \log\bar{n}(|x'|) - \log\bar{n}(R)$, respectively, which are denoted again by the same letters $n^\varepsilon(x,t)$ and $n_0^\varepsilon(x)$. Then (1.3) becomes

$$(1.5)\quad \begin{cases} \left(\dfrac{\partial}{\partial t} - (\nabla\phi^\varepsilon \times \mathbf{e}) \cdot \nabla\right)(\Delta\phi^\varepsilon - n^\varepsilon) \\ \qquad = c_2\Delta^2\phi^\varepsilon, \\ \varepsilon\left(\dfrac{\partial}{\partial t} - (\nabla\phi^\varepsilon \times \mathbf{e}) \cdot \nabla\right)n^\varepsilon \\ \qquad = -\dfrac{1}{\bar{n}}\dfrac{\partial^2}{\partial x_3^2}(\phi^\varepsilon - n^\varepsilon) \quad (x \in \Omega,\ t > 0), \end{cases}$$

while (1.4) remains unchanged.

Putting $\varepsilon = 0$ in (1.5) implies

$$(1.6)\quad \begin{cases} \left(\dfrac{\partial}{\partial t} - (\nabla\phi^0 \times \mathbf{e}) \cdot \nabla\right)(\Delta\phi^0 - n^0) \\ \qquad = c_2\Delta^2\phi^0, \\ \dfrac{1}{\bar{n}}\dfrac{\partial^2}{\partial x_3^2}(\phi^0 - n^0) = 0 \quad (x \in \Omega,\ t > 0). \end{cases}$$

We seek a solution of (1.6) satisfying an additional condition

$$(1.7)\quad \overline{n^0}(x',t) \equiv \frac{1}{2L}\int_{-L}^{L} n^0(x', x_3, t)\,\mathrm{d}x_3 = 0.$$

The aim of this paper is to establish the unique existence of a strong global in time solution to the initial boundary value problem for (1.6), (1.7) and the convergence of $(\phi^\varepsilon, n^\varepsilon)$ to $(\phi^0, n^0)$ as $\varepsilon$ tends to zero on some time interval, which corresponds to the vanishing resistivity of Hasegawa–Wakatani equations.

We consider these problems in Sobolev spaces $W_2^l(\Omega)$ $(l = 0, 1, 2, \dots)$ defined as follows. Let $\Omega$ be a domain in $\mathbf{R}^m$ $(m \in \mathbf{N})$. By $W_2^l(\Omega)$ we denote the space of functions $u(x)$, $x \in \Omega$, equipped with the finite norm

$$\|u\|^2_{W_2^l(\Omega)} = \sum_{|k| \le l} \|D_x^k u\|^2_{L^2(\Omega)}.$$

Here $D_x^k u = \partial^{|k|}u/\partial x_1^{k_1} \dots \partial x_m^{k_m}$ is the generalized derivative of order $|k| = k_1 + k_2 + \dots + k_m$ for a multi-index $k = (k_1, k_2, \dots, k_m)$. For $1 \le p < \infty$, we denote by $\|\cdot\|_{L^p(\Omega)}$ the norm of the Lebesgue space $L^p(\Omega)$. For simplicity, let us denote $\|\cdot\|_{L^2(\Omega)}$ by $\|\cdot\|$.

Let $T > 0$ and the anisotropic Sobolev space $W_2^{2l,l}(Q_T)$ $(Q_T \equiv \Omega \times (0,T),\ l = 0, 1, 2, \dots)$ is defined as $L^2(0,T; W_2^{2l}(\Omega)) \cap L^2(\Omega; W_2^l(0,T))$, equipped with the finite norm

$$\|u\|^2_{W_2^{2l,l}(Q_T)} = \|u\|^2_{W_2^{2l,0}(Q_T)} + \|u\|^2_{W_2^{0,l}(Q_T)} \equiv \int_0^T \|u\|^2_{W_2^{2l}(\Omega)}\,\mathrm{d}t + \int_\Omega \|u\|^2_{W_2^l(0,T)}\,\mathrm{d}x.$$

Our first result for problem (1.5), (1.4) is as follows:

**Theorem 1.1.** *Let $\varepsilon > 0$ and $\bar{n}(|x'|) \in W_2^2(\omega)$ satisfy $\bar{n}(|x'|) \ge n_*$ with a positive constant $n_*$. Assume that $(\phi_0^\varepsilon, n_0^\varepsilon) \in W_2^4(\Omega) \times W_2^2(\Omega)$ satisfies the compatibility conditions*

$$(1.8)\quad \begin{cases} \phi_0^\varepsilon(x) = \Delta\phi_0^\varepsilon(x) = n_0^\varepsilon(x) = 0 \quad (x \in \Gamma), \\ \phi_0^\varepsilon,\ n_0^\varepsilon, \quad \textit{periodic in the } x_3\textit{-direction.} \end{cases}$$

*Then there exists a unique solution $(\phi^\varepsilon, n^\varepsilon)$ to problem (1.5), (1.4) on some time interval $[0,T]$ such that $(\phi^\varepsilon, n^\varepsilon) \in L^2(0,T; W_2^4(\Omega)) \times W_2^{2,1}(Q_T)$, $\partial\phi^\varepsilon/\partial t \in L^2(0,T; W_2^2(\Omega))$. Here $T$ is a constant independent of $\varepsilon$.*

Next, it is clear that the second equation of (1.6) and the periodicity condition in $x_3$ imply $(\phi^0 - n^0)(x', x_3, t) = -f(x',t)$ for any smooth function $f(x',t)$, and hence $(\overline{\phi^0} - \overline{n^0})(x',t) = -f(x',t)$. This together with (1.7) yields $n^0(x,t) = \phi^0(x,t) - \overline{\phi^0}(x',t)$. Therefore it is obvious that the problem (1.6), (1.7), (1.4) with $\varepsilon = 0$ is equivalent to problem

$$(1.9)\quad \begin{cases} \left(\dfrac{\partial}{\partial t} - (\nabla\phi^0 \times \mathbf{e}) \cdot \nabla\right)(\Delta\phi^0 - \phi^0 + \overline{\phi^0}) \\ \qquad = c_2\Delta^2\phi^0 \quad (x \in \Omega,\ t > 0), \\ \phi^0(x,0) = \phi_0^0(x) \quad (x \in \Omega), \\ \phi^0(x,t) = \Delta\phi^0(x,t) = 0 \quad (x \in \Gamma,\ t > 0), \\ \phi^0, \quad \text{periodic in the } x_3\text{-direction.} \end{cases}$$

It is to be noted that the equation in (1.9) is similar to the Hasegawa–Mima equation (1.2) with an higher order correction term.

Our second result is the following.

**Theorem 1.2.** *Let $T$ be any positive number. Assume that $\phi_0^0 \in W_2^3(\Omega)$ satisfies the compatibility conditions* (1.8) *with $\varepsilon = 0$. Then there exists a unique solution $\phi^0$ to the problem* (1.9) *on $[0,T]$ such that $\phi^0 \in L^2(0,T;W_2^4(\Omega))$, $\partial\phi^0/\partial t \in L^2(0,T;W_2^2(\Omega))$.*

For this solution $\phi^0$ let $n^0(x,t) = \phi^0(x,t) - \overline{\phi^0}(x',t)$ and $n_0^0(x) = \phi_0^0(x) - \overline{\phi^0}(x',0)$. Then it is easily seen that $(\phi^0, n^0)$ satisfies (1.6), (1.7) and (1.4) with $\varepsilon = 0$.

Finally the following is our main result.

**Theorem 1.3.** *Let $(\phi^\varepsilon, n^\varepsilon)$ and $(\phi^0, n^0)$ be the solutions established in Theorems* 1.1 *and* 1.2*, respectively. If the initial data $(\phi_0^\varepsilon, n_0^\varepsilon) \to (\phi_0^0, n_0^0)$ as $\varepsilon \to 0$ in $W_2^3(\Omega) \times W_2^2(\Omega)$, then $(\phi^\varepsilon, n^\varepsilon) \to (\phi^0, n^0)$ as $\varepsilon \to 0$ in $L^2(0,T;W_2^4(\Omega)) \times W_2^{2,1}(Q_T)$ and $\Delta\phi^\varepsilon - n^\varepsilon \to \Delta\phi^0 - n^0$ as $\varepsilon \to 0$ in $W_2^{2,1}(Q_T)$ on the same time interval $[0,T]$ as in Theorem* 1.1.

Here we give only brief proofs, whose details will be published elsewhere [6].

This paper is organized as follows. In §2 we prove Theorem 1.1 from our result in [5] and the *a priori* estimates for problem (1.5), (1.4). In §3 Theorem 1.2 is proved through the local in time existence and *a priori* estimates. In §4 we give a proof of Theorem 1.3 by virtue of *a priori* estimates, Theorems 1.1 and 1.2.

**2. Proof of Theorem 1.1.** For the initial boundary value problem (1.1) for $x \in \Omega$, $t > 0$ and (1.4), we have the following existence theorem in [5]:

**Theorem 2.1.** *Let $\bar{n}(|x'|) \in W_2^2(\omega)$ satisfy $\bar{n}(|x'|) \geq n_*$ with a positive constant $n_*$. Assume that $(\phi_0, n_0) \in W_2^4(\Omega) \times W_2^2(\Omega)$ satisfies the compatibility conditions* (1.8)*. Then there exists a unique solution $(\phi, n)$ to problem* (1.1) *for $x \in \Omega$, $t > 0$ and* (1.4) *on some interval $[0, T^*]$ such that $(\phi, n) \in L^2(0,T^*;W_2^4(\Omega)) \times W_2^{2,1}(Q_{T^*})$, $\partial\phi/\partial t \in L^2(0,T^*;W_2^2(\Omega))$.*

We denote the solution $(\phi, n)$ established in Theorem 2.1 in the case of $c_1 = 1/\varepsilon$ by $(\phi^\varepsilon, n^\varepsilon)$. Then it is easy to see that $(\phi^\varepsilon, n^\varepsilon)$ is also the solution of the problem (1.5), (1.4). Since $T^*$ in Theorem 2.1 may depend on $\varepsilon$, to complete the proof of Theorem 1.1, it is sufficient to show that $T^*$ can be taken independently of $\varepsilon$.

We proceed to get *a priori* estimates of the solution $(\phi^\varepsilon, n^\varepsilon)$. Let it belong to $(L^2(0,T;W_2^4(\Omega)) \cap W_2^1(0,T;W_2^2(\Omega))) \times W_2^{2,1}(Q_T)$ for $T > 0$.

First we prove

**Lemma 2.1.** *For any $t \in [0,T]$*

$$(2.1)\quad \|\nabla\phi^\varepsilon(t)\|^2 + \|n^\varepsilon(t)\|^2 + c_2\int_0^t \|\Delta\phi^\varepsilon(\tau)\|^2\,\mathrm{d}\tau \leq \|\nabla\phi_0^\varepsilon\|^2 + \|n_0^\varepsilon\|^2,$$

$$(2.2)\quad \int_0^t \left\|\frac{\partial(\phi^\varepsilon - n^\varepsilon)}{\partial x_3}(\tau)\right\|^2 \mathrm{d}\tau \leq c\varepsilon(\|\nabla\phi_0^\varepsilon\|^2 + \|n_0^\varepsilon\|^2).$$

*Here $c$ is a constant independent of $\varepsilon$.*

*Proof.* Multiplying the first equation of (1.5) by $\phi^\varepsilon$ and integrating over $\Omega$, we have, by virtue of integration by parts,

$$(2.3)\quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\phi^\varepsilon(t)\|^2 + c_2\|\Delta\phi^\varepsilon(t)\|^2 = -\int_\Omega \frac{\partial n^\varepsilon}{\partial t}\,\phi^\varepsilon\,\mathrm{d}x.$$

Multiplying the second equation of (1.5) by $\phi^\varepsilon - n^\varepsilon$ and integrating over $\Omega$, we have

$$(2.4)\quad \varepsilon\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|n^\varepsilon(t)\|^2 + \left\|\left(\frac{1}{\bar{n}}\right)^{\frac{1}{2}}\frac{\partial(\phi^\varepsilon - n^\varepsilon)}{\partial x_3}(t)\right\|^2 = \varepsilon\int_\Omega \frac{\partial n^\varepsilon}{\partial t}\,\phi^\varepsilon\,\mathrm{d}x.$$

Adding (2.4) and (2.3) multiplied by $\varepsilon$ and integrating it over $[0,T]$, we have (2.1) and (2.2). □

Next we prove

**Lemma 2.2.** *There exists a positive constant $T$ independent of $\varepsilon$ such that the estimate*

$$(2.5)\quad \varepsilon(\|\nabla n^\varepsilon(t)\|^2 + \|\Delta\phi^\varepsilon(t)\|^2 + \|\Delta n^\varepsilon(t)\|^2 + \|\nabla\Delta\phi^\varepsilon(t)\|^2) + \int_0^t \Bigg(c_2\varepsilon(\|\nabla\Delta\phi^\varepsilon(\tau)\|^2 + \|\Delta^2\phi^\varepsilon(\tau)\|^2) + \left\|\frac{\partial\nabla(\phi^\varepsilon - n^\varepsilon)}{\partial x_3}(\tau)\right\|^2 + \left\|\frac{\partial\Delta(\phi^\varepsilon - n^\varepsilon)}{\partial x_3}(\tau)\right\|^2\Bigg)\mathrm{d}\tau$$
$$\leq \varepsilon\left[\left(\frac{1}{S_0^* + \|\nabla\phi_0^\varepsilon\|^2 + \|n_0^\varepsilon\|^2} - ct\right)^{-1} - (\|\nabla\phi_0^\varepsilon\|^2 + \|n_0^\varepsilon\|^2)\right]$$

*holds for any $t \in [0,T)$. Here $c$ is a constant independent of $\varepsilon$ and $S_0^* \equiv \|\nabla n_0^\varepsilon\|^2 + \|\Delta\phi_0^\varepsilon\|^2 + \|\Delta n_0^\varepsilon\|^2 + \|\nabla\Delta\phi_0^\varepsilon\|^2 + c(\|\nabla\phi_0^\varepsilon\|^2 + \|n_0^\varepsilon\|^2 + 1)$.*

*Proof.* In the similar way to Lemma 2.1, multiplying the first and the second equations of (1.5) by $\Delta\phi^\varepsilon$ and $\Delta(\phi^\varepsilon - n^\varepsilon)$, respectively, and integrating over $\Omega$, we have

$$(2.6)\quad \varepsilon\biggl(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla n^\varepsilon(t)\|^2+\|\Delta\phi^\varepsilon(t)\|^2)
+\frac{3c_2}{4}\|\nabla\Delta\phi^\varepsilon(t)\|^2\biggr)+\frac{1}{2}\biggl\|\Bigl(\frac{1}{\bar n}\Bigr)^{\frac12}\frac{\partial\nabla(\phi^\varepsilon-n^\varepsilon)}{\partial x_3}(t)\biggr\|^2
\le\varepsilon\biggl(\frac{c_2}{4}\|\Delta\phi^\varepsilon(t)\|^2+c\Bigl(\|\Delta n^\varepsilon(t)\|^4+\|\nabla n^\varepsilon(t)\|^4\Bigr)\biggr)
+c\biggl\|\frac{\partial(\phi^\varepsilon-n^\varepsilon)}{\partial x_3}(t)\biggr\|^2.$$

Next multiply the first equation of (1.5) by $\Delta^2\phi^\varepsilon$ and integrate it over $\Omega$, apply Laplacian $\Delta$ to the second equation of (1.5), multiply it by $\Delta(\phi^\varepsilon-n^\varepsilon)$ and integrate it over $\Omega$. From these we have

$$(2.7)\quad \varepsilon\biggl(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Delta n^\varepsilon(t)\|^2+\|\nabla\Delta\phi^\varepsilon(t)\|^2)
+\frac{c_2}{2}\|\Delta^2\phi^\varepsilon(t)\|^2\biggr)+\frac{1}{4}\biggl\|\Bigl(\frac{1}{\bar n}\Bigr)^{\frac12}\frac{\partial\Delta(\phi^\varepsilon-n^\varepsilon)}{\partial x_3}(t)\biggr\|^2
\le\varepsilon\biggl(\frac{c_2}{4}\|\nabla\Delta\phi^\varepsilon(t)\|^2+c(\|\nabla\Delta\phi^\varepsilon(t)\|^4+\|\Delta n^\varepsilon(t)\|^4
+\|\nabla n^\varepsilon(t)\|^4)\biggr)+c\biggl\|\Bigl(\frac{1}{\bar n}\Bigr)^{\frac12}\frac{\partial\nabla(\phi^\varepsilon-n^\varepsilon)}{\partial x_3}(t)\biggr\|^2
+c\biggl\|\frac{\partial(\phi^\varepsilon-n^\varepsilon)}{\partial x_3}(t)\biggr\|^2+c(\|\nabla\phi_0^\varepsilon\|^2+\|n_0^\varepsilon\|^2)^2.$$

Adding (2.7) and (2.6) multiplied by $c$ yields

$$(2.8)\quad \frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}(c\|\nabla n^\varepsilon(t)\|^2+c\|\Delta\phi^\varepsilon(t)\|^2
+\|\Delta n^\varepsilon(t)\|^2+\|\nabla\Delta\phi^\varepsilon(t)\|^2)
+\frac{\varepsilon c_2}{2}(c\|\nabla\Delta\phi^\varepsilon(t)\|^2+\|\Delta^2\phi^\varepsilon(t)\|^2)
+c\biggl\|\Bigl(\frac{1}{\bar n}\Bigr)^{\frac12}\frac{\partial\nabla(\phi^\varepsilon-n^\varepsilon)}{\partial x_3}(t)\biggr\|^2
+\frac{1}{4}\biggl\|\Bigl(\frac{1}{\bar n}\Bigr)^{\frac12}\frac{\partial\Delta(\phi^\varepsilon-n^\varepsilon)}{\partial x_3}(t)\biggr\|^2
\le c\varepsilon(\|\nabla\Delta\phi^\varepsilon(t)\|^4+\|\Delta n^\varepsilon(t)\|^4
+\|\nabla n^\varepsilon(t)\|^4+(\|\nabla\phi_0^\varepsilon\|^2+\|n_0^\varepsilon\|^2)^2)
+c\biggl\|\frac{\partial(\phi^\varepsilon-n^\varepsilon)}{\partial x_3}(t)\biggr\|^2+c\varepsilon\|\Delta\phi^\varepsilon(t)\|^2.$$

Putting $S(t)\equiv\|\nabla n^\varepsilon(t)\|^2+\|\Delta\phi^\varepsilon(t)\|^2+\|\Delta n^\varepsilon(t)\|^2+\|\nabla\Delta\phi^\varepsilon(t)\|^2$ and integrating (2.8) over $[0,t]$, we have with the help of (2.1), (2.2)

$$S(t)\le c\int_0^t S^2(\tau)\,\mathrm{d}\tau+c(\|\nabla\phi_0^\varepsilon\|^2+\|n_0^\varepsilon\|^2+1)
+c(\|\nabla\phi_0^\varepsilon\|^2+\|n_0^\varepsilon\|^2)^2t+S(0)
\equiv c\int_0^t S^2(\tau)\,\mathrm{d}\tau+C(t)\equiv S^*(t).$$

Then $S^*(t)$ satisfies the differential inequality

$$\frac{\mathrm{d}S^*(t)}{\mathrm{d}t}\le c\Bigl(S^*(t)+\|\nabla\phi_0^\varepsilon\|^2+\|n_0^\varepsilon\|^2\Bigr)^2,$$

from which it follows

$$S^*(t)\le\biggl(\frac{1}{S_0^*+\|\nabla\phi_0^\varepsilon\|^2+\|n_0^\varepsilon\|^2}-ct\biggr)^{-1}
-\Bigl(\|\nabla\phi_0^\varepsilon\|^2+\|n_0^\varepsilon\|^2\Bigr),\quad S_0^*=C(0).$$

Now we take $T$ in such a way that $T=[c(S_0^*+\|\nabla\phi_0^\varepsilon\|^2+\|n_0^\varepsilon\|^2)]^{-1}$. Then we conclude (2.5) on $[0,T)$ from this inequality and (2.8). □

By the standard arguments based upon the *a priori* estimates in Lemmas 2.1 and 2.2 the solution can be extended up to $T$ indicated in the proof of Lemma 2.2. Thus the proof of Theorem 1.1 is complete.

**3. Proof of Theorem 1.2.** The proof of Theorem 1.2 is divided into two parts. First we prove the following theorem on the local in time existence by the method of successive approximations in §3.1.

**Theorem 3.1.** *Assume that $\phi_0^0\in W_2^3(\Omega)$ satisfies the compatibility conditions*

$$\phi_0^0=\Delta\phi_0^0=0 \textit{ on } \Gamma,\ \phi_0^0,\ \textit{periodic in the } x_3\textit{-direction.}$$

*Then there exists a unique solution $\phi^0$ to the problem* (1.9) *on some interval* $[0,T'']$ *such that* $\phi^0\in L^2(0,T'';W_2^4(\Omega))$, $\partial\phi^0/\partial t\in L^2(0,T'';W_2^2(\Omega))$.

Second we prove the following theorem on the global in time existence with the help of *a priori* estimates in §3.2.

**Theorem 3.2.** *Assume that $\phi_0^0\in W_2^3(\Omega)$ satisfies the compatibility conditions*

$$\phi_0^0=\Delta\phi_0^0=0 \textit{ on } \Gamma,\ \phi_0^0,\ \textit{periodic in the } x_3\textit{-direction.}$$

*Then for any positive number $T$ there exists a unique solution $\phi^0$ to the problem* (1.9) *on* $[0,T]$ *such that* $\phi^0\in L^2(0,T;W_2^4(\Omega))$, $\partial\phi^0/\partial t\in L^2(0,T;W_2^2(\Omega))$.

**3.1. Proof of Theorem 3.1.** The following lemmas are well-known (see, for example, [7,8]).

**Lemma 3.1.** *Let $0<T<\infty$. Assume that $\psi_0\in W_2^1(\Omega)$ satisfies the compatibility conditions*

$\psi_0 = 0$ *on* $\Gamma$, $\psi_0$, *periodic in the* $x_3$*-direction.*

*Then for any* $g \in L^2(Q_T)$ *there exists a unique solution* $\psi \in W_2^{2,1}(Q_T)$ *to problem*

$$\begin{cases} \dfrac{\partial \psi}{\partial t} - c_2 \Delta \psi = g \quad (x \in \Omega,\ 0 < t < T), \\ \psi(x,0) = \psi_0(x) \quad (x \in \Omega), \\ \psi(x,t) = 0 \quad (x \in \Gamma,\ 0 < t < T), \\ \psi, \quad \text{periodic in the } x_3\text{-direction.} \end{cases}$$

*Moreover, this solution satisfies the inequality*

$$\|\psi\|_{W_2^{2,1}(Q_T)} \le c(\|\psi_0\|_{W_2^1(\Omega)} + \|g\|_{L^2(Q_T)}).$$

**Lemma 3.2.** *Assume that* $\psi \in W_2^{2,1}(Q_T)$. *Then problem*

$$\begin{cases} \Delta \phi - \phi = \psi \quad (x \in \Omega,\ 0 < t < T), \\ \phi(x,t) = 0 \quad (x \in \Gamma,\ 0 < t < T), \\ \phi, \quad \text{periodic in the } x_3\text{-direction} \end{cases}$$

*has a unique solution* $\phi \in L^2(0,T;W_2^4(\Omega)) \cap W_2^1(0,T;W_2^2(\Omega))$, *which satisfies the inequality*

$$\|\phi\|_{L^2(0,T;W_2^4(\Omega))} + \|\phi\|_{W_2^1(0,T;W_2^2(\Omega))} \le c\|\psi\|_{W_2^{2,1}(Q_T)}.$$

Let us reduce problem (1.9) to the problem with zero initial data. According to Lemmas 3.1 and 3.2, there exists $\phi^* \in L^2(0,T;W_2^4(\Omega)) \cap W_2^1(0,T;W_2^2(\Omega))$ satisfying the equations

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - c_2 \Delta\right)(\Delta \phi^* - \phi^*) = 0 \quad (x \in \Omega,\ 0 < t < T), \\ \phi^*(x,0) = \phi_0^0(x) \quad (x \in \Omega), \\ \phi^*(x,t) = \Delta \phi^*(x,t) = 0 \quad (x \in \Gamma,\ 0 < t < T), \\ \phi^*, \quad \text{periodic in the } x_3\text{-direction} \end{cases}$$

and the inequalities

$$\|\phi^*\|_{L^2(0,T;W_2^4(\Omega))} + \|\phi^*\|_{W_2^1(0,T;W_2^2(\Omega))} \le c\|\phi_0^0\|_{W_2^3(\Omega)}. \tag{3.1}$$

By putting $\Phi \equiv \phi^0 - \phi^*$, the problem (1.9) is equivalent to the problem

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - c_2 \Delta\right)(\Delta \Phi - \Phi) \\ \quad = (\nabla(\Phi + \phi^*) \times \mathbf{e}) \cdot \nabla(\Delta(\Phi + \phi^*) \\ \qquad - (\Phi + \phi^*) + (\overline{\Phi} + \overline{\phi^*})) + c_2 \Delta(\Phi + \phi^*) \\ \qquad - \dfrac{\partial}{\partial t}(\overline{\Phi} + \overline{\phi^*}) \quad (x \in \Omega,\ 0 < t < T), \\ \Phi(x,0) = 0 \quad (x \in \Omega), \\ \Phi(x,t) = \Delta \Phi(x,t) = 0 \quad (x \in \Gamma,\ 0 < t < T), \\ \Phi, \quad \text{periodic in the } x_3\text{-direction.} \end{cases}$$

This problem is easily solved by the method of successive approximations with the help of Lemmas 3.1, 3.2 and (3.1).

Thus the proof of Theorem 3.1 is complete.

**3.2. Proof of Theorem 3.2.** In the similar way as in Lemma 2.2 we can show the following *a priori* estimates of the solution $\phi^0$ established in §3.1. Let $T$ be an arbitrary positive number and $\phi^0$ be a solution of problem (1.9) belonging to $L^2(0,T;W_2^4(\Omega)) \cap W_2^1(0,T;W_2^2(\Omega))$.

**Lemma 3.3.** *Let* $\widetilde{\phi^0} = \phi^0 - \overline{\phi^0}$. *For any* $t \in [0,T]$

$$\begin{aligned} &\|\nabla \phi^0(t)\|^2 + \|\widetilde{\phi^0}(t)\|^2 + 2c_2 \int_0^t \|\Delta \phi^0(\tau)\|^2 \,\mathrm{d}\tau \\ &\quad = \|\nabla \phi_0^0\|^2 + \|\widetilde{\phi_0^0}\|^2 \equiv c^*, \\ &\|\Delta \phi^0(t)\|^2 + \|\nabla \widetilde{\phi^0}(t)\|^2 + c_2 \int_0^t \|\nabla \Delta \phi^0(\tau)\|^2 \,\mathrm{d}\tau \\ &\quad \le \|\Delta \phi_0^0\|^2 + \|\nabla \widetilde{\phi_0^0}\|^2 + c^{**} t^{1/4} \equiv c_0^{**} + c^{**} t^{1/4}, \\ &\|\nabla \Delta \phi^0(t)\|^2 + \|\Delta \widetilde{\phi^0}(t)\|^2 + c_2 \int_0^t \|\Delta^2 \phi^0(\tau)\|^2 \,\mathrm{d}\tau \\ &\quad \le \|\nabla \Delta \phi_0^0\|^2 + \|\Delta \widetilde{\phi_0^0}\|^2 + c_0^{***} + c^{***} t, \end{aligned}$$

*where* $c^{**}$ *is a constant depending on* $c^*$, *and* $c_0^{***}$ *and* $c^{***}$ *are constants depending on* $c^*, c_0^{**}$ *and* $c^*, c^{**}$, *respectively.*

By the standard arguments with the help of the *a priori* estimates in Lemma 3.3 the solution $\phi^0$ established in Theorem 3.1 can be extended to any time interval $[0,T]$. Thus the proof of Theorem 3.2 is complete.

**4. Proof of Theorem 1.3.** Subtracting (1.6) from (1.5) and denoting by $\phi \equiv \phi^\varepsilon - \phi^0$, $n \equiv n^\varepsilon - n^0$, we have

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - (\nabla \phi^\varepsilon \times \mathbf{e}) \cdot \nabla\right)(\Delta \phi - n) \\ \qquad - (\nabla \phi \times \mathbf{e}) \cdot \nabla(\Delta \phi^0 - n^0) = c_2 \Delta^2 \phi, \\ \varepsilon\left(\dfrac{\partial}{\partial t} - (\nabla \phi^\varepsilon \times \mathbf{e}) \cdot \nabla\right) n \\ \quad = -\varepsilon\left(\dfrac{\partial}{\partial t} - (\nabla \phi^\varepsilon \times \mathbf{e}) \cdot \nabla\right) n^0 \\ \qquad - \dfrac{1}{\bar{n}} \dfrac{\partial^2(\phi - n)}{\partial x_3^2} \quad (x \in \Omega,\ 0 < t < T), \\ \phi(x,0) = \phi_0^\varepsilon - \phi_0^0,\ n(x,0) = n_0^\varepsilon - n_0^0 \\ \qquad\qquad (x \in \Omega), \\ \phi(x,t) = \Delta \phi(x,t) = n(x,t) = 0 \\ \qquad\qquad (x \in \Gamma,\ 0 < t < T), \\ \phi,\ n, \quad \text{periodic in the } x_3\text{-direction.} \end{cases} \tag{4.1}$$

By virtue of Lemmas 2.1, 2.2 and 3.3, the following lemma is derived from (4.1). Here we denote by $c$ a constant independent of $t$ and by $C(t)$ a constant dependent on both $t$ and the bounds of $\phi^\varepsilon$, $n^\varepsilon$, $\phi^0$, $n^0$, which may differ at each occurrence.

**Lemma 4.1.** *For any* $t \in [0,T]$

$$\varepsilon\left(\|\nabla\phi(t)\|^2 + \|n(t)\|^2 + \int_0^t \|\Delta\phi(\tau)\|^2\,\mathrm{d}\tau\right) + \int_0^t \left\|\frac{\partial(\phi-n)}{\partial x_3}(\tau)\right\|^2 \mathrm{d}\tau \le \varepsilon C(t)\left(\|\nabla\phi(0)\|^2 + \|n(0)\|^2\right) + \varepsilon^2 C(t),$$

$$\varepsilon\left(\|\Delta\phi(t)\|^2 + \|\nabla n(t)\|^2 + \int_0^t \|\nabla\Delta\phi(\tau)\|^2 \mathrm{d}\tau\right) + \int_0^t \left\|\frac{\partial\nabla(\phi-n)}{\partial x_3}(\tau)\right\|^2 \mathrm{d}\tau \le \varepsilon C(t)(\|\nabla\phi(0)\|^2_{W_2^1(\Omega)} + \|n(0)\|^2_{W_2^1(\Omega)}) + \varepsilon^2 C(t),$$

$$\varepsilon\left(\|\nabla\Delta\phi(t)\|^2 + \|\Delta n(t)\|^2 + \int_0^t \|\Delta^2\phi(\tau)\|^2 \mathrm{d}\tau\right) + \int_0^t \left\|\frac{\partial\Delta(\phi-n)}{\partial x_3}(\tau)\right\|^2 \mathrm{d}\tau \le \varepsilon C(t)(\|\nabla\phi(0)\|^2_{W_2^2(\Omega)} + \|n(0)\|^2_{W_2^2(\Omega)}) + \varepsilon^2 C(t),$$

$$\int_0^t \left\|\frac{\partial(\Delta\phi - n)}{\partial\tau}(\tau)\right\|^2 \mathrm{d}\tau \le C(t)(\|\nabla\phi(0)\|^2_{W_2^2(\Omega)} + \|n(0)\|^2_{W_2^2(\Omega)}) + \varepsilon C(t),$$

$$\varepsilon^2 \int_0^t \left\|\frac{\partial n}{\partial\tau}(\tau)\right\|^2 \mathrm{d}\tau \le \varepsilon C(t)\left(\|\nabla\phi(0)\|^2 + \|n(0)\|^2\right) + \varepsilon^2 C(t)(\|\nabla\phi(0)\|^2_{W_2^2(\Omega)} + \|n(0)\|^2_{W_2^2(\Omega)} + \varepsilon + 1).$$

From Lemma 4.1, it is easy to see that if the initial data $(\phi_0^\varepsilon, n_0^\varepsilon) \to (\phi_0^0, n_0^0)$ as $\varepsilon \to 0$ in $W_2^3(\Omega) \times W_2^2(\Omega)$, then $(\phi^\varepsilon, n^\varepsilon) \to (\phi^0, n^0)$ as $\varepsilon \to 0$ in $L^2(0,T; W_2^4(\Omega)) \times W_2^{2,1}(Q_T)$ and $\Delta\phi^\varepsilon - n^\varepsilon \to \Delta\phi^0 - n^0$ as $\varepsilon \to 0$ in $W_2^{2,1}(Q_T)$. Finally we remark that Lemma 4.1 holds so far as $(\phi^\varepsilon, n^\varepsilon)$ exists, that means the time interval $[0,T]$ is the same as in Theorem 1.1. Thus the proof of Theorem 1.3 is complete.

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