CMC-1 trinoids in hyperbolic 3-space and metrics of constant curvature one with conical singularities on the 2-sphere

By Shoichi FUJIMORI,[*1)] Yu KAWAKAMI,[*2)] Masatoshi KOKUBU,[*3)] Wayne ROSSMAN,[*4)] Masaaki UMEHARA[*5)] and Kotaro YAMADA[*6)]

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Abstract: CMC-1 trinoids (i.e. constant mean curvature one immersed surfaces of genus zero with three regular embedded ends) in hyperbolic 3-space H^3 are irreducible generically, and the irreducible ones have been classified. However, the reducible case has not yet been fully treated, so here we give an explicit description of CMC-1 trinoids in H^3 that includes the reducible case.

Key words: Constant mean curvature; spherical metrics; conical singularities; trinoids.

1. Introduction. Let H^3 denote the hyperbolic 3-space of constant sectional curvature -1.

A *CMC*-1 *trinoid* in H^3 is a complete immersed constant mean curvature one surface of genus zero with three regular embedded ends. There are CMC-1 trinoids with horospherical ends (i.e. regular embedded ends which are asymptotic to a horosphere). However, an irreducible trinoid admits only catenoidal ends. The last two authors [9] gave a classification of those CMC-1 trinoids in H^3 that are irreducible. In particular, they showed that the moduli space of irreducible CMC-1 trinoids in H^3 (i.e. the quotient space of such immersions by the rigid motions of H^3) corresponds to a certain open dense subset of the set of irreducible spherical (i.e. constant curvature 1) metrics with three conical singularities (see Section 2). The paper [9] also investigated the reducible case, but had not obtained a complete classification there.

After that, Bobenko, Pavlyukevich, and Springborn [1] developed a representation formula for CMC-1 surfaces in H^3 in terms of holomorphic spinors and derived explicit parametrizations for irreducible CMC-1 trinoids in H^3 in terms of hypergeometric functions. The crucial step in [1] was a direct reduction of the ordinary differential equation that produces CMC-1 trinoids into a Fuchsian differential equation with three regular singularities, and we call this *BPS-reduction*. On the other hand, Daniel [2] gave an alternative proof of the classification theorem for irreducible CMC-1 trinoids, by applying Riemann's classical work on minimal surfaces in $\boldsymbol{R}^3$ bounded by three straight lines.

After the work [9] on the irreducible case, Furuta and Hattori [4] gave a full classification of spherical metrics with three conical singularities, using a purely geometric method. Later, Eremenko [3] proved it using hypergeometric equations. In this paper, using the argument in [3] and the BPS-reduction, we describe a complete classification of reducible CMC-1 trinoids in H^3.

2. Preliminaries. Let M^2 be a 2-manifold, and consider a CMC-1 immersion $f : M^2 \to H^3$. The existence of such an immersion implies orientability of M^2. By the existence of isothermal coordinates, there is a unique complex structure on M^2 such that the metric ds_f^2 induced by f is conformal (i.e. ds_f^2 is Hermitian). In this situation, there exists a holomorphic immersion (called a *null lift* of f)

$$F : \tilde{M}^2 \to \mathrm{SL}(2, \boldsymbol{C})$$

defined on the universal cover $\tilde{M}^2$ of M^2 so that:

- F is a *null* holomorphic map, namely, $F_z := dF/dz$ is of rank less than 2 on each local complex coordinate $(U; z)$ of M^2.

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*1) Department of Mathematics, Faculty of Science, Okayama University, Okayama 700-8530, Japan.
*2) Graduate School of Science and Engineering, Yamaguchi University, Yamaguchi 753-8512, Japan.
*3) Department of Mathematics, School of Engineering, Tokyo Denki University, Tokyo 101-8457, Japan.
*4) Department of Mathematics, Faculty of Science, Kobe University, Rokko, Kobe, Hyogo 657-8501, Japan.
*5) Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Tokyo 152-8552, Japan.
*6) Department of Mathematics, Tokyo Institute of Technology, Tokyo 152-8551, Japan.

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- $f \circ \pi = \hat{\pi} \circ F$, where $\pi : \tilde{M}^2 \to M^2$ is the covering projection and
$$\hat{\pi} : \mathrm{SL}(2, \boldsymbol{C}) \to H^3 = \mathrm{SL}(2, \boldsymbol{C})/\,\mathrm{SU}(2)$$
is the canonical projection.

Then there exist a meromorphic function g and a holomorphic 1-form ω on $\tilde{M}^2$ such that

$$F^{-1}dF = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix}\omega, \tag{1}$$

and the first fundamental form ds_f^2 of f satisfies
$$ds_f^2 = (1 + |g|^2)^2|\omega|^2.$$
The second fundamental form of f is given by
$$h := -Q - \bar{Q} + ds_f^2 \quad (Q := \omega dg),$$
where the holomorphic 2-differential Q on M^2 is called the *Hopf differential* of f. The set of zeros of Q corresponds to the set of umbilics of f. We set

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}. \tag{2}$$

Since $\det(dF) = 0$, one can easily show via (1) that

$$g = -\frac{dF_{12}}{dF_{11}} = -\frac{dF_{22}}{dF_{21}}. \tag{3}$$

With $\pi_1(M^2)$ denoting the covering transformation group on the universal cover $\tilde{M}^2$, for each $\tau \in \pi_1(M^2)$, there exists $\rho(\tau) \in \mathrm{SU}(2)$ such that

$$F \circ \tau = F\rho(\tau), \tag{4}$$

which gives a representation (i.e. a group homomorphism) $\rho : \pi_1(M^2) \to \mathrm{SU}(2)$ satisfying

$$g \circ \tau^{-1} = \frac{a_{11}g + a_{12}}{a_{21}g + a_{22}} =: \rho(\tau) \star g, \tag{5}$$

for each $\tau \in \pi_1(M^2)$, where $\rho(\tau) = (a_{ij})_{i,j=1,2}$.

Definition 1. A representation $\rho : \pi_1(M^2) \to \mathrm{SU}(2)$ is called *reducible* if $\rho(\pi_1(M^2))$ is abelian and otherwise is called *irreducible*. A CMC-1 immersion $f : M^2 \to H^3$ is called *irreducible* (*reducible*) if the induced representation ρ is irreducible (reducible).

The meromorphic function (cf. [8])
$$G := \frac{dF_{11}}{dF_{21}} = \frac{dF_{12}}{dF_{22}}$$
is well-defined on M^2, and is called the *hyperbolic Gauss map* of f.

We now consider a CMC-1 immersion f satisfying the following properties:

(a) The metric ds_f^2 induced by f is complete and of finite total curvature.

By (a), there exists a closed Riemann surface $\bar{M}^2$ such that M^2 is bi-holomorphic to $\bar{M}^2 \setminus \{p_1, \ldots, p_n\}$, where $p_1, \ldots, p_n$ are distinct points of $\bar{M}^2$ called the *ends* of f. Then, the Hopf differential Q has at most a pole at each of $p_1, \ldots, p_n$.

Now, we suppose the second condition:

(b) All the ends $p_1, \ldots, p_n$ of f are properly embedded, namely, there is a neighborhood U_j of p_j in $\bar{M}^2$ such that the restriction $f|_{U_j \setminus \{p_j\}}$ is a proper embedding, for each $j = 1, \ldots, n$.

Then, the condition (b) implies that G has at most a pole at each end p_j ($j = 1, \ldots, n$), namely, the ends $p_1, \ldots, p_n$ are all regular ends.

Definition 2 [7]**.** Let $\bar{M}^2$ be a closed Riemann surface. Let $d\sigma^2$ be a conformal metric on $\bar{M}^2 \setminus \{p_1, \ldots, p_n\}$, where $p_1, \ldots, p_n$ are distinct points. Then $d\sigma^2$ has a *conical singularity* of order μ_j at p_j if $\mu_j > -1$ and $d\sigma^2/|z|^{2\mu_j}$ is positive definite at p_j, where z is a local coordinate so that $z = 0$ at p_j. $2\pi(1 + \mu_j)$ is called the *conical angle* of $d\sigma^2$ at p_j.

We set $\bar{M}^2 = S^2$ and consider conformal metrics that have exactly three conical singularities at $0, 1, \infty$ on $S^2 = \boldsymbol{C} \cup \{\infty\}$. We denote by $\mathcal{M}_3(S^2)$ the set of such metrics having constant curvature 1 on $M^2 := \boldsymbol{C} \setminus \{0, 1\}$, namely, $\mathcal{M}_3(S^2)$ can be identified with the moduli space of conformal metrics of constant curvature 1 with three conical singularities. We fix a metric $d\sigma^2 \in \mathcal{M}_3(S^2)$, and then there exists a developing map
$$g : \tilde{M}^2 \to S^2 = \boldsymbol{C} \cup \{\infty\}$$
so that $d\sigma^2 = 4dgd\bar{g}/(1 + |g|^2)^2$, where $\tilde{M}^2$ is the universal cover of $M^2 (= \boldsymbol{C} \setminus \{0, 1\})$. Then there is a representation ([9, (2.15) and Lemma 2.2])

$$\rho : \pi_1(M^2) \to \mathrm{SU}(2) \tag{6}$$

satisfying (5). The metric $d\sigma^2$ is called *irreducible* if ρ is irreducible.

We return to the previous situation of CMC-1 surfaces. Let K be the Gaussian curvature of the CMC-1 immersion f. Then

$$d\sigma_f^2 := (-K)ds_f^2 = \frac{4\,dg\,d\bar{g}}{(1 + |g|^2)^2}. \tag{7}$$

This relation implies that $d\sigma_f^2$ has constant curvature 1 wherever $d\sigma_f^2$ is positive definite. Moreover [8],
$$ds_f^2 d\sigma_f^2 = 4Q\bar{Q}$$

implies that $d\sigma_f^2$ has a conical singularity at a zero q of Q, and the conical order of $d\sigma_f^2$ at q equals Q's order there. The condition (a) implies that $d\sigma_f^2$ has also a conical singularity at each end p_j.

Definition 3. Let $f : M^2 \to H^3$ be a CMC-1 immersion satisfying conditions (a) and (b). Then f is called a *CMC-1 n-noid* if $\bar{M}^2$ is conformally equivalent to the 2-sphere S^2. An end p of a CMC-1 n-noid is called *catenoidal* if Q has a pole of order 2 at p. A CMC-1 n-noid is called *catenoidal* if all ends are catenoidal.

Let f be a CMC-1 n-noid. When $n = 1$, f is congruent to the horosphere. When $n = 2$, f is congruent to a catenoid cousin or a warped catenoid cousin (cf. [6]).

So it is natural to consider the case $n = 3$. Since the three ends are embedded, the Osserman-type inequality [8] implies $\deg(G) = 2$. We call a CMC-1 3-noid a *trinoid* (or a CMC-1 trinoid). We denote by $\mathcal{M}_3(H^3)$ the set of congruence classes of trinoids. We now fix a trinoid f. As shown in [5], there are only two possibilities:

(i) Q has poles of order 2 at p_1, p_2, p_3.

(ii) Q has at most poles of order 2 at p_1, p_2, p_3, but at least one of the p_j has a pole of order 1.

As CMC-1 trinoids satisfying (i) are catenoidal, irreducible trinoids are catenoidal (see [9]). CMC-1 immersions satisfying (ii) have been classified in [5, Theorems 4.5–4.7]. So from now on we consider just the case (i). Without loss of generality we may assume $p_1 = 0$, $p_2 = 1$, $p_3 = \infty$. As mentioned above, the metric $d\sigma_f^2$ given by (7) has conical singularities at the zeros of Q and the three ends p_1, p_2, p_3. We denote by $\beta_j(> -1)$ the order of $d\sigma_f^2$ at p_j, and by

$$B_j := \pi(1 + \beta_j)(> 0) \quad (j = 1, 2, 3)$$

the half of the conical angle of $d\sigma_f^2$ at p_j ($j = 1, 2, 3$). The group $\rho(\pi_1(M^2))$ is generated by three monodromy matrices ρ_1, ρ_2, ρ_3 which represent loops surrounding $z = 0$, 1, ∞. Each ρ_j ($j = 1, 2, 3$) has eigenvalues $-\exp(\pm iB_j)$. Then we have (cf. [9])

$$(8) \qquad 2Q = \frac{c_3 z^2 + (c_2 - c_3 - c_1)z + c_1}{z^2(z-1)^2}\, dz^2,$$

where $c_j := -\beta_j(\beta_j + 2)/2$ does not vanish by (i) (i.e. $B_j \neq \pi$) for $j = 1$, 2, 3, and

$$(9) \qquad \frac{(c_1)^2 + (c_2)^2 + (c_3)^2}{2} \neq c_1c_2 + c_2c_3 + c_3c_1.$$

We denote by q_1, q_2 the two roots of the equation

$$(10) \qquad c_3 z^2 + (c_2 - c_3 - c_1)z + c_1 = 0.$$

Since $c_3 \neq 0$, the Hopf differential Q has exactly two zeros at q_1 and q_2. In fact, (9) is equivalent to the condition $q_1 \neq q_2$ (i.e. the discriminant of (10) does not vanish). As shown in [9], the condition (b) implies that G does not branch at the three ends 0, 1, ∞, but has exactly two branch points at q_1, q_2. Since G is of degree 2 and has the ambiguity of Möbius transformations, we may set (cf. [9])

$$(11) \qquad G := z + \frac{(q_1 - q_2)^2}{2(2z - q_1 - q_2)}.$$

Take a solution $F : \tilde{M}^2 \to \mathrm{SL}(2, \boldsymbol{C})$ of the ordinary differential equation

$$(12) \qquad dFF^{-1} = \begin{pmatrix} G & -G^2 \\ 1 & -G \end{pmatrix} \frac{Q}{dG}.$$

If the image $\rho(\pi_1(M^2))$ of the representation ρ of F is conjugate to a subgroup of SU(2), then $f = \hat{\pi}(Fa)$ gives a CMC-1 trinoid for a suitable choice of $a \in \mathrm{SL}(2, \boldsymbol{C})$ (cf. (4)). We denote by $\mathcal{M}_{B_1,B_2,B_3}(H^3)$ (resp. $\mathcal{M}_{B_1,B_2,B_3}(S^2)$) the congruence classes of trinoids f satisfying (i) (resp. of the metrics $d\sigma^2$ of constant curvature 1) such that $d\sigma_f^2$ (resp. $d\sigma^2$) has conical angle $2B_j(\neq 2\pi)$ at each p_j.

Fact 1 [9]. *For each B_1, B_2, $B_3 \in (0, \infty)$, $\mathcal{M}_{B_1,B_2,B_3}(H^3)$ (resp. $\mathcal{M}_{B_1,B_2,B_3}(S^2)$) consists of a unique irreducible element if it satisfies* (9) (*resp. no condition*) *and*

$$(13) \qquad \cos^2 B_1 + \cos^2 B_2 + \cos^2 B_3 + 2\cos B_1 \cos B_2 \cos B_3 < 1.$$

Conversely, any irreducible trinoids (*resp. any irreducible metrics in $\mathcal{M}_3(S^2)$*) *are so obtained.*

In particular, there is a unique catenoidal trinoid f such that

- *the hyperbolic Gauss map G is given by* (11),
- *the Hopf differential Q is given by* (8),
- *$d\sigma_f^2$ has conical angle $2B_j$ at each end p_j.*

Fig. 1, left (resp. right) is an irreducible trinoid (resp. a cutaway view of an irreducible trinoid) for $B_1 = B_2 = B_3 (= B)$ with $B < \pi$ (resp. $B > \pi$).

Remark 1. Since the hyperbolic Gauss map G changes under a rigid motion of H^3, the above trinoid f is uniquely determined without the ambiguity of isometries of H^3 (cf. [9, Appendix B]). After [9], Bobenko, Pavlyukevich and Springborn [1] gave a different proof, whose underlying idea also

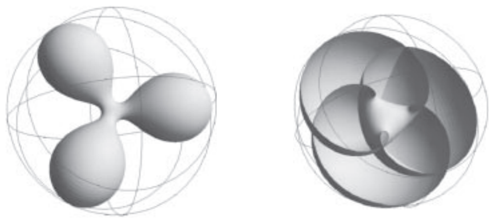

Fig. 1. Trinoids with $B_1 = B_2 = B_3$.

appears in the next section. Also, Daniel [2] gave an alternative proof of this fact (see the introduction).

For each B_j $(j = 1, 2, 3)$ there exists a unique real number $\hat{B}_j \in [0, \pi]$ such that $\cos B_j = \cos \hat{B}_j$, since $\cos t = \cos(2\pi - t)$ for $t \in [0, 2\pi)$. By definition, it holds that $B_j \geq \hat{B}_j$. Without loss of generality, we may assume that $\hat{B}_1 \leq \hat{B}_2 \leq \hat{B}_3$. We now set $B'_1 := \hat{B}_1$, and for $j = 2, 3$,

$$B'_j := \begin{cases} \hat{B}_j & \text{if } \hat{B}_2 + \hat{B}_3 \leq \pi, \\ \pi - \hat{B}_j & \text{if } \hat{B}_2 + \hat{B}_3 > \pi. \end{cases}$$

Then we have that

$$0 \leq B'_1 + B'_2,\ B'_1 + B'_3,\ B'_2 + B'_3 \leq \pi, \tag{14}$$

and the condition (13) is equivalent to

$$\cos^2 B'_1 + \cos^2 B'_2 + \cos^2 B'_3 + 2 \cos B'_1 \cos B'_2 \cos B'_3 < 1,$$

which is equivalent to the condition

$$\cos \frac{B'_1 + B'_2 + B'_3}{2} \cos \frac{-B'_1 + B'_2 + B'_3}{2} \times \cos \frac{B'_1 - B'_2 + B'_3}{2} \cos \frac{B'_1 + B'_2 - B'_3}{2} < 0.$$

By (14), this then reduces to the condition

$$B'_1 + B'_2 + B'_3 > \pi. \tag{15}$$

The condition (13) (or equivalently (15)) implies $B_j \notin \pi \boldsymbol{Z}$ $(j = 1, 2, 3)$, and is the same condition as in [9], [4] or [3] that there exists an irreducible metric in $\mathcal{M}_3(S^2)$ with three conical angles $2B_1$, $2B_2$, $2B_3$.

3. Reducible trinoids. Let α be a 2×2-matrix valued meromorphic 1-form on $\boldsymbol{C} \cup \{\infty\}$. Consider an ordinary differential equation

$$dE E^{-1} = \alpha, \tag{16}$$

which is called a *Fuchsian differential equation* if it admits only regular singularities. For example, the equation (12) with G, Q satisfying (11) and (8) is a Fuchsian differential equation with regular singularities at $z = 0,\ 1,\ \infty,\ q_1,\ q_2$. Let $p_1, \dots, p_n \in \boldsymbol{C} \cup \{\infty\}$ be the regular singularities of the equation (16). We denote by $\tilde{M}^2$ the universal cover of

$$M^2 := \boldsymbol{C} \cup \{\infty\} \setminus \{p_1, \dots, p_n\}.$$

Then there exists a solution $E : \tilde{M}^2 \to \mathrm{GL}(2, \boldsymbol{C})$ of (16). Since α is defined on M^2, there exists a representation $\gamma : \pi_1(M^2) \to \mathrm{GL}(2, \boldsymbol{C})$ such that $E \circ \tau = E\gamma(\tau)$. Let

$$\mathrm{GL}(2, \boldsymbol{C}) \ni a \mapsto [a] \in \mathrm{PGL}(2, \boldsymbol{C}) = \mathrm{PSL}(2, \boldsymbol{C})$$

be the canonical projection. Then

$$h_1 := -E_{12}/E_{11}, \quad h_2 := -E_{22}/E_{21}$$

satisfy (see (5) for the definition of $\star$)

$$h_i \circ \tau^{-1} = \gamma(\tau) \star h_i \quad (\tau \in \pi_1(M^2),\ i = 1, 2),$$

where $E = (E_{jk})_{j,k=1,2}$. Thus the functions h_i $(i = 1, 2)$ induce a common group homomorphism $[\gamma] : \pi_1(M^2) \to \mathrm{PGL}(2, \boldsymbol{C})$ which is called the *monodromy representation* of the equation (16). In particular, the representation $[\rho]$ for F as in (12) is just the monodromy representation.

Definition 4. Let $r(z), s(z)$ be meromorphic functions on $\boldsymbol{C} \cup \{\infty\}$ and

$$X'' + rX' + sX = 0 \tag{17}$$

be an ordinary differential equation with regular singularities at $z = p_1, \dots, p_n$, where $X' = dX/dz$. Then there exists a pair of solutions $w_1, w_2 : \tilde{M}^2 \to \boldsymbol{C}$ which are linearly independent, and $\{w_1, w_2\}$ is called a fundamental system of solutions. There exists a representation $\gamma : \pi_1(M^2) \to \mathrm{GL}(2, \boldsymbol{C})$ for each fundamental system $\{w_1, w_2\}$, such that

$$(w_1 \circ \tau, w_2 \circ \tau) = (w_1, w_2)\gamma(\tau),$$

where (w_1, w_2) is a row vector. As a monodromy of the function $-w_2/w_1$, the induced homomorphism $[\gamma] : \pi_1(M^2) \to \mathrm{PGL}(2, \boldsymbol{C})$ is called the *monodromy representation* of the equation (17).

To give a complete classification of trinoids, the following reduction given in [1] is crucial: Let F be a null lift of the catenoidal trinoid f whose hyperbolic Gauss map G and Hopf differential Q are given by (11) and (8), respectively. In the expression (12), we can write

$$\begin{pmatrix} G & -G^2 \\ 1 & -G \end{pmatrix} \frac{Q}{dG} = \begin{pmatrix} \mathbf{P}_1 \mathbf{P}_2 & (\mathbf{P}_1)^2 \\ -(\mathbf{P}_2)^2 & -\mathbf{P}_1 \mathbf{P}_2 \end{pmatrix} dz,$$

where $\mathbf{P}_i := \dfrac{p_i^0}{z} + \dfrac{p_i^1}{z-1} + p_i^\infty$ and p_i^0, p_i^1, p_i^∞ $(i=1,2)$ are constants depending only on B_1, B_2, B_3. In [1], the matrix $\Phi := D^{-1}F$ is defined by

$$D := \sqrt{z-1}\begin{pmatrix} \mathbf{P}_1 & \alpha_1 z+\beta_1 \\ -\mathbf{P}_2 & \alpha_2 z+\beta_2 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ \dfrac{k}{z(z-1)} & \dfrac{1}{z-1} \end{pmatrix}\begin{pmatrix} \vartheta & 0 \\ 1 & 1 \end{pmatrix},$$

where α_j, β_j $(j=1,2)$, k and ϑ are all real constants depending only on B_1, B_2, B_3. Then there exist 2×2 matrices A_0, A_1 with real coefficients such that

$$d\Phi\Phi^{-1} = \left(\frac{A_0}{z} + \frac{A_1}{z-1}\right)dz. \tag{18}$$

We call (18) the *BPS-reduction* of (12). (This reduction does not work if f has a horospherical end, but such trinoids would be in the case (ii) mentioned before.) By (4), it holds that $\Phi\circ\tau = \Phi\rho(\tau)$ for each $\tau\in\pi_1(M^2)$. Obviously (18) has three regular singularities at $z=0,1,\infty$. Since A_0 and A_1 are both constant real matrices, it is well-known that there exist real numbers a, b, c such that the monodromy representation of the ordinary differential equation (called the *hypergeometric equation*)

$$z(1-z)X'' + (c-(a+b+1)z)X' - abX = 0 \tag{19}$$

is conjugate to that of (18) (i.e. $[\rho]$). On the other hand, if we express F as in (2), then $X = F_{11}$, F_{12} satisfy the ordinary differential equation (cf. [5, p. 32])

$$X'' - (\log(\hat{Q}/G'))'X' + \hat{Q}X = 0, \tag{20}$$

where $Q = \hat{Q}(z)dz^2$ and $G' = dG/dz$. Thus the monodromy representation of (20) with respect to (F_{11}, F_{12}) is equal to that of F. In particular, the monodromy representation of (20) is conjugate to that of (19). Hence, these two ordinary differential equations have the same *exponent* (i.e. the difference of the two solutions of the indicial equation) at each regular singularity. Since (20) has the exponent B_1/π, B_2/π, B_3/π at $z=0,1,\infty$, respectively, we have

$$\pm B_1 = \pi(1-c), \quad \pm B_2 = \pi(a-b), \quad \pm B_3 = \pi(c-a-b),$$

which is the same set of relations as in [3, (4)]. This implies the classification of catenoidal trinoids reduces to that of metrics in $\mathcal{M}_3(S^2)$. In particular, the classification results for reducible metrics in $\mathcal{M}_3(S^2)$ given in Furuta-Hattori [4] and Eremenko [3, Theorem 2] yield the following assertion.

Theorem. *Suppose B_1/π is an integer, and $B_j \neq \pi$ $(j=1,2,3)$. Then $\mathcal{M}_{B_1,B_2,B_3}(H^3)$ (resp. $\mathcal{M}_{B_1,B_2,B_3}(S^2)$) is non-empty if and only if B_1, B_2, B_3 satisfy* (9) *(resp. no condition) and one of the following two conditions*:

(C_1) *$B_2, B_3 \notin \pi\boldsymbol{Z}$, but either $|B_2-B_3|/\pi$ or $(B_2+B_3)/\pi$ is an integer m of opposite parity from B_1/π, and $\pi m \le B_1 - \pi$. In this case, $\mathcal{M}_{B_1,B_2,B_3}(H^3)$ (resp. $\mathcal{M}_{B_1,B_2,B_3}(S^2)$) is 1-dimensional.*

(C_2) *$B_2, B_3 \in \pi\boldsymbol{Z}$, and $(B_1+B_2+B_3)/\pi$ is odd, and each of B_1, B_2, B_3 is less than the sum of the others. In this case, $\mathcal{M}_{B_1,B_2,B_3}(H^3)$ (resp. $\mathcal{M}_{B_1,B_2,B_3}(S^2)$) is 3-dimensional.*

Corollary 1. *A catenoidal trinoid f is irreducible if and only if B_1/π, B_2/π, B_3/π are all non-integers, and f is reducible if and only if at least one of B_1/π, B_2/π, B_3/π is an integer.*

Proof. A trinoid f is irreducible if the representation ρ as in (4) is irreducible. The representation ρ coincides with that of the corresponding metric in $\mathcal{M}_3(S^2)$. The corresponding assertion for metrics in $\mathcal{M}_3(S^2)$ is proved in [9, Lemma 3.1]. □

Remark 2. Reducibility is equivalent to at least one of B_1/π, B_2/π, B_3/π being an integer. This cannot be proved purely algebraically, as there are diagonal matrices ρ_1, ρ_2, ρ_3 in SU(2) with $\rho_1\rho_2\rho_3 = \mathrm{id}$ so that no eigenvalues of ρ_1, ρ_2, ρ_3 are ± 1.

Remark 3. Eremenko [3, Theorem 2] asserts the uniqueness of $d\sigma^2 \in \mathcal{M}_{B_1,B_2,B_3}(S^2)$ with prescribed conical angles. This is correct in the irreducible case, but if $B_1 \in \pi\boldsymbol{Z}$, then the metric has a non-trivial deformation preserving its conical angles: A metric $d\tau^2 \in \mathcal{M}_3(S^2)$ has the same conical angles as those of $d\sigma^2$ if and only if each developing map of $d\tau^2$ is given by $k = a\star h$ for suitable $a\in \mathrm{SL}(2,\boldsymbol{C})$, where h is a developing map of $d\sigma^2$. So $\mathcal{M}_{B_1,B_2,B_3}(S^2)$ can be identified with the set

$$\{\hat{\pi}(a);\ a(\operatorname{Im}\rho)a^{-1} \subset \mathrm{SU}(2),\ a\in \mathrm{SL}(2,\boldsymbol{C})\} (\subset H^3),$$

where $\hat{\pi} : \mathrm{SL}(2,\boldsymbol{C}) \to H^3$ is the canonical projection and $\operatorname{Im}\rho$ is the image of ρ as in (6). Then $\mathcal{M}_{B_1,B_2,B_3}(S^2) = H^3$ if B_j/π $(j=1,2,3)$ are all integers, and $\mathcal{M}_{B_1,B_2,B_3}(S^2)$ is a geodesic line in H^3 if one of B_j/π $(j=1,2,3)$ is not an integer (cf. [9]). A metric $d\sigma^2$ in $\mathcal{M}_{B_1,B_2,B_3}(S^2)$ is called *symmetric* if the metric is invariant under an anti-holomorphic involution. We denote by $\hat{\mathcal{M}}_{B_1,B_2,B_3}(S^2)$ the subset

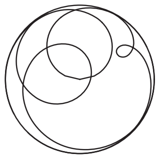

Fig. 2. A profile curve of a non-symmetric trinoid with $B_1 = B_2 = B_3 = 3\pi$. (The outer circle represents the ideal boundary of H^3.)

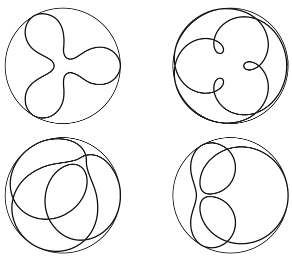

Fig. 3. Profile curves of trinoids of different types.

consisting of symmetric metrics in $\mathcal{M}_{B_1,B_2,B_3}(S^2)$. If B_j/π $(j = 1, 2, 3)$ are all integers, $\hat{\mathcal{M}}_{B_1,B_2,B_3}(S^2)$ consists of a hyperbolic plane in H^3. If one of B_j/π $(j = 1, 2, 3)$ is not an integer, $\hat{\mathcal{M}}_{B_1,B_2,B_3}(S^2)$ coincides with $\mathcal{M}_{B_1,B_2,B_3}(S^2)$ (cf. [9]). A metric $d\sigma^2$ in $\hat{\mathcal{M}}_{B_1,B_2,B_3}(S^2)$ with conical angles $2B_1$, $2B_2$ and $2B_3$ can be regarded as a doubling of the generalized spherical triangle with angles B_1, B_2 and B_3. Using this, Furuta-Hattori [4] gave two operations in $\hat{\mathcal{M}}_3(S^2)$ for distinct $\{i, j, k\} = \{1, 2, 3\}$:

$$(B_i, B_j, B_k) \mapsto (B_i + \pi, B_j + \pi, B_k),$$
$$(B_i, B_j, B_k) \mapsto (\pi - B_i, B_j + \pi, B_k),$$

with the second operation allowed only when $B_i < \pi$. The first operation is attaching a closed hemisphere in S^2 to the edge B_iB_j of the spherical triangle $\triangle B_iB_jB_k$. The second operation is attaching a geodesic 2-gon of equi-angles $\pi - B_i$ to the edge B_iB_j so that the initial vertex B_i becomes an interior point of an edge of the new triangle. Conditions (C_1) and (C_2) are invariant under these two operations. Moreover, the three angles (B_1, B_2, B_3) satisfying conditions (C_1) and (C_2) are obtained from a given initial data (B_1', B_2', B_3') by these two operations. Furuta-Hattori proved this using a geometric argument. On the other hand, Eremenko found (C_1) and (C_2) from the viewpoint of hypergeometric equations. We remark that spherical triangles of arbitrary angles B_1, B_2, $B_3 \in (0, \infty)$ were investigated by Felix Klein in 1933 (see the end of [9]). The trinoid shown in Fig. 2 is not symmetric, although there does exist a symmetric trinoid with the same conical angles and dihedral symmetry.

Finally, we group the surfaces by the signatures of c_1, c_2, c_3. For example, a trinoid f is said to be of type $(+,+,+)$ if c_1, c_2, c_3 are all positive, and of type $(-,+,+)$ if one of c_1, c_2, c_3 is negative and the other two are positive, etc. As remarked in [6], by numerical experiment, it seems that the four types $(+,+,+)$, $(-,+,+)$, $(-,-,+)$ and $(-,-,-)$ have distinct regular homotopy types (see Fig. 3). Surfaces of type $(+,+,+)$ have absolute total curvature less than 8π, and it seems that only surfaces in this class can be embedded.

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