Nonseparability of Banach spaces of bounded harmonic functions on Riemann surfaces

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Abstract: The separability of certain seminormed spaces of harmonic functions on Riemann surfaces will be considered. An application of the result obtained in the above to some inverse inclusion problem in the classification theory of Riemann surfaces will be appended.

Key words: Banach space; Dirichlet integral; Green function; harmonic function; Hilbert space; separable.

1. Introduction. A subset T of a subset S of a seminormed space X equipped with a seminorm $\|\cdot\|_X$ is said to be dense in S if for an arbitrarily chosen point $s \in S$ and for any positive number $\varepsilon > 0$ there exists a point $t \in T$ such that $\|s - t\|_X < \varepsilon$. A subset S of X is referred to as being *separable* if S contains a countable dense subset T of S. Here of course the case $S = X$ is not excluded. If S is not separable, then we say that S is nonseparable. We will discuss the separability of some concrete seminormed spaces of harmonic functions on Riemann surfaces.

We denote by $H(R)$ the linear space of harmonic functions on an open (i.e. noncompact) Riemann surface R. We will consider two linear subspaces of $H(R)$. The first is the linear subspace $HB(R)$ of $H(R)$ consisting of bounded harmonic functions u on R. Then the space $HB(R)$ forms a Banach space equipped with the supremum norm $\|u\|_\infty := \sup_R |u|$. Concerning this space we have the following result.

Theorem 1.1. *The Banach space $HB(R)$ is nonseparable unless it is of finite dimension. In other words, the Banach space $HB(R)$ is separable if and only if $HB(R)$ is finite dimensional.*

Fix an arbitrary point $o \in R$ and consider the normalized subspace $HB(R, o)$ of $HB(R)$ given by the set $\{u \in HB(R) : u(o) = 0\}$. As a closed subspace of the Banach space $HB(R)$ the space $HB(R; o)$ is also a Banach space with respect to the same supnorm. Clearly $\dim HB(R) = \dim HB(R; o) + 1$, and, $HB(R)$ and $HB(R; o)$ are simultaneously separable or nonseparable. Then

Corollary to Theorem 1.1. *The Banach space $HB(R; o)$ is nonseparable unless it is of finite dimension. In other words, the Banach space $HB(R; o)$ is separable if and only if $HB(R; o)$ is finite dimensional.*

As the second linear subspace of $H(R)$ we consider the space $HD(R)$ of Dirichlet finite harmonic functions u on R, i.e. harmonic functions u on R whose Dirichlet integral $D(u) := \int_R du \wedge *du$ is finite. Observe that $D(u)^{1/2}$ is only a seminorm on $HD(R)$ and we obtain a seminormed space $HD(R)$ with respect to the seminorm $D(\cdot)^{1/2}$. Contrary to the above Theorem 1.1, we obtain this time the following clear conclusion.

Theorem 1.2. *The seminormed space $HD(R)$ is always separable.*

The normalized subspace $HD(R; o)$ of $HD(R)$ given by $\{u \in HD(R) : u(o) = 0\}$ forms a Hilbert space whose inner product is given by the mutual Dirichlet integral $D(u, v) := \int_R du \wedge *dv$ for u and v in $HD(R)$. Similarly as in the case of $HB(R)$ we also have $\dim HD(R) = \dim HD(R; o) + 1$. Once the above Theorem 1.2 is taken for granted, then we can trivially admit the following claim.

Corollary to Theorem 1.2. *The Hilbert space $HD(R; o)$ is always separable.*

We apply the above results to the so called inverse inclusion problem in the classification theory of Riemann surfaces. Concretely we give a simple proof using the above facts to the following result of Masaoka ([7–9]):

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Theorem M. *The phenomenon* $HB(R) = HD(R)$ *occurs if and only if* $\dim HB(R) = \dim HD(R)$ *and the common value is finite.*

Thus far three proofs including the original proof of Masaoka have been given to the above result. The original first proof of Masaoka ([7], see also [8] and in particular [9]) is the constructive one: when $HB(R) = HD(R)$ is of infinite dimension, one can construct, contradictorily, a bounded Dirichlet infinite harmonic function using the Doob generalization [3] to the case of Martin boundary setting of the classical Douglas characterization [4] of an L^1-function on the unit circle to be the boundary function of an HD-function on the unit disc. The proof is thus quite far from being simple. As the second proof, the present author gave one [10] using his characterization [11] of capacitary functions on the Royden harmonic boundary, which is relatively simple but still slightly distant from being trivial. The third one [12] is also given recently by the present author based upon the observation that $HB(R)$ is not reflexive as a Banach space unless it is of finite dimension while the Hilbert space $HD(R;o)$ is of course reflexive, which may be said to be almost trivial. The one being given below in this paper is further simpler and probably we can say it is the simplest or at least there may be no simpler one among other possible proofs.

As usual we denote by $\mathbf{R}$ the real number field and by $\mathbf{N}$ the set of strictly positive integers $\{1, 2, 3, \cdots\}$. Hereafter in this paper to avoid the triviality we always assume the *hyperbolicity* of the Riemann surface R in the sense that R carries the Green function since otherwise spaces $HB(R)$ and $HD(R)$ are reduced to the trivial space $\mathbf{R}$.

2. Proof of Theorem 1.1. We begin this section by recalling the definition of Stonean spaces (cf. e.g. [15]). A compact Hausdorff space Ω is EXTREMELY DISCONNECTED if the closure of any open subset of Ω is again open so that Ω is a fortiori totally disconnected. In other words, Ω is extremely disconnected if every point of Ω has a base of the neighborhood system at the point consisting of clopen (i.e. closed and open simultaneously) neighborhoods of the point. Stone [17] proved that an extremely disconnected compact Hausdorff space Ω is characterized by the property of $C(\Omega)$ being a BOUNDED COMPLETE LATTICE: every bounded subset of $C(\Omega)$ has a supremum in $C(\Omega)$ relative to the natural ordering for real functions. By virtue of this characterization by Stone, we say that Ω is a *Stonean space* if it is an extremely disconnected compact Hausdorff space, or equivalently, if it is a compact Hausdorff space Ω whose $C(\Omega)$ is a bounded complete lattice. Stonean spaces are used in this paper in the following representation of $HB(R)$: There exists a unique (up to homeomorphisms) Stonean space δ such that

$$HB(R) = C(\delta) \tag{2.1}$$

in the sense that the Banach space $HB(R)$ is isometrically (linear) isomorphic to the Banach space $C(\delta)$ with the supnorm on δ. There are more than one proofs for (2.1) (cf. e.g. [14]) but probably the simplest is to take δ as the Wiener harmonic boundary $\delta = \delta R$ of R (cf. e.g. [2,6,16], etc.). By understanding the dimension $\dim X$ of a linear space X is either the number $n \in \mathbf{N}$ of elements in its finite base, if it exists, or ∞, if it does not exist, and also the number $\#Y$ of a set Y is either the number $n \in \mathbf{N}$ of elements in Y when Y is a finite set or ∞ and we can easily see (cf. the next paragraph below) that

$$\dim HB(R) = \#\delta. \tag{2.2}$$

Hence, if $\dim HB(R) = n \in \mathbf{N}$, then δ consists of n points and $C(\delta) = \mathbf{R}^n$, which is separable. The proof is, thus, over if we show that $C(\delta)$ is nonseparable if $\#\delta = \infty$.

The assumption $\#\delta = \infty$ implies the existence of at least one accumulation point of δ, say ζ, in δ. Since δ is a clopen neighborhood of ζ in the Stonean space δ, we can find a point $\zeta_1 \in \delta$ and a clopen neighborhood $V_1 \subset \delta$ of ζ_1 such that $\zeta \in \delta \setminus V_1$. Since $\delta \setminus V_1$ is a clopen neighborhood of ζ, we can find a point $\zeta_2 \in \delta \setminus V_1$ and a clopen neighborhood $V_2 \subset \delta \setminus V_1$ of ζ_2 such that $\zeta \in \delta \setminus (V_1 \cup V_2)$. Since $\delta \setminus (V_1 \cup V_2)$ is a clopen neighborhood of ζ, we can find a point $\zeta_3 \in \delta \setminus (V_1 \cup V_2)$ and a clopen neighborhood $V_3 \subset \delta \setminus (V_1 \cup V_2)$ of ζ_3 such that $\zeta \in \delta \setminus (V_1 \cup V_2 \cup V_3)$. Repeating this procedure we can choose a sequence $(\zeta_j)_{j\in\mathbf{N}}$ of mutually distinct points ζ_j in δ ($j \in \mathbf{N}$) and a sequence $(V_j)_{j\in\mathbf{N}}$ of mutually disjoint clopen neighborhood V_j of ζ_j ($j \in \mathbf{N}$) in δ.

Let $\chi_j = \chi_{V_j}$ be the characteristic function of V_j on δ. Since V_j is clopen, $\chi_j \in C(\delta)$ ($j \in \mathbf{N}$). Let I be the open interval $(0, 1) \subset \mathbf{R}$ and

$$\lambda = 0.\lambda_1\lambda_2 \cdots \lambda_n \cdots \tag{2.3}$$

be the infinite dyadic fractional expression of $\lambda \in I$ so that $\lambda_j \in \{0,1\}$ for every $j \in \mathbf{N}$ and there are at least one $\lambda_j = 0$ and infinitely many $\lambda_j = 1$. Since δ is Stonean, we can define

$$(2.4)\qquad e_\lambda := \sup\left\{\sum_{j=1}^{n} \lambda_j \chi_j : n \in \mathbf{N}\right\}$$

because $0 \leqq \sum_{j=1}^{n} \lambda_j \chi_j \leqq 1$ on δ. Then set

$$E := \{e_\lambda : \lambda \in I\}.$$

Observe that $\sum_{j=1}^{n} \lambda_j \chi_j$ forms an increasing sequence for $n \in \mathbf{N}$. If $\lambda_k = 0$, then we have

$$0 \leqq \sum_{j=1}^{n} \lambda_j \chi_j \leqq 1 - \chi_k$$

for every $n \in \mathbf{N}$ so that we deduce $0 \leqq e_\lambda \leqq 1 - \chi_k$ or $e_\lambda | V_k = 0 = \lambda_k$. If $\lambda_k = 1$, then

$$\chi_k \leqq \sum_{j=1}^{n} \lambda_j \chi_j \leqq 1$$

for every $n \in \mathbf{N}$ with $n \geqq k$ so that we infer that $\chi_k \leqq e_\lambda \leqq 1$ or $e_\lambda | V_k = 1 = \lambda_k$. Thus we have seen that

$$(2.5)\qquad e_\lambda | V_j = \lambda_j \quad (j \in \mathbf{N}).$$

This proves that $e_\lambda \neq e_\mu$ for $\lambda \neq \mu$ and the cardinal number

$$(2.6)\qquad \operatorname{card} E = \operatorname{card} I = \aleph.$$

Take any subset $F \subset C(\delta)$ which is dense in $C(\delta)$. We can choose and then fix an $f_\lambda \in F$ for each $\lambda \in I$ such that $\|f_\lambda - e_\lambda\|_\infty < 1/4$. For each couple $(\lambda, \mu) \in I \times I$ with $\lambda \neq \mu$, there is a $j \in \mathbf{N}$ such that $|\lambda_j - \mu_j| = 1$, where

$$\mu = 0.\mu_1\mu_2 \cdots \mu_n \cdots$$

is the infinite dyadic fractional expression of μ. Observe that expressing $f_\lambda - f_\mu$ as the sum of $(e_\lambda - e_\mu)$ and $\{(f_\lambda - e_\lambda) - (f_\mu - e_\mu)\}$ and applying the trianle inequality, $\|f_\lambda - f_\mu\|_\infty$ is seen to be greater than or equal to $\|e_\lambda - e_\mu\|_\infty - \|f_\lambda - e_\lambda\|_\infty - \|f_\mu - e_\mu\|_\infty$ which is estimated from below by $|\lambda_j - \mu_j| - 1/4 - 1/4 = 1/2$ so that

$$\|f_\lambda - f_\mu\|_\infty \geqq 1/2,$$

which assures $f_\lambda \neq f_\mu$ on δ. This proves that the mapping $e_\lambda \mapsto f_\lambda : E \to F$ is injective so that $\operatorname{card} F \geqq \operatorname{card} E$ and (2.6) yields that any dense subset F in $C(\delta)$ cannot be countable. We have thus shown that if $\dim C(R) = \infty$ or $\#\delta = \infty$, then $C(\delta)$ is not separable. $\square$

In the above proof, instead of (2.1), we can also use another representation $HB(R) = L^\infty(\delta, \omega)$ with the harmonic measure ω on the Wiener harmonic boundary δ of R, by which we can replace the topological consideration on δ by the measure theoretic one on δ. But it is hard to tell which is simpler or easier.

We turn to the proof of Corollary to Theorem 1.1. As already stated we have

$$\dim HB(R) = \dim HB(R; o) + 1$$

and thus $HB(R)$ is of finite dimension if and only if $HB(R; o)$ is finite dimensional. Let F be a countable dense subset in $HB(R)$. Then the set $G = \{f - f(o) : f \in F\}$ is also countable. Take any $v \in HB(R; o)$ and any ε. We can find an $f \in F$ with $\|v - f\|_\infty < \varepsilon/2$. Then $|f(o)| = |v(o) - f(o)| \leqq \|v - f\|_\infty < \varepsilon/2$. Since $f - f(o) \in G$ and

$$\|v - (f - f(o))\|_\infty \leqq \|v - f\|_\infty + |f(o)| < \varepsilon,$$

which shows that G is countable dense in $HB(R; o)$. Conversely, let G be a countable dense subset in $HB(R; o)$ and let F be a subset of $HB(R)$ given by $F := \{g + r : g \in G, r \in \mathbf{Q}\}$, where $\mathbf{Q}$ is the set of rational numbers in $\mathbf{R}$ so that F is again countable. For any $u \in HB(R)$ and any $\varepsilon > 0$, we can find a $g \in G$ and an $r \in \mathbf{Q}$ such that

$$\|(u - u(o)) - g\|_\infty < \varepsilon/2 \text{ and } |u(o) - r| < \varepsilon/2.$$

Then by $u - (g + r) = \{(u - u(o)) - g\} + (u(o) - r)$ we see at once that $\|u - (g + r)\|_\infty < \varepsilon$, which shows that F is dense in $HB(R)$. $\square$

3. Proof of Theorem 1.2. We start this section by fixing some notation. When $Z_1, Z_2, \cdots, Z_m$ are m abstract sets, we form the product set of them as the set of $z := (z_1, z_2, \cdots, z_m)$ with $z_j \in Z_j$ $(1 \leqq j \leqq m)$, which we denote by

$$\bigotimes_{1 \leqq j \leqq m} Z_j = Z_1 \otimes Z_2 \otimes \cdots \otimes Z_m.$$

Let X_j be a Hilbert space with $\|\cdot\|_j$ as its norm and $(\cdot, \cdot)_j$ as its inner product ($j \in \mathbf{N}$). We denote by

$$X := \bigotimes_{j \in \mathbf{N}} X_j$$

the set of $x := (x_1, x_2, \cdots)$ $(x_j \in X_j$ $(j \in \mathbf{N}))$ with

$$\sum_{j \in \mathbf{N}} \|x_j\|_j^2 < +\infty.$$

For the above x and $y = (y_1, y_2, \cdots)$ in X and for $c \in \mathbf{R}$ we define the addition and scalar multiplication componentwise:

$$x + y = (x_1 + y_1, x_2 + y_2, \cdots)$$

and

$$cx = (cx_1, cx_2, \cdots),$$

which make X a linear space. The norm $\|x\|$ of $x \in X$ is given by

$$\|x\|^2 := \sum_{j\in\mathbf{N}} \|x_j\|_j^2,$$

which induces the inner product (x, y) on X as

$$(x, y) := \sum_{j\in\mathbf{N}} (x_j, y_j)_j,$$

which makes X a new Hilbert space. Identifying $x = (x_1, \cdots, x_m, 0, 0, \cdots)$ (i.e. $x = (x_1, x_2, \cdots)$ with $x_j = 0$ for $j > m$) in X with $(x_1, \cdots, x_m)$ in $\bigotimes_{1\leqq j\leqq m} X_j$ we will view that

$$\bigotimes_{1\leqq j\leqq m} X_j \subset \bigotimes_{j\in\mathbf{N}} X_j$$

as a closed linear subspace. In case Z_j is a mere subset of X_j, we can also view that

$$\bigotimes_{1\leqq j\leqq m} Z_j \subset \bigotimes_{j\in\mathbf{N}} X_j.$$

We turn to the separability question. It is clear that if each component of $\bigotimes_{1\leqq j\leqq m} X_j$ is separable, then $\bigotimes_{1\leqq j\leqq m} X_j$ is separable and vice versa. One step further we have

Fact 3.1. *If every Hilbert space X_j is separable for $j \in \mathbf{N}$, then the product Hilbert space $\bigotimes_{j\in\mathbf{N}} X_j$ is also separable.*

Proof. Take a countable dense subset $Z_j \subset X_j$ for each $j \in \mathbf{N}$ and set

$$Z := \bigcup_{m\in\mathbf{N}} \left(\bigotimes_{1\leqq j\leqq m} Z_j \right) \subset \bigotimes_{j\in\mathbf{N}} X_j.$$

Then Z is countable as the countable union of countable sets $\bigotimes_{1\leqq j\leqq m} Z_j$. Given an arbitrary point $x = (x_1, x_2, \cdots)$ in $X := \bigotimes_{j\in\mathbf{N}} X_j$ and any positive number $\varepsilon > 0$. We can find an $m \in \mathbf{N}$ such that $\|x - x(m)\| < \varepsilon/2$, where $x(m) = (x_1, x_2, \cdots, x_m)$ in $\bigotimes_{1\leqq j\leqq m} X_j$. As is easily seen, $\bigotimes_{1\leqq j\leqq m} Z_j$ is dense in $\bigotimes_{1\leqq j\leqq m} X_j$, we can find a $z(m) := (z_1, z_2, \cdots, z_m)$ in $\bigotimes_{1\leqq j\leqq m} Z_j \subset Z$ such that $\|x(m) - z(m)\| < \varepsilon/2$ so that $\|x - z(m)\| < \varepsilon$ with $z(m) \in Z$. Then Z is a countable dense subset of X, i.e. X is separable. □

We take the Hilbert space $\Gamma(R)$ of square integrable 1-forms α on an open Riemann surface R (cf. [1]), i.e. if $\alpha = adx + bdy$ is the local expression of α on a parametric disc $(U, z = x + iy)$, then coefficients $a(z)$ and $b(z)$ belongs to the Lebesgue space $L^2(U)$ and the square of the norm of α is given by

$$\int_R \alpha \wedge *\alpha = \int_R (a(z)^2 + b(z)^2)dxdy < \infty.$$

We will ascertain that $\Gamma(R)$ is separable. Let

$$R = \cup_{j\in\mathbf{N}} \overline{S}_j$$

be the triangular decomposition of R so that each S_j is piecewise analytic simply connected Jordan region, $S_j \cap S_k = \emptyset$ $(j \neq k)$, and $\{S_j\}_{j\in\mathbf{N}}$ is locally finite. Hence, by renumbering $\{S_j\}_{j\in\mathbf{N}}$ if necessary, if R_n is the interior of $\cup_{1\leqq j\leqq n} \overline{S}_j$, then $\{R_n\}_{n\in\mathbf{N}}$ is an exhaustion of R by piecewise smooth subregions R_n $(n \in \mathbf{N})$. Then

$$\alpha \mapsto (\alpha|S_1, \alpha|S_2. \cdots)$$

gives an isometric injective linear mapping of $\Gamma(R)$ to $\bigotimes_{j\in\mathbf{N}} \Gamma(S_j)$ and we can view that

$$\Gamma(R) \subset \bigotimes_{j\in\mathbf{N}} \Gamma(S_j).$$

The mapping $\alpha = adx + bdy \mapsto (a, b)$ of $\Gamma(S_j)$ to $L^2(S_j) \otimes L^2(S_j)$ is an isometric isomorphism for each $j \in \mathbf{N}$. Hence the separability of $L^2(S_j)$ implies that of $L^2(S_j) \otimes L^2(S_j)$ and hence that of $\Gamma(S_j)$ for each $j \in \mathbf{N}$. Fact 3.1 then assures the separability of $\bigotimes_{j\in\mathbf{N}} \Gamma(S_j)$ and, as its subset, $\Gamma(R)$ is separable. Let

$$dHD(R) := \{du : u \in HD(R)\},$$

which is a closed subspace of $\Gamma(R)$. Therefore $dHD(R)$ is separable. The mapping $u \mapsto du$ of $HD(R; o)$ to $dHD(R)$ is a bijective linear isomorphism and isometric since

$$D(u) = \int_R du \wedge *du = \|du\|^2_{\Gamma(R)}.$$

This proves that $HD(R : o)$ is also separable. Let S be a countable dense subset of $HD(R; o)$. Choose an arbitrary $u \in HD(R)$ and take any $\varepsilon > 0$. Then $u - u(o) \in HD(R; o)$ and we can find an $s \in S$ such that $D((u - u(o)) - s)^{1/2} < \varepsilon$. But

$$D((u - u(o)) - s)^{1/2} = D(u - s)^{1/2}$$

shows that S is also a countable dense subset of $HD(R)$ so that $HD(R)$ is also separable. This simultaneously completes proofs of both of Theorem 1.2 and the corollary to it. □

4. Application: Proof of Theorem M. In addition to two classes $HB(R)$ and $HD(R)$ it is occasionally convenient to consider one more space $HBD(R) := HB(R) \cap HD(R)$. With respect to the combined norm

$$\|u\|_{BD} := \|u\|_\infty + \sqrt{D(u)}$$

the new space is also given as follows:

$$HBD(R) := \{u \in H(R) : \|u\|_{BD} < +\infty\}.$$

Both of the convergence in $\|\cdot\|_\infty$ and $\sqrt{D(\cdot)}$ yield the local uniform convergence and a fortiori we can conclude that $HBD(R)$ is a Banach space equipped with the norm $\|\cdot\|_{BD}$. We can also consider the normalized class

$$HBD(R; o) = \{u \in HBD(R) : u(o) = 0\},$$

which is a closed subspace of $HBD(R)$ so that it is again a Banach space equipped with the above combined norm $\|\cdot\|_{BD}$. We can also say that

$$HBD(R; o) = HB(R; o) \cap HD(R; o).$$

Recall that Theorem M to be proven below maintains the following class identity

$$HB(R) = HD(R) \tag{4.1}$$

is equivalent to the dimensional identity

$$\dim HB(R) = \dim HD(R) < \infty \tag{4.2}$$

as linear spaces. The implication (4.2) ⇒ (4.1) has long been known (cf. e.g. [13,16]) and actually its proof is straightforward. Thus this part has nothing to do with our present application of Theorems 1.1 and 1.2. However, just for the completeness sake, we insert its proof here. For this we use the Virtanen-Royden theorem (cf. e.g. [16]) that the space $HBD(R)$, as a subset of $HD(R)$, is dense in the seminormed space $HD(R)$: $\overline{HBD(R)} = HD(R)$. Since $HBD(R)$ is a subspace of $HB(R)$ which is of finite dimension, we must have $\dim HBD(R) < \infty$. Then $\overline{HBD(R)} = HBD(R)$ and hence $HD(R) = HBD(R) \subset HB(R)$. Under the circumstance $HD(R) \subset HB(R)$, (4.2) must imply $HD(R) = HB(R)$: (4.1).

The essential part of Theorem M is thus the implication (4.1) ⇒ (4.2). Clearly (4.1) is equivalent to the class identity

$$HB(R; o) = HD(R; o). \tag{4.3}$$

Since $\dim HX(R) = \dim HX(R; o) + 1$ for $X = B$ or D, (4.2) is equivalent to the dimensional identity

$$\dim HB(R; o) = \dim HD(R; o) < \infty. \tag{4.4}$$

Here the work of proving (4.1) ⇒ (4.2) is identical with the task of showing (4.3) ⇒ (4.4). Therefore we can say that the main part of Theorem M is, in essence, the implication (4.3) ⇒ (4.4), which we are going to prove. Of course (4.3) merely means the set identity or at the most the identity as linear spaces and saying nothing about Banach space structures at this point. Hence we can at least maintain the validity of the first identity in (4.4): $\dim HB(R; o) = \dim HD(R; o)$. The point here is to show the finiteness of them. Therefore what we really have to show is the following assertion.

Claim 4.5. *The relation* (4.3) *implies that*

$$\dim HB(R; o) < \infty.$$

Proof. Observe that (4.3) means that

$$HB(R; o) = HD(R; o) = HBD(R; o). \tag{4.6}$$

We denote the Banach space $(HB(R; o), \|\cdot\|_\infty)$ by $(X, \|\cdot\|_X)$, the Banach space $(HD(R; o), D(\cdot)^{1/2})$ by $(Y, \|\cdot\|_Y)$, and finally the last third Banach space $(HBD(R; o), \|\cdot\|_{BD})$ by $(Z, \|\cdot\|_Z)$. Let $T_1 : Z \to X$ be the linear operator given by the the identity and $T_2 : Z \to Y$ the linear operator given also by the identity. For every $z \in Z$, we have

$$\|T_1 z\|_X = \|z\|_X \leqq \|z\|_X + \|z\|_Y = \|z\|_Z$$

so that the operator norm $\|T_1\| \leqq 1$, and similarly

$$\|T_2 z\|_Y = \|z\|_Y \leqq \|z\|_X + \|z\|_Y = \|z\|_Z$$

and therefore $\|T_2\| \leqq 1$. By the Banach open mapping principle (cf. e.g. [5,18]), we see that $T_1^{-1} : X \to Z$ and $T_2^{-1} : Y \to Z$ are also bounded, too. Then, we see that

$$T := T_2 \circ T_1^{-1} : X \to Y$$

and the inverse of T, i.e.

$$T^{-1} = T_1 \circ T_2^{-1} : Y \to X$$

are also bounded so that

$$K := \max\bigl(\|T\|, \|T^{-1}\|\bigr) \in [1, +\infty)$$

and we have

$$\|T^{-1}(Tx)\|_X \leqq K\|Tx\|_Y \leqq K^2\|x\|_X$$

for every $x \in X$, which establishes

$$K^{-1}\|u\|_\infty \leqq D(u)^{1/2} \leqq K\|u\|_\infty \tag{4.7}$$

for every u in $X = HBD(R; o)$. Since $HD(R; o)$ is separable by Theorem 1.2, we must conclude with (4.7) and (4.3) the separability of $HB(R; o)$. However Theorem 1.1 says that $HB(R; o)$ is separable if and only if $\dim HB(R; o) < \infty$. $\square$

In the above proof of the fact that Banach spaces $HB(R; o)$ and $HD(R; o)$ are homeomorphically linear isomorphic we can avoid the use of the Banach space $HBD(R; o)$ by observing that each convergence in norm $\|u\|_\infty$ or $\sqrt{D(u)}$ in $HB(R; o)$ or $HD(R; o)$ implies the local uniform convergence on R. However which is simpler belongs to the matter of taste.

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