Comparability of clopen sets in a zero-dimensional dynamical system

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(Communicated by Masaki KASHIWARA, M.J.A., Sept. 12, 2011)

Abstract: Let φ be a homeomorphism on a totally disconnected, compact metric space X. We introduce a binary relation on the family of clopen subsets of X, which is described in terms of the φ-invariant probability measures. We show that φ is uniquely ergodic if and only if any two clopen subsets of X are comparable with respect to the binary relation.

Key words: Unique ergodicity; totally ordered group; countable Hopf-equivalence; ordered Bratteli diagram; Bratteli-Vershik system.

1. Introduction. Let φ be a homeomorphism on a totally disconnected, compact metric space X. Let M_φ denote the set of φ-invariant probability measures. For clopen sets $A, B \subset X$, we write $A \geq B$ either if $\mu(A) > \mu(B)$ for all $\mu \in M_\varphi$, or if $\mu(A) = \mu(B)$ for all $\mu \in M_\varphi$. If φ is minimal, then $A \geq B$ induces an embedding of B into A via finite or countable Hopf-equivalence [9]. The embedding plays significant roles in analyses of orbit structures of Cantor minimal systems [8,9,11] and also in those for locally compact Cantor minimal systems [13]. We refer the reader to [14,16] for other facts concerning Hopf-equivalence.

Another important object in analyses of the orbit structures is ordered group. Let G_φ denote the quotient group of the abelian group $C(X, \mathbf{Z})$ of integer-valued continuous functions on X by a subgroup:

$$Z_\varphi = \{f \in C(X, \mathbf{Z}) | \int_X f d\mu = 0 \text{ for all } \mu \in M_\varphi\}.$$

Let

$$G_\varphi^+ = \{[f] \in G_\varphi | f \geq 0\},$$

where $[f]$ is the equivalence class of $f \in C(X, \mathbf{Z})$. If φ is minimal, then the ordered group (G_φ, G_φ^+) with the canonical order unit is a complete invariant for orbit equivalence [7].

If φ is uniquely ergodic, then any clopen subsets of X are comparable (with respect to $\geq$). As is mentioned above, if in addition φ is minimal, then one of any two clopen subsets of X is embedded into the other clopen subset via countable Hopf-equivalence. These facts may lead us to have questions:

- does a non-uniquely ergodic system always have incomparable clopen sets?
- does a non-uniquely ergodic system always have a pair of clopen sets neither of which is embedded into the other clopen set via countable Hopf-equivalence?

The goal of this paper is to give an affirmative answer to these questions in the following way.

Theorem 1.1. *The following are equivalent*:

(i) φ *is uniquely ergodic*;

(ii) *any two clopen subsets of* X *are comparable*;

(iii) *the ordered group* (G_φ, G_φ^+) *is totally ordered.*

By presenting some examples, we also demonstrate in Section 4 that, in general, neither of the conditions:

- the quotient group $K^0(X, \varphi)$ of $C(X, \mathbf{Z})$ by the coboundary subgroup:

$$B_\varphi = \{f \circ \varphi - f | f \in C(X, \mathbf{Z})\}$$

  is totally ordered;
- one of any two clopen subsets of X is embedded into the other via countable Hopf-equivalence

is equivalent to any condition of Theorem 1.1.

Throughout this paper, we freely use terminology concerning (partially) ordered group, dimension group, ordered Bratteli diagram, tail equivalence relation, Bratteli-Vershik system and etc. See for precise definitions of them [3–5,7,10,12,15].

2. Preliminaries. Put

$$K^0(X, \varphi)^+ = \{[f] \in K^0(X, \varphi) | f \geq 0\}.$$

2010 Mathematics Subject Classification. Primary 37B05; Secondary 37A55.

doi: 10.3792/pjaa.87.123

The quotient group of $K^0(X,\varphi)$ by a subgroup Z_φ/B_φ is order isomorphic to G_φ. If any point in X is chain recurrent for φ, then $(K^0(X,\varphi),K^0(X,\varphi)^+)$ becomes an ordered group; see [2]. This fact is proved also in [16] by means of finite Hopf-equivalence. If φ is minimal (resp. almost minimal), then $(K^0(X,\varphi),K^0(X,\varphi)^+)$ becomes a simple (resp. almost simple) dimension group; see [12] (resp. [3]). In each of these cases, $(K^0(X,\varphi),K^0(X,\varphi)^+)$ with the canonical order unit $[\chi_X]$ is a complete invariant for strong orbit equivalence; see [3,7], where χ_X is the characteristic function of X.

Suppose that φ has a unique minimal set. By [12, Theorem 1.1], any point in X is chain recurrent for φ. Given $\mu\in M_\varphi$, define a state τ_μ on $(K^0(X,\varphi),[\chi_X])$ by for $f\in C(X,\mathbf{Z})$,

$$\tau_\mu([f])=\int_X f d\mu.$$

The map $\mu\mapsto\tau_\mu$ is a bijection between M_φ and the set of states on $(K^0(X,\varphi),[\chi_X])$; see [12, Theorem 5.5].

Proposition 2.1. *(G_φ,G_φ^+) is an ordered group.*

Proof. Suppose that $[f]\in G_\varphi^+\cap(-G_\varphi^+)$ with $f\in C(X,\mathbf{Z})$. There are nonnegative $g_1,g_2\in C(X,\mathbf{Z})$ such that $f-g_1,f+g_2\in Z_\varphi$. Since for all $\mu\in M_\varphi$,

$$0=\int_X(g_1+g_2)d\mu\geq\int_X g_1 d\mu\geq 0,$$

we obtain $[f]=[g_1]=0$, i.e. $G_\varphi^+\cap(-G_\varphi^+)=\{0\}$. Other requirements for (G_φ,G_φ^+) to be an ordered group are readily verified. □

Definition 2.2. Clopen sets $A,B\subset X$ are said to be *countably Hopf-equivalent* if there exist $\{n_i\in\mathbf{Z}|i\in\mathbf{Z}^+\}$ and disjoint unions

$$A=\bigcup_{i\in\mathbf{N}}A_i\cup\{x_0\}\text{ and }B=\bigcup_{i\in\mathbf{N}}B_i\cup\{y_0\}$$

of nonempty clopen sets A_i,B_i and singletons $\{x_0\},\{y_0\}$ such that

- $\varphi^{n_0}(x_0)=y_0$;
- $\varphi^{n_i}(A_i)=B_i$ for every $i\in\mathbf{N}$;
- the map $\alpha:A\to B$ defined by

$$\alpha(x)=\begin{cases}\varphi^{n_i}(x) & \text{if } x\in A_i \text{ and } i\in\mathbf{N};\\ y_0 & \text{if } x=x_0\end{cases}$$

is a homeomorphism.

We shall refer to α as a *countable equivalence map* from A onto B.

Lemma 2.3. *Suppose that φ is minimal. Let $A,B\subset X$ be clopen. Put*

$$D_\varphi=\{[\chi_C]\in G_\varphi|C\subset X\ \textit{is clopen.}\}.$$

Then, the following are equivalent:

(a) $A\geq B$;

(b) *B is countably Hopf-equivalent to a clopen subset of A;*

(c) $[\chi_A]-[\chi_B]\in D_\varphi$.

Proof. By [9, Proposition 2.6], (a) is equivalent to (b). If $\alpha:B\to\alpha(B)\subset A$ is a countable equivalence map, then

$$[\chi_A]-[\chi_B]=[\chi_A]-[\chi_{\alpha(B)}]=[\chi_{A\setminus\alpha(B)}]\in D_\varphi.$$

Hence, (b) implies (c). If $[\chi_A]-[\chi_B]=[\chi_C]$ for some clopen set $C\subset X$, then $\mu(A)-\mu(B)=\mu(C)\geq 0$ for all $\mu\in M_\varphi$. Then, the minimality of φ implies $A\geq B$. Hence, (c) implies (a). This completes the proof. □

3. A proof of Theorem 1.1. (ii) $\Rightarrow$ (iii): We first show that φ must have a unique minimal set on which any φ-invariant probability measure is supported. Let $Y\subset X$ be a minimal set and $\mu\in M_\varphi$ be supported on Y. Suppose $\nu\in M_\varphi\setminus\{\mu\}$. Assume that $\nu(A)>0$ for a clopen set $A\subset X\setminus Y$. Define $\nu'\in M_\varphi$ by for a Borel set $U\subset X$,

$$\nu'(U)=\frac{\nu(U\setminus Y)}{\nu(X\setminus Y)}.$$

By regularity, there exists a clopen set B containing Y such that $\nu'(B)<\nu'(A)$. However,

$$\mu(B)=1>0=\mu(A).$$

This contradicts (ii).

In the remainder of this proof, we tacitly use Lemma 2.3. The fact proved in the preceding paragraph allows us to assume the minimality of φ. Given $a\in G_\varphi$, choose

$$\{a_i,b_j\in D_\varphi\setminus\{0\}|1\leq i\leq n,1\leq j\leq m\}$$

so that

$$a=a_1+a_2+\cdots+a_n-b_1-b_2-\cdots-b_m.$$

The following procedure, consisting of at most m steps, determines $a\geq 0$ or $a\leq 0$.

Step 1. If

$$\sum_{i=1}^n a_i-b_1\leq 0,$$

then $a \le 0$, and the procedure ends. Otherwise, there is k_1 for which

$$c_{k_1} := \sum_{i=1}^{k_1} a_i - b_1 \in D_\varphi \setminus \{0\};$$
$$a = c_{k_1} + a_{k_1+1} + \cdots + a_n - b_2 - b_3 - \cdots - b_m.$$

By this operation, the number of terms b_i decreases by one. We may write

$$a = a_{k_1} + a_{k_1+1} + \cdots + a_n - b_2 - b_3 - \cdots - b_m.$$

Step 2. If

$$\sum_{i=k_1}^{n} a_i - b_2 \le 0,$$

then $a \le 0$, and the procedure ends. Otherwise, there is $k_2 \ge k_1$ for which

$$c_{k_2} := \sum_{i=k_1}^{k_2} a_i - b_2 \in D_\varphi \setminus \{0\};$$
$$a = c_{k_2} + a_{k_2+1} + \cdots + a_n - b_3 - b_4 - \cdots - b_m.$$

By this operation, the number of terms b_i decreases by one. We may write

$$a = a_{k_2} + a_{k_2+1} + \cdots + a_n - b_3 - b_4 - \cdots - b_m.$$

Now, it is clear how we should execute each step. The procedure necessarily ends by Step m. We obtain $a \ge 0$ exactly when the procedure ends at Step m.

(iii) $\Rightarrow$ (i): Assume the existence of a clopen set $A \subset X$ such that

$$c_2 := \inf_{\mu \in M_\varphi} \int_X \chi_A d\mu < \sup_{\mu \in M_\varphi} \int_X \chi_A d\mu =: c_1.$$

Since M_φ is compact, there exist $\mu_i \in M_\varphi$, $i = 1, 2$, such that for each $i = 1, 2$,

$$c_i = \int_X \chi_A d\mu_i.$$

Take $m, n \in \mathbf{N}$ so that

$$c_2 < \frac{n}{m} < c_1.$$

Then,

$$\int_X (m\chi_A - n) d\mu_1 > 0;$$
$$\int_X (m\chi_A - n) d\mu_2 < 0.$$

This contradicts (iii), completing the proof of Theorem 1.1.

The proof of (ii) $\Rightarrow$ (iii) developed above is based on an idea implied in the first paragraph of [6, Subsection 5.4]. G. Elliott showed in [6] that given an AF-algebra A, any two projections in the AF-algebra A are comparable in the sense of Murray and von Neumann if and only if the dimension group associated with the AF-algebra A is totally ordered. The author believes that this result would not immediately lead to Theorem 1.1.

4. Examples. We first provide an example of a non-uniquely ergodic, minimal homeomorphism having incomparable clopen sets. Since $\mathbf{Q}^2$ with the strict ordering is a simple dimension group, there exists a properly ordered Bratteli diagram B such that $K^0(X_B, \lambda_B)$ is order isomorphic to $\mathbf{Q}^2$ by an isomorphism ι mapping the canonical order unit $[\chi_{X_B}]$ to $(1, 1)$, where (X_B, λ_B) is the Bratteli-Vershik system associated with the properly ordered Bratteli diagram B. See for details [4,7,12]. See also [1, 7.7.3]. The homeomorphism λ_B has exactly two ergodic probability measures. The measures μ_i correspond to states $\tau_i : \mathbf{Q}^2 \to \mathbf{Q}$ $(i = 1, 2)$ which are the projections to the i-th coordinate. By [9, Lemma 2.4], there exist clopen sets $C, D \subset X_B$ such that

$$\iota([\chi_C]) = \left(\frac{2}{3}, \frac{1}{2}\right) \text{ and } \iota([\chi_D]) = \left(\frac{1}{2}, \frac{2}{3}\right).$$

Since

$$\mu_1(C) = \frac{2}{3} > \frac{1}{2} = \mu_1(D);$$
$$\mu_2(C) = \frac{1}{2} < \frac{2}{3} = \mu_2(D),$$

the clopen sets C and D are incomparable.

Let V and E denote the vertex set and the edge set of the properly ordered Bratteli diagram B, respectively. The set V is decomposed into pairwise disjoint, finite subsets V_0 (a singleton), $V_1, V_2, \ldots$. The set E is also decomposed into pairwise disjoint, finite subsets $E_1, E_2, \ldots$ so that for each $i \in \mathbf{N}$, each edge in E_i starts from V_{i-1} and terminates at V_i. Since the Bratteli diagram (V, E) is simple, by telescoping B if necessary, we may assume that for each $i \in \mathbf{N}$, there exists an edge from a given vertex in V_{i-1} to a given vertex in V_i. For each $i \in \mathbf{N}$, set

$$V_i' = V_i \cup \{v_i\}$$

with an additional vertex v_i. For each integer $i \geq 2$, add edges to E_i, denoting by E'_i the resulting set, so that in E'_i,

(a) at least two edges exist from v_{i-1} to a given vertex in V_i;
(b) only one edge exists from v_{i-1} to v_i;
(c) there exist no edges from V_{i-1} to v_i.

Put an additional edge e_1 from v_0 to v_1, where $V_0 = \{v_0\}$. Set

$$E'_1 = E_1 \cup \{e_1\}.$$

We obtain a Bratteli diagram (V', E'), where

$$V' = V_0 \cup \bigcup_{i=1}^{\infty} V'_i \text{ and } E' = \bigcup_{i=1}^{\infty} E'_i.$$

For each $i = 1, 2$, extend the measure μ_i on X_B to a measure μ'_i on $X_{(V',E')}$ by assigning each cylinder set $C \subset X_{(V',E')} \setminus X_B$ terminating at $\bigcup_{i=1}^{\infty} V_i$ the μ_i-measure of a cylinder subset of X_B terminating at the range vertex of C; see [15, Lemma 4.4]. By adding more edges to each E'_i with $i \geq 2$ which start from v_{i-1} and terminate at V_i if necessary, we may assume that each μ'_i is infinite. The properties of the Bratteli diagram (V', E') allow us to put a partial order $\geq'$ on E' so that $B' = (V', E', \geq')$ becomes an almost simple, ordered Bratteli diagram. This implies that the associated Bratteli-Vershik system $(X_{B'}, \lambda_{B'})$ is almost minimal; see [3]. Since each μ'_i is invariant under the tail equivalence relation on $X_{(V',E')}$, it is also $\lambda_{B'}$-invariant. The homeomorphism $\lambda_{B'}$ is uniquely ergodic, because there exists a one-to-one correspondence between the set of $\lambda_{B'}$-invariant measures on $X_{B'}$ which are finite on any clopen set disjoint from a fixed point z of $\lambda_{B'}$ and the set of λ_B-invariant finite measures on X_B; see [15, Lemma 4.4]. The unique invariant probability measure is the point mass concentrated on z. Let $C, D \subset X_B$ be as in the preceding paragraph. Observe that

$$\mu'_1(C) = \frac{2}{3} > \frac{1}{2} = \mu'_1(D);$$
$$\mu'_2(C) = \frac{1}{2} < \frac{2}{3} = \mu'_2(D).$$

It follows from these inequalities that neither C nor D is embedded into the other via countable Hopf-equivalence. Let F denote a subgroup:

$$\{[f] \in K^0(X_{B'}, \lambda_{B'}) | z \notin \mathrm{supp}(f), f \in C(X, \mathbf{Z})\},$$

where

$$\mathrm{supp}(f) = \{x \in X_{B'} | f(x) \neq 0\}.$$

Define group homomorphisms $\rho_i : F \to \mathbf{R}$ $(i = 1, 2)$ by

$$\rho_i([f]) = \int_{X_{B'}} f d\mu'_i.$$

Observe that for any $a \in F \cap K^0(X_{B'}, \lambda_{B'})^+$,

$$0 \leq \rho_i(a) < \infty.$$

If the equivalence classes of C and D are comparable in $K^0(X_{B'}, \lambda_{B'})$, then we may obtain a contradiction to the above inequalities. Hence, $K^0(X_{B'}, \lambda_{B'})$ is not totally ordered.

Acknowledgement. Although the previous version of this paper was rejected by a journal, the referee gave the author useful comments, which made a proof of Theorem 1.1 much shorter. The author appreciates it.

References

[1] B. Blackadar, *K-theory for operator algebras*, second edition, Mathematical Sciences Research Institute Publications, 5, Cambridge Univ. Press, Cambridge, 1998.

[2] M. Boyle and D. Handelman, Orbit equivalence, flow equivalence and ordered cohomology, Israel J. Math. **95** (1996), 169–210.

[3] A. I. Danilenko, Strong orbit equivalence of locally compact Cantor minimal systems, Internat. J. Math. **12** (2001), no. 1, 113–123.

[4] F. Durand, B. Host and C. Skau, Substitutional dynamical systems, Bratteli diagrams and dimension groups, Ergodic Theory Dynam. Systems **19** (1999), no. 4, 953–993.

[5] E. G. Effros, *Dimensions and C^*-algebras*, CBMS Regional Conference Series in Mathematics, 46, Conf. Board Math. Sci., Washington, DC, 1981.

[6] G. A. Elliott, On totally ordered groups, and K_0, in *Ring theory* (*Proc. Conf., Univ. Waterloo, Waterloo,* 1978), 1–49, Lecture Notes in Math., 734 Springer, Berlin.

[7] T. Giordano, I. F. Putnam and C. F. Skau, Topological orbit equivalence and C^*-crossed products, J. Reine Angew. Math. **469** (1995), 51–111.

[8] T. Giordano, I. F. Putnam and C. F. Skau, Full groups of Cantor minimal systems, Israel J. Math. **111** (1999), 285–320.

[9] E. Glasner and B. Weiss, Weak orbit equivalence of Cantor minimal systems, Internat. J. Math. **6** (1995), 559–579.

[10] T. Hamachi, M. S. Keane and M. K. Roychowdhury, Finitary orbit equivalence and measured Bratteli diagrams, Colloq. Math. **110** (2008), no. 2, 363–

382.
[11] T. Hamachi, M. S. Keane and H. Yuasa, Universally measure preserving homeomorphisms of Cantor minimal systems, J. Anal. Math., **113** (2011), 1–15.
[12] R. H. Herman, I. F. Putnam and C. F. Skau, Ordered Bratteli diagrams, dimension groups and topological dynamics, Internat. J. Math. **3** (1992), no. 6, 827–864.
[13] H. Matui, Topological orbit equivalence of locally compact Cantor minimal systems, Ergodic Theory Dynam. Systems **22** (2002), no. 6, 1871–1903.
[14] H. Yuasa, Hopf-equivalences for zero-dimensional dynamical systems, Japan. J. Math. (N.S.) **28** (2002), no. 2, 299–312.
[15] H. Yuasa, Invariant measures for the subshifts arising from non-primitive substitutions, J. Anal. Math. **102** (2007), 143–180.
[16] H. Yuasa, Not finitely but countably Hopf-equivalent clopen sets in a Cantor minimal system, Topol. Methods Nonlinear Anal. **33** (2009), no. 2, 355–371.