# Recurrence/transience criteria for skew product diffusion processes

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**Abstract:** We give recurrence/transience criteria for skew products of one dimensional diffusion process and the spherical Brownian motion with respect to a positive continuous additive functional of the former one dimensional diffusion process. Further we give recurrence/transience criteria for their time changed processes.

**Key words:** Recurrence/transience criteria; skew product.

## 1. Recurrence/transience criteria.

Let $(\mathcal{E}, \mathcal{F})$, $\{T_t, t > 0\}$, and $\mathbf{M}$ be a regular Dirichlet form on $L^2(\Xi; m)$, the corresponding Markovian semigroup on $L^2(\Xi; m)$, and the Markov process associated with $(\mathcal{E}, \mathcal{F})$, respectively. Here $\Xi$ is a locally compact separable metric space and $m$ is a positive Radon measure on $\Xi$ such that $\mathrm{supp}[m] = \Xi$. $\mathbf{M}$ is called recurrent [resp. transient] if

$$\int_0^\infty T_t f(\xi)\, dt = 0 \text{ or } \infty \text{ [resp. } < \infty] \quad m\text{-a.e. } \xi \in \Xi,$$

for any $f \in L^1_+(\Xi; m)$, where $L^1_+(\Xi; m)$ is the set of nonnegative functions in $L^1(\Xi; m)$. In this paper we give a simple necessary and sufficient condition for which $\mathbf{M}$ is recurrent/transient when $\mathbf{M}$ is a skew product of a one dimensional diffusion process R and the spherical Brownian motion $\Theta$ with respect to a positive continuous additive functional of R. We denote by X the skew product as above. Further we show that the same condition is necessary and sufficient for which a time changed process Y of X is recurrent/transient.

In [5], we treated the skew product X and the time changed process Y. We discussed there their Feller properties and gave the Dirichlet forms corresponding to X and Y. When the support of the Radon measure $\mu$ corresponding to the time change does not coincide with $I$, the Dirichlet form $\mathcal{E}^{\mathrm{Y}}$ corresponding to Y is represented as a symmetric bilinear form possessing jump terms (see Theorem 5.6 of [5]). It is interesting that Theorem 2 below is valid for Markov processes corresponding to such Dirichlet forms.

Let $s^{\mathrm{R}}$ be a continuous strictly increasing function on an open interval $I = (l_1, l_2)$, and $m^{\mathrm{R}}$ be a right continuous nondecreasing function on $I$, where $-\infty \le l_1 < l_2 \le \infty$. We denote by $\mathrm{R} = [R_t, P^{\mathrm{R}}_r]$ a one dimensional diffusion process on $I$ with scale function $s^{\mathrm{R}}$, speed measure $m^{\mathrm{R}}$ and no killing measure. Throughout this paper, we assume that $\mathrm{supp}[m^{\mathrm{R}}] = I$ and both of the end points $l_i$, $i = 1, 2$, are entrance or natural in the sense of Feller [2]. We also denote by $\Theta = [\Theta_t, P^{\Theta}_\theta]$ the spherical Brownian motion on $S^{d-1} \subset \mathbf{R}^d$ with generator $(1/2)\Delta$, $\Delta$ being the spherical Laplacian on $S^{d-1}$. Let $\nu$ be a Radon measure on $I$ satisfying $\mathrm{supp}[\nu] = I$, and set $\mathbf{f}(t) = \int_I l^{\mathrm{R}}(t, r)\, d\nu(r)$, where $l^{\mathrm{R}}(t, r)$ is the local time of R. We denote by X the skew product of R and $\Theta$ with respect to the positive continuous additive functional $\mathbf{f}(t)$, that is,

$$\mathrm{X} = [X_t = (R_t, \Theta_{\mathbf{f}(t)}),\ P^{\mathrm{X}}_{(r,\theta)} = P^{\mathrm{R}}_r \otimes P^{\Theta}_\theta].$$

Our first result is as follows.

**Theorem 1.** *The skew product* X *is recurrent [resp. transient] if and only if* $s^{\mathrm{R}}(l_1) = -\infty$ *and* $s^{\mathrm{R}}(l_2) = \infty$ *[resp.* $s^{\mathrm{R}}(l_1) > -\infty$ *or* $s^{\mathrm{R}}(l_2) < \infty$*].*

Fukushima and Oshima obtained a recurrent criterion for the skew product of recurrent diffusions (see Theorem 7.2 of [3]). By using their criterion, we can show that the skew product X is recurrent if $s^{\mathrm{R}}(l_1) = -\infty$ and $s^{\mathrm{R}}(l_2) = \infty$ (see Remark 5 below). In this paper we however give a direct proof of our necessary and sufficient condition by means of eigenfunction expansion for transition probability density of X.

We next turn to a time changed process of X. Let $\mu$ be a non-trivial Radon measure on $I$ and set

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$$\mathbf{g}(t) = \int_I l^{\mathrm{R}}(t,r)\,d\mu(r), \quad t > 0.$$

We denote by $\tau(t)$ the right continuous inverse of $\mathbf{g}(t)$. We consider the time changed process $\mathrm{Y} = [Y_t = (R_{\tau(t)}, \Theta_{\mathbf{f}(\tau(t))}),\ P^{\mathrm{Y}}_{(r,\theta)} = P^{\mathrm{R}}_r \otimes P^{\Theta}_\theta]$. Combining some results of [4] with Theorem 1, we obtain the following

**Theorem 2.** *The time changed process* Y *is recurrent [resp. transient] if and only if* $s^{\mathrm{R}}(l_1) = -\infty$ *and* $s^{\mathrm{R}}(l_2) = \infty$ *[resp.* $s^{\mathrm{R}}(l_1) > -\infty$ *or* $s^{\mathrm{R}}(l_2) < \infty$*].*

**2. Proof of Theorem 1.** The semigroup $\{p^{\mathrm{X}}_t,\ t > 0\}$ corresponding to the skew product X is given in [5], that is,

$$(1)\quad \begin{aligned} p^{\mathrm{X}}_t f(r,\theta) &= E^{P^{\mathrm{R}}_r \otimes P^{\Theta}_\theta}[f(R_t, \Theta_{\mathbf{f}(t)})] \\ &= \int_{S^{d-1}} E^{P^{\mathrm{R}}_r}\left[f(R_t,\varphi)\, p^{\Theta}(\mathbf{f}(t),\theta,\varphi)\right] dm^{\Theta}(\varphi) \\ &= \sum_{n=0}^{\infty}\sum_{l=1}^{\kappa(n)} S^l_n(\theta) \int_{S^{d-1}} S^l_n(\varphi) \\ &\qquad \times E^{P^{\mathrm{R}}_r}[f(R_t,\varphi) e^{-\gamma_n \mathbf{f}(t)}]\, dm^{\Theta}(\varphi), \end{aligned}$$

for $t > 0$, $(r,\theta) \in I \times S^{d-1}$, and $f \in C_b(I \times S^{d-1})$. Here $dm^{\Theta}$ is the surface element of $S^{d-1}$ so that $\int_{S^{d-1}} dm^{\Theta} = 2\pi^{d/2}/\Gamma(d/2)$ is the total area of the sphere $S^{d-1}$; $p^{\Theta}(t,\theta,\eta)$ is the transition probability density of $\Theta$; $S^l_n$ $(n = 0,1,2,\cdots, l = 1,2,\cdots,\kappa(n))$ are spherical harmonics; $\gamma_n = n(n+d-2)/2$ and $\kappa(n) = (2n+d-2)\cdot(n+d-3)!/n!(d-2)!$. When $d = 2$, (1) is reduced to the following

$$(2)\quad \begin{aligned} &p^{\mathrm{X}}_t f(r,\theta) \\ &= \frac{1}{2\pi}\int_{S^1} E^{P^{\mathrm{R}}_r}[f(R_t,\varphi)]\, dm^{\Theta}(\varphi) \\ &\quad + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{S^1} \cos n(\theta-\varphi) \\ &\qquad \times E^{P^{\mathrm{R}}_r}[f(R_t,\varphi)e^{-n^2\mathbf{f}(t)/2}]\, dm^{\Theta}(\varphi). \end{aligned}$$

We summarize some properties of spherical harmonics. Let $C^{\beta}_n[x]$ be the Gegenbauer polynomial of degree $n$ and order $\beta$, that is, it is defined by the generating function

$$(3)\quad (1-2xt+t^2)^{-\beta} = \sum_{n=0}^{\infty} C^{\beta}_n[x] t^n, \quad \beta \neq 0,\ |x| < 1,\ |t| < 1.$$

We can take $x = 1$ in (3) so that

$$(4)\quad (1-t)^{-2\beta} = \sum_{n=0}^{\infty} C^{\beta}_n[1] t^n,\ \beta \neq 0,\ |t| < 1,$$

from which

$$(5)\quad 2\beta t(1-t)^{-2\beta-1} = \sum_{n=0}^{\infty} C^{\beta}_n[1] n t^n,\ \beta \neq 0,\ |t| < 1.$$

Here we note the following

$$(6)\quad \sum_{l=1}^{\kappa(n)} S^l_n(\xi)^2 = \frac{\kappa(n)\Gamma(d/2)}{2\pi^{d/2}}, \quad d \geq 2,\ n = 0,1,2,\cdots,\ \xi \in S^{d-1}.$$

When $n = 0$, (6) is obvious because $\kappa(0) = 1$ and $S^1_0(\xi) = \{2\pi^{d/2}/\Gamma(d/2)\}^{-1/2}$. Let $d = 2$ and $n \geq 1$. Then $\kappa(n) = 2$, $S^1_n(\xi) = \pi^{-1/2}\cos n\xi$, and $S^2_n(\xi) = \pi^{-1/2}\sin n\xi$. Therefore we have (6). Let $d \geq 3$ and $n \geq 1$. By means of [1] (see (2) on p. 243),

$$(7)\quad \frac{C^{d/2-1}_n[(\xi,\eta)]}{C^{d/2-1}_n[1]} = \frac{2\pi^{d/2}}{\kappa(n)\Gamma(d/2)}\sum_{l=1}^{\kappa(n)} S^l_n(\xi)S^l_n(\eta),\ \xi,\eta \in S^{d-1},$$

where $(\xi,\eta)$ is the inner product. We get (6) from (7).

We also note the following

$$(8)\quad \sum_{n=0}^{\infty}\kappa(n)t^n = (1+t)(1-t)^{1-d},\ d \geq 2,\ |t| < 1.$$

Here is the proof. When $d = 2$, (8) is obvious since $\kappa(0) = 1$ and $\kappa(n) = 2$ for $n \geq 1$. Let $d \geq 3$. By virtue of [1] (see (29) on p. 236),

$$\kappa(n) = C^{d/2-1}_n[1](2n+d-2)/(d-2),\ n = 0,1,2,\cdots.$$

Combining this with (4) and (5), we get

$$\begin{aligned} \sum_{n=0}^{\infty}\kappa(n)t^n &= \frac{1}{d-2}\sum_{n=0}^{\infty} C^{d/2-1}_n[1](2n+d-2)t^n \\ &= 2t(1-t)^{-d+1} + (1-t)^{-d+2} \\ &= (1+t)(1-t)^{1-d}, \end{aligned}$$

which shows (8).

Now we set

$$\begin{aligned} &Q_T f(r,\theta) \\ &= \int_T^{\infty}\Big\{p^X_t f(r,\theta) \\ &\quad - S^1_0(\theta)\int_{S^{d-1}} S^1_0(\varphi)E^{P^{\mathrm{R}}_r}[f(R_t,\varphi)]\, dm^{\Theta}(\varphi)\Big\}\, dt \\ &= \int_T^{\infty}\Big\{\sum_{n=1}^{\infty}\sum_{l=1}^{\kappa(n)} S^l_n(\theta) \\ &\quad \times \int_{S^{d-1}} S^l_n(\varphi)E^{P^{\mathrm{R}}_r}[f(R_t,\varphi)e^{-\gamma_n\mathbf{f}(t)}]\, dm^{\Theta}(\varphi)\Big\}\, dt. \end{aligned}$$

**Lemma 3.** *For* $(r,\theta) \in I \times S^{d-1}$, $f \in C_b(I \times S^{d-1})$*, and* $T > 0$,

$$|Q_T f(r,\theta)| < \infty.$$

*Proof.* Let us fix $(r,\theta) \in I \times S^{d-1}$, $f \in C_b(I \times S^{d-1})$, and $T > 0$, arbitrarily. We set

$$\tilde{f}(\xi) = \int_{S^{d-1}} |f(\xi,\varphi)|\, dm^{\Theta}(\varphi),\ \xi \in I.$$

By using Schwarz' inequality and (6) and by Fubini's theorem, we find that

$$\begin{aligned} &|Q_T f(r,\theta)| \\ &\le \frac{\Gamma(d/2)}{2\pi^{d/2}} \sum_{n=1}^{\infty} \kappa(n) \int_T^{\infty} E^{P_r^{\mathrm{R}}}[\tilde{f}(R_t) e^{-\gamma_n \mathbf{f}(t)}]\, dt \\ &\le \frac{\Gamma(d/2)}{2\pi^{d/2}} E^{P_r^{\mathrm{R}}}\biggl[\int_T^{\infty} \tilde{f}(R_t) e^{-\mathbf{f}(t)/4} \\ &\qquad \times \sum_{n=1}^{\infty} \kappa(n) e^{-n\mathbf{f}(t)/4}\, dt\biggr], \end{aligned}$$

where we used $\gamma_n/2 \ge n/4 \ge 1/4$ for $n = 1, 2, \cdots$. Combining this with (8), we get

$$\begin{aligned} (9)\quad &|Q_T f(r,\theta)| \\ &\le \frac{\Gamma(d/2)}{2\pi^{d/2}} E^{P_r^{\mathrm{R}}}\biggl[\int_T^{\infty} \tilde{f}(R_t) e^{-\mathbf{f}(t)/4} \\ &\qquad \times (1 + e^{-\mathbf{f}(t)/4})(1 - e^{-\mathbf{f}(t)/4})^{1-d}\, dt\biggr] \\ &\le \frac{\Gamma(d/2)}{\pi^{d/2}} E^{P_r^{\mathrm{R}}}\biggl[(1 - e^{-\mathbf{f}(T)/4})^{1-d} \\ &\qquad \times \int_T^{\infty} \tilde{f}(\mathrm{R}_t) e^{-\mathbf{f}(t)/4}\, dt\biggr]. \end{aligned}$$

Since $P_r^{\mathrm{R}}(\mathbf{f}(T) > 0) = 1$,

$$(10)\qquad (1 - e^{-\mathbf{f}(T)/4})^{1-d} < \infty,\ P_r^{\mathrm{R}}\text{-a.e.}$$

We claim the following

$$(11)\qquad \int_T^{\infty} \tilde{f}(R_t) e^{-\mathbf{f}(t)/4}\, dt < \infty,\ P_r^{\mathrm{R}}\text{-a.e.}$$

Here is the proof of (11). Let $\tilde{\mathrm{R}}$ be the one dimensional diffusion process on $I$ with the scale function $s^{\mathrm{R}}$, the speed measure $m^{\mathrm{R}}$ and the killing measure $\nu/4$. We denote by $\tilde{p}(t,r,\xi)$ and $\tilde{G}_o$ the transition probability density of $\tilde{\mathrm{R}}$ and the 0 order Green operator corresponding to $\tilde{\mathrm{R}}$, respectively. We note that there exists the 0 order Green operator because the killing measure $\nu/4$ is not null. Therefore

$$\begin{aligned} &E^{P_r^{\mathrm{R}}}\biggl[\int_T^{\infty} \tilde{f}(R_t) e^{-\mathbf{f}(t)/4}\, dt\biggr] \\ &= \int_T^{\infty} \int_{\xi \in I} \tilde{f}(\xi)\tilde{p}(t,r,\xi)\, dm^{\mathrm{R}}(\xi) dt \\ &\le \tilde{G}_o \tilde{f}(r) < \infty, \end{aligned}$$

which shows (11).

By means of (9), (10), and (11), and by the monotone's convergence theorem, we see that

$$\begin{aligned} &\limsup_{T\to\infty} |Q_T f(r,\theta)| \\ &\le \frac{\Gamma(d/2)}{\pi^{d/2}} E^{P_r^{\mathrm{R}}}\biggl[\lim_{T\to\infty} (1 - e^{-\mathbf{f}(T)/4})^{1-d} \\ &\qquad \times \int_T^{\infty} \tilde{f}(R_t) e^{-\mathbf{f}(t)/4}\, dt\biggr] \\ &= 0. \end{aligned}$$

This implies the conclusion of the lemma. □

By means of (1), we see that $|p_t^{\mathrm{X}} f(r,\theta)| \le \sup_{(r,\theta)\in I\times S^{d-1}} |f(r,\theta)|$. Therefore we have the following lemma.

**Lemma 4.** *For* $(r,\theta) \in I \times S^{d-1}$, $f \in C_b(I \times S^{d-1})$ *and* $T > 0$,

$$\int_0^T |p_t^{\mathrm{X}} f(r,\theta)|\, dt < \infty.$$

Now we show Theorem 1. We fix an $f \in L^1_+(I \times S^{d-1}, m^{\mathrm{R}} \otimes m^{\Theta}) \cap C_b(I \times S^{d-1})$ such that $m^{\mathrm{R}} \otimes m^{\Theta}\{(r,\theta) \in I \times S^{d-1} :\ f(r,\theta) > 0\} > 0$. By means of Lemmas 3 and 4,

$$(12)\qquad \int_0^{\infty} p_t^{\mathrm{X}} f(r,\theta)\, dt = \infty$$

if and only if

$$S_0^1(\theta) \int_T^{\infty} \int_{S^{d-1}} S_0^1(\varphi) E^{P_r^{\mathrm{R}}}[f(R_t,\varphi)]\, dm^{\Theta}(\varphi)\, dt = \infty,$$

for $T > 0$. Since $S_0^1(\theta) = \{\Gamma(d/2)/2\pi^{d/2}\}^{1/2}$, (12) holds true if and only if

$$\begin{aligned} (13)\quad &\int_T^{\infty} \int_{S^{d-1}} E^{P_r^{\mathrm{R}}}[f(R_t,\varphi)]\, dm^{\Theta}(\varphi)\, dt \\ &= \int_T^{\infty} E^{P_r^{\mathrm{R}}}[f^*(R_t)]\, dt = \infty, \end{aligned}$$

for $T > 0$, where $f^*(\xi) = \int_{S^{d-1}} f(\xi,\varphi)\, dm^{\Theta}(\varphi)$. Since $E^{P_r^{\mathrm{R}}}[f^*(R_t)]$ is the semigroup of R, (13) holds true if and only if R is recurrent, or, $s^{\mathrm{R}}(l_1) = -\infty$ and $s^{\mathrm{R}}(l_2) = \infty$.

Obviously the semigroups corresponding to R and $\Theta$ are irreducible. Therefore by means of

Theorem 7.2 of [3], $\{p_t^{\mathrm{X}}, t > 0\}$ is irreducible. Combining this with Lemma 1.6.4 of [4], we find that X is recurrent or transient. Thus $s^{\mathrm{R}}(l_1) = -\infty$ and $s^{\mathrm{R}}(l_2) = \infty$ [resp. $s^{\mathrm{R}}(l_1) > -\infty$ or $s^{\mathrm{R}}(l_2) < \infty$] if and only if X is recurrent [resp. transient]. □

**Remark 5.** Assume that R is recurrent, that is, $s^{\mathrm{R}}(l_1) = -\infty$ and $s^{\mathrm{R}}(l_2) = \infty$. Since $\Theta$ is recurrent and $m^{\Theta}(S^{d-1}) < \infty$, by virtue of Theorem 7.2 of [3], X is recurrent.

**Remark 6.** In the same way as in the next section, we can directly derive the irreducibility of X from (1).

**3. Proof of Theorem 2.** First we note that Y is irreducible. It is enough to show that for any Borel sets $A_i \subset I$, $B_i \subset S^{d-1}$ satisfying $\mu(A_i) > 0$, $m^{\Theta}(B_i) > 0$ $(i = 1, 2)$, there exists a $t > 0$ such that

$$(14) \quad \int_{A_1 \times B_1} P^{\mathrm{Y}}_{(r,\theta)}(Y_t \in A_2 \times B_2)\, d\mu \otimes dm^{\Theta}(r, \theta) > 0.$$

The semigroup $\{p_t^{\mathrm{Y}},\ t > 0\}$ corresponding to Y is given by (1) with $t$ replaced by $\tau_t$, that is,

$$\begin{aligned} p_t^{\mathrm{Y}} f(r, \theta) &= E^{P_r^{\mathrm{R}} \otimes P_\theta^{\Theta}}[f(R_{\tau_t}, \Theta_{\mathbf{f}(\tau_t)})] \\ &= \int_{S^{d-1}} E^{P_r^{\mathrm{R}}}\left[f(R_{\tau_t}, \varphi)\, p^{\Theta}(\mathbf{f}(\tau_t), \theta, \varphi)\right] dm^{\Theta}(\varphi), \end{aligned}$$

for $t > 0$, $(r, \theta) \in \Lambda \times S^{d-1}$ with $\Lambda = \mathrm{supp}[\mu]$, and $f \in C_b(I \times S^{d-1})|_{\Lambda \times S^{d-1}}$. Therefore

$$\begin{aligned} (15) \quad & \int_{A_1 \times B_1} P^{\mathrm{Y}}_{(r,\theta)}(Y_t \in A_2 \times B_2)\, d\mu \otimes dm^{\Theta}(r, \theta) \\ &= \int_{r \in A_1,\ \theta \in B_1,\ \phi \in B_2} E^{P_r^{\mathrm{R}}}[I_{A_2}(R_{\tau_t}) P^{\Theta}(\mathbf{f}(\tau_t), \theta, \varphi)] \\ &\quad \times d\mu(r) dm^{\Theta}(\theta) dm^{\Theta}(\varphi). \end{aligned}$$

Since $[R_{\tau_t}, P_r^{\mathrm{R}}]$ is a one dimensional generalized diffusion process on $I$, we see that

$$(16) \qquad P_r^{\mathrm{R}}(R_{\tau_t} \in A_2) > 0,\ r \in A_1,\ t > 0.$$

We also see that

$$(17) \qquad P_r^{\mathrm{R}}(\mathbf{f}(\tau_t) > 0) = 1,\ r \in A_1,\ t > 0,$$

and hence

$$(18) \qquad \begin{aligned} & P_r^{\mathrm{R}}(p^{\Theta}(\mathbf{f}(\tau_t), \theta, \varphi) > 0) = 1, \\ & r \in A_1,\ t > 0,\ \theta, \varphi \in S^{d-1}. \end{aligned}$$

(14) follows from (15)–(18).

Since Y is irreducible, Y is recurrent or transient by Lemma 1.6.4 of [4]. By means of Theorem 6.2.3 of [4], Y is recurrent [resp. transient] if X is recurrent [resp. transient]. Combining this with Theorem 1, we obtain Theorem 2. □

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