A simplification of the proof of Bol's conjecture on sextactic points

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Abstract: In a previous work with Thorbergsson, it was proved that a simple closed curve in the real projective plane $\mathbf{P}^2$ that is not null-homotopic has at least three sextactic points. This assertion was conjectured by Gerrit Bol. That proof used an axiomatic approach called 'intrinsic conic system'. The purpose of this paper is to give a more elementary proof.

Key words: Sextactic points; affine curvature; inflection points; affine evolute.

1. Introduction. In the real projective plane $\mathbf{P}^2$, one can consider their osculating conics and sextactic points of curves. Choose five points on a curve γ in a neighborhood of a point p that are not inflection points. There is a unique regular conic passing through the five points. Letting the five points all converge to p, and so the conics converge to a uniquely defined regular conic that is called the *osculating conic* of γ in p. The osculating conic meets with multiplicity at least five in p. If it meets with multiplicity at least six in p, then p is called a *sextactic point*. In this paper, we shall assume curves are all C^∞-regular. It should be pointed out that the assertions in this paper are more generally true for curves that are only C^4 with essentially the same proofs (see Remark 2 below).

Theorem. (Thorbergsson-Umehara [12]). *Let γ be a simple closed curve in $\mathbf{P}^2$ that is not null-homotopic. Then γ has at least three sextactic points.*

The theorem was stated as a problem by Bol in [4] on page 43. A proof of the theorem under rather strong genericity assumptions on the inflection points was given by Fabricius-Bjerre in [6]. In this paper, a simple arc means an arc without self-intersections. To prove the theorem, it is sufficient to show the following assertion, as explained in the next section. (We denote by $|\mathbf{a}, \mathbf{b}|$ the determinant of the square 2-matrix $(\mathbf{a}, \mathbf{b})$ for $\mathbf{a}, \mathbf{b} \in \boldsymbol{R}^2$.)

Proposition. *Let $\gamma : [0,1] \to \boldsymbol{R}^2$ be a closed curve in the affine plane $\boldsymbol{R}^2$. Suppose that $|\dot\gamma(t), \ddot\gamma(t)|$ is positive on $(0,1)$ and vanishes at $t = 0, 1$. Then there exists a sextactic point of γ on $(0,1)$.*

This proposition is a generlization of [12, Proposition 5.1] where the same assertion has been proved when γ has no self-intersections. The proof of [12, Proposition 5.1] is not elementary and is accomplished by introducing a powerful tool to find sextactic points called the '*intrinsic conic system*'. However, by using the tool, the following assertion is also proved.

Fact [12]. *The total number of sextactic and inflection points on a simple closed curve in $\mathbf{P}^2$ which is null-homotopic is at least four.*

Here, *inflection points* are defined as zeros of the function $|\dot\gamma, \ddot\gamma|$. This assertion does not follow from the proposition, since the number of inflection points may not be even (cf. [12, Example B.4]). The fact above is a generalization of the classical fact that *a closed strictly convex curve has at least six sextactic points*. A refinement of this assertion is given in [13, Theorem 1.2]. Historical remarks on sextactic points are written in [12] (see also [10, p73]).

In this paper, we show that the proposition can be proved much more easily.

2. Proof of Theorem. Before proving the proposition, we shall observe that the theorem follows from the proposition according to [12]: Let $\gamma : [0,1] \to \mathbf{P}^2$ be a simple closed curve as in the theorem. Of importance for us is the well-known classical result of Möbius [9] that a simple closed curve in $\mathbf{P}^2$ that is not null-homotopic has at least three inflection points. (An elementary proof is in [11]. A refinement of this fact when the curve has a suitable convexity is given in [14].) Then we can take three subarcs

$$\gamma_i : [0,1] \to \mathbf{P}^2 \qquad (i = 1, 2, 3)$$

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of γ such that they have no inflection points except when $t=0,1$ and then each image of γ_i lies in an affine plane. Here, we applied the fact that *any simple closed arc* $\sigma:[0,1]\to\mathbf{P}^2$ *having no inflection points on the interval* $(0,1)$ *lies in an affine plane*. This assertion is a key lemma of the tennis ball theorem which assets that *a simple closed curve having at most two inflection points lies in an affine plane*. (see [1,2,12, Appendix A] and also [10, p100]). Now we can apply the proposition for each γ_i and can find a sextactic point on each open arc $\gamma_i|_{(0,1)}$, which proves the assertion.

Now, we give a proof of the proposition. Let γ be a curve as in the proposition and $\mu:(0,1)\to\boldsymbol{R}$ the *affine curvature function* of γ, that is,

$$\mu=\frac{12|\dot\gamma,\ddot\gamma|\,|\ddot\gamma,\gamma^{(3)}|+3|\dot\gamma,\ddot\gamma|\,|\dot\gamma,\gamma^{(4)}|-5|\dot\gamma,\gamma^{(3)}|^2}{9|\dot\gamma,\ddot\gamma|^{8/3}},$$

which is independent of the choice of a parametrization t of γ. The critical points of $\mu(t)$ correspond to the sextactic points (cf. [5, p12]). To prove the proposition, we prepare the following assertion.

Lemma. *Let* $\gamma:[0,1]\to\boldsymbol{R}^2$ *be a curve such that* $|\dot\gamma(t),\ddot\gamma(t)|$ *is positive for* $t>0$ *and vanishes at* $t=0$. *Then for each positive integer* n, *there exists* $t_n\in(0,1/n)$ *such that* $\mu(t_n)\le -n$. *In particular,* $\inf_{t\in(0,x]}\mu(t)$ *diverges to* $-\infty$ *as* $x\to 0$.

Remark 1. Fabricius-Bjerre [6] observed that $\mu(t)$ diverges to $-\infty$ if $|\dot\gamma(0),\gamma^{(3)}(0)|$ does not vanish. More generally, as pointed out in [12, p79], $\lim_{t\to 0}\mu(t)=-\infty$ holds unless all derivatives of $\gamma(t)$ at $t=0$ vanish. The lemma is a generalization of those results. In general, $\mu(t)$ may not be bounded from above (see the example below).

Let $\gamma(t)$ be a curve without inflection points in affine plane, then

$$\mathbf{n}(t):=|\dot\gamma,\ddot\gamma|^{-3/2}\ddot\gamma-\frac{1}{3}|\dot\gamma,\ddot\gamma|^{-5/2}|\dot\gamma,\gamma^{(3)}|\dot\gamma$$

is called the *affine normal vector field* along γ. The vector $\mathbf{n}(t)$ points the direction of the locus of middle points of chords of the curve parallel to the tangent line at $\gamma(t)$ (cf. [3, p6] or [5, p9]). Then the envelop generated by affine normal lines of γ is given by

$$\sigma(t):=\gamma(t)+\frac{1}{\mu(t)}\mathbf{n}(t),$$

which is called the *affine evolute* or the *affine caustic* of γ. (Several properties of affine evolutes are given in Giblin-Sapiro [7] and Izumiya-Sano [8].) As a consequence, we get the following assertion.

Corollary. *Let* γ *be as in the assumption of the lemma. Then the affine evolute* $\sigma(t)$ *of* $\gamma(t)$ *accumulates to the inflection point* $\gamma(0)$. *Moreover,* $\sigma(t)$ *converges to* $\gamma(0)$ *as* $t\to 0$, *if at least one of* $\gamma^{(m)}(0)$ $(m\ge 3)$ *does not vanish.*

The proposition follows from the lemma immediately, since the affine curvature function μ of γ as in the proposition must have at least one local maximum point on $(0,1)$.

We now prove the lemma. Let (x,y) be the canonical coordinate system of $\boldsymbol{R}^2$. Without loss of generality, we may assume that the arc γ can be locally expressed as a graph $y=f(x)$ of $\boldsymbol{R}^2$ defined on $(0,\varepsilon)$ for some fixed $\varepsilon>0$ such that $x=0$ is the inflection point satisfying $f(0)=\dot f(0)=0$. Then the condition $|\dot\gamma(t),\ddot\gamma(t)|>0$ implies $\ddot f(x)>0$ if $x>0$. Since $x=0$ is an inflection point, $\ddot f(x)$ tends to zero as $x\to 0$. The affine curvature function of γ is expressed by

$$\mu(x)=-\frac{5f^{(3)}(x)^2-3f^{(4)}(x)\ddot f(x)}{9\ddot f(x)^{8/3}},$$

which can be rewritten as (cf. [3, p14, (83)] or [5, p8, (3.21)])

$$(1)\qquad \mu(x)=-\frac{\ddot\varphi_f(x)}{2},\qquad \varphi_f(x):=\ddot f(x)^{-2/3}.$$

Since $\varphi(x)>0$ and $\lim_{x\to 0+0}\varphi(x)=\infty$, the proof of the lemma reduces to the following assertion.

Sub-lemma. *Let* $\varphi(x)$ *be a positive-valued* C^2*-function defined on* $(0,1]$. *If* $\varphi(x)$ *diverges to* ∞ *as* $x\to 0$, *then for each positive integer* n, *there exists* $x_n\in(0,\frac{1}{n}]$ *such that* $\ddot\varphi(x_n)>n$.

Proof. If the assertion fails, then $\ddot\varphi(x)\le n$ holds for $x\in(0,\frac{1}{n}]$. Integrating this inequality twice on the interval $[x,\frac{1}{n}]$, we have

$$\varphi(x)\le\frac{nx^2}{2}+\left(\dot\varphi\left(\frac{1}{n}\right)-1\right)x+c_n,$$

where

$$c_n:=\frac{1}{2n}-\frac{1}{n}\dot\varphi\left(\frac{1}{n}\right)+\varphi\left(\frac{1}{n}\right).$$

If x tends to zero, the right-hand side converges to the constant c_n, which contradicts the fact that $\varphi(x)\to\infty$. □

Example. We set

$$f(x):=\int_0^x\int_0^s e^{-h(t)/t^2}dtds,\quad h(x):=2-\sin\frac{1}{x^2}.$$

Since $h \geq 1$, $f(x)$ is C^∞ at $x = 0$. Moreover, $x = 0$ is an isolated inflection point, since $\ddot{f}(x) = e^{-h(x)/x^2}$. By a straightforward calculation, we have that

$$\frac{f^{(3)}}{2} = -\frac{\cos z + x^2(\sin z - 2)}{x^5 e^{h(x)/x^2}},$$
$$\frac{f^{(4)}}{4} = \frac{\cos^2 z - x^2(\sin z + 4\cos z - \sin 2z) + o(x^2)}{x^{10} e^{h(x)/x^2}},$$

where $z = 1/x^2$ and $o(x^2)$ is the higher order term than x^2. We set

$$\mu_0(x) := \frac{x^{10} e^{2h(x)/x^2}}{4} (5 f^{(3)}(x)^2 - 3 f^{(4)}(x) \ddot{f}(x)),$$

then we have that

$$\mu_0 = 1 + \cos 2z + x^2(3\sin z - 8\cos z + 2\sin 2z) + o(x^2).$$

In particular, it holds that

$$\mu_0(x)|_{z=3\pi/2} = -3x^2 + o(x^2).$$

The relation $\mu = -4x^{-10} e^{2h(x)/3x^2} \mu_0/9$ yields that the sequence $\{\mu(x_n)\}_{n=1,2,\ldots}$ diverges to ∞ if we set

$$x_n := \frac{1}{\sqrt{2\pi n + 3\pi/2}} \qquad (n = 1, 2, \ldots).$$

Thus, we get the following two relations

$$\lim_{x\to 0} \sup_{|t|\leq|x|} \mu(x) = \infty, \quad \lim_{x\to 0} \inf_{|t|\leq|x|} \mu(x) = -\infty.$$

Remark 2. If one wishes to prove the theorem for a C^4-regular curve as in [12], a sextactic point should be defined as a point whose osculating conic is locally contained in the closure of one side of the curve, since the sexctactic point cannot be characterized as a critical point of the affine curvature function. In fact, *a point on an arc which attains local maximum or minimum of the affine curvature function is a sextactic point in this sense*: Let $f(x)$ be a C^4-function defined on the interval $(-\delta, \delta)$ $(\delta > 0)$. We consider the case that $x = 0$ attains the maximum of the affine curvature function $\mu(x)$, namely, it holds that $\mu(x) \leq \mu(0)$ for $|x| < \delta$. We may assume that $f(0) = \dot{f}(0) = 0$ and $\ddot{f}(0) > 0$. Let $y = g(x)$ be the local graph of the osculating conic at $x = 0$. Then the affine curvature function of $g(x)$ is the constant $\mu(0)$. By (1), it holds that

$$(2) \qquad -\ddot{\varphi}_f(x) = 2\mu(x) \leq 2\mu(0) = -\ddot{\varphi}_g(x).$$

Since the osculating conic meets the curve with multiplicity 5, it holds that

$$f^{(i)}(0) = g^{(i)}(0) \qquad (0 \leq i \leq 4).$$

Thus, integrating (2) on the interval between 0 and x twice, we have $\varphi_f(x) \geq \varphi_g(x)$, namely, it holds that

$$\ddot{g}(x) \geq \ddot{f}(x) \qquad (|x| < \delta).$$

Again, integrating this twice, we get the inequality $g(x) \geq f(x)$ on $(-\delta, \delta)$, that is, $x = 0$ satisfies the definition of sextactic point as above.

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