Binding numbers of fractional k-deleted graphs

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Abstract: Let k be an integer with $k \geq 2$. We show that if G be a graph such that $|G| > 4k + 1 - 4\sqrt{k-1}$ and $bind(G) > \dfrac{(2k-1)(n-1)}{k(n-2)}$, then G is a fractional k-deleted graph. We also show that in the case where k is even, if G be a graph such that $|G| > 4k + 1 - 4\sqrt{k}$ and $bind(G) > \dfrac{(2k-1)(n-1)}{k(n-2)+1}$, then G is a fractional k-deleted graph.

Key words: Binding number; fractional factor.

1. Introduction. In this paper, we consider only finite, simple, undirected graphs with no loops and no multiple edges.

Let $G = (V(G), E(G))$ be a graph. For $x \in V(G)$, $N_G(x)$ denotes the set of vertices adjacent to x in G, and $\deg_G(x)$ denotes the degree of x in G. We let $\delta(G)$ denote the minimum of $\deg_G(x)$ as x ranges over $V(G)$.

For $X \subseteq V(G)$, we let $N_G(X)$ denote the union of $N_G(x)$ as x ranges over X. The binding number $bind(G)$ of G is defined as

$$bind(G) = \min\left\{\frac{|N_G(X)|}{|X|}\,\middle|\, \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G)\right\}.$$

For an integer $k \geq 1$, a subgraph F of G such that $V(F) = V(G)$ and $\deg_F(x) = k$ for all $x \in V(F)$ is called a k-factor of G. We can find theorems concerning the relation between the binding number and the existence of k-factors in [1].

For $x \in V(G)$, $E(x)$ denotes the set of edges incident with x. For an integer $k \geq 1$, a fractional k-factor is a function h that assigns a real number in $[0, 1]$ to each edge of a graph G so that for each vertex x we have $\deg_G^h(x) = k$, where $\deg_G^h(x) = \sum_{e\in E(x)} h(e)$ is the fractional degree of x in G. A graph G is a fractional k-deleted graph if there exists a fractional k-factor for the subgraph obtained by deleting an arbitrary edge of G.

The following theorem was proved by Zhou in [4].

Theorem A. *Let k be an integer with $k \geq 2$. Let G be a graph of order n with $n \geq 4k - 5$, and suppose that*

$$bind(G) > \frac{(2k-1)(n-1)}{k(n-2)}.$$

Then G is a fractional k-deleted graph.

The purpose of this paper is to weaken the condition on the order of G in Theorem A.

Theorem 1. *Let k be an integer with $k \geq 2$. Let G be a graph of order n with $n > 4k + 1 - 4\sqrt{k-1}$, and suppose that*

$$bind(G) > \frac{(2k-1)(n-1)}{k(n-2)}.$$

Then G is a fractional k-deleted graph.

Moreover, in the case where k is even, we can relax the binding number condition as follows:

Theorem 2. *Let k be an even integer with $k \geq 2$. Let G be a graph of order n with $n > 4k + 1 - 4\sqrt{k}$, and suppose that*

$$bind(G) > \frac{(2k-1)(n-1)}{k(n-2)+1}.$$

Then G is a fractional k-deleted graph.

The following example shows that the bound on the order of G in Theorem 1 is best possible.

Example 3. *Let h be an even non-negative integer, and set $k = h^2 + 1$, $n = 4h^2 - 4h + 5$ and $a = 2h^2 - 3h + 3$. Let H be a complete graph of order a,*

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and let I be a cycle of order $n-a$ to the power of $h/2$. Let G be the graph obtained from $H \cup I$ by adding an edge joining vertices x and y for arbitrary $x \in V(H)$ and $y \in V(I)$. Then $|G| = n = 4k + 1 - 4\sqrt{k-1}$, and, by $n > 2k+1$, we see

$$bind(G) = \frac{n-1}{n-a-h} = 2 > \frac{(2k-1)(n-1)}{k(n-2)}.$$

On the other hand, G is not a fractional k-deleted graph (if we apply Theorem B in Section 2 with $S = V(H)$ and $T = V(I)$, then we get $\theta(S,T) = 1 < 2$).

The bound on the order of G in Theorem 2 is also best possible.

Example 4. *Let h be an even non-negative integer, and set $k = h^2$, $n = 4h^2 - 4h + 1$ and $a = 2h^2 - 3h + 1$. Let H be a complete graph of order a, and let I be a cycle of order $n-a$ to the power of $h/2$. Let G be the graph obtained from $H \cup I$ by adding an edge joining vertices x and y for arbitrary $x \in V(H)$ and $y \in V(I)$. Then $|G| = n = 4k + 1 - 4\sqrt{k}$, and, by $n > 2k-1$, we see*

$$bind(G) = \frac{n-1}{n-a-h} = 2 > \frac{(2k-1)(n-1)}{k(n-2)+1},$$

On the other hand, as in the prceeding paragraph, we see G is not a fractional k-deleted graph.

The following example shows that the bounding number condition in Theorem 2 is best possible.

Example 5. *Let k be an even non-negative integer, and let r be an integer with $r \geq \left\lceil \frac{4k+1-4\sqrt{k}}{2k-1} \right\rceil$. Set $l = \frac{kr}{2}$, $m = kr - r$ and $n = m + 2l$. Let H be a complete graph of order m, and I be the union of l complete graphs of order 2. Let G be the graph obtained from $H \cup I$ by adding an edge joining vertices x and y for arbitrary $x \in V(H)$ and $y \in V(I)$. Then $|G| = n = 2kr - r$, and*

$$bind(G) = \frac{n-1}{kr-1} = \frac{(2k-1)(n-1)}{k(n-2)+1}.$$

On the other hand, G is not a fractional k-deleted graph (if we apply Theorem B in Section 2 with $S = V(H)$ and $T = V(I)$, then we get $\theta(S,T) = 1 < 2$).

Our notation is standard possibly except the following

Let G be a graph. For A, $B \subseteq V(G)$ with $A \cap B = \emptyset$, $E(A,B)$ denotes the set of those edges of G which join a vertex in A and a vertex in B. For $A \subseteq V(G)$, the graph obtained from G by deleting all vertices in A together with the edges incident with them is denoted by $G - A$. For a subset T of $V(G)$, we often identify a induced subgraph on T of G with its vertex set T.

2. Preliminary results. In this section, we state preliminary results, which we use in the proof of the theorems.

First we give the following numerical result which is applied in the proof of theorems.

Lemma 2.1. *Let a, b and c be integers such that $a \geq 2$, $2 \leq b \leq a-1$, $c = 0$ or 1, and let x and y be nonnegative integers. Suppose that*

$$x \leq \frac{(a-b)y+c}{2a-b} \tag{2.1}$$

and

$$x > \frac{(a-1)y+c}{2a-1} + 1 - b. \tag{2.2}$$

Then $y \leq 4a + 1 - 4\sqrt{a-c}$.

Proof. By (2.1) and (2.2),

$$\frac{(a-1)y+c}{2a-1} + 1 - b < \frac{(a-b)y+c}{2a-b},$$

and hence

$$y < 4a - 2b - 2 + \frac{b+c}{a} \leq 4a - 2b - 1 < 4a - 2.$$

Thus $\frac{y}{2a-1} < 2$, and this implies

$$\frac{y-2c}{2a-1} < 2. \tag{2.3}$$

By (2.2), $2x > y - \frac{y-2c}{2a-1} + 2 - 2b$, and this together with (2.3) implies $2x > y - 2b$, thus

$$x \geq \frac{y-2b+1}{2}. \tag{2.4}$$

By (2.1) and (2.4),

$$y \leq 4a + 1 - 2\Big(b + \frac{a-c}{b}\Big) \leq 4a + 1 - 4\sqrt{a-c}, \tag{2.5}$$

as desired. □

The following lemma concerning the binding number and the minimum degree is well known.

Lemma 2.2 [3]. *Let G be a graph of order n with bind$(G) > c$. Then $\delta(G) > n - \frac{n-1}{c}$.*

Let k be an integer, and let G be a graph. For $S, T \subseteq V(G)$ with $S \cap T = \emptyset$, we define $\theta(S,T)$ by

$$\theta(S,T) = k|S| + \Sigma_{y\in T}(\deg_{G-S}(y) - k),$$

and we define $\varepsilon(S,T)$ by

$$\varepsilon(S,T) = \begin{cases} 2 & (T \text{ is not independent}), \\ 1 & (T \text{ is independent}, \\ & \text{ and } |E(T, V(G)-S-T)| \geq 1), \\ 0 & (\text{otherwise}). \end{cases}$$

The following theorem is essential for our proof.

Theorem B [2]. *Let G be a graph. G is a fractional k-deleted graph if and only if $\theta(S, T) \geq \varepsilon(S, T)$ for arbitrary S, $T \subseteq V(G)$ with $S \cap T = \emptyset$.*

By the definition of $\varepsilon(S,T)$, we obtain the following lemma easily.

Lemma 2.3. $\varepsilon(S,T) \leq \min\{2, |T|\}$.

Throughout the rest of this section, let k be an integer with $k \geq 2$, and let G be a graph such that $|G| > 4k + 1 - 4\sqrt{k}$, and bind$(G) > \frac{(2k-1)(n-1)}{k(n-2)+1}$, and G is not a fractional k-deleted graph. Then, by Theorem B, there exist $S, T \subseteq V(G)$ with $S \cap T = \emptyset$ such that $\theta(S,T) < \varepsilon(S,T)$.

By Lemma 2.3,

$$\theta(S,T) \leq 1, \tag{2.6}$$

$$\text{and } \theta(S,T) \leq |T| - 1. \tag{2.7}$$

We set $n = |G|$ and $h = \min\{\deg_{G-S}(y) \mid y \in T\}$. Note that we have

$$|T| \leq n - |S|. \tag{2.8}$$

Under these assumptions, we prove the following three claims.

Claim 2.4. $h \leq k-1$.

Proof. By Lemma 2.2,

$$\delta(G) > \frac{(k-1)n}{2k-1} + 1.$$

Since $4k + 1 - 4\sqrt{k} \geq 2k - 1$, $n > 2k - 1$. Hence $\delta(G) > k$, thus

$$\delta(G) \geq k+1. \tag{2.9}$$

To prove this claim, we assume that $h \geq k$. Then $\sum_{y\in T}(\deg_{G-S}(y) - k) \geq 0$, this together with (2.6) implies $k|S| \leq 1$, thus $S = \emptyset$. Hence, we have $\theta(S,T) = \sum_{y\in T}(\deg_G(y) - k) \geq |T|$ by (2.9), which contradicts (2.7). □

Claim 2.5. $|S| \leq \frac{(k-h)n+1}{2k-h}$. *If k is even, then $|S| \leq \frac{(k-h)n}{2k-h}$.*

Proof. By (2.6),

$$k|S| + \sum_{y\in T}(\deg_{G-S}(y) - k) \leq 1, \tag{2.10}$$

and hence

$$k|S| + (h-k)|T| \leq 1. \tag{2.11}$$

By (2.8), (2.11), and Claim 2.4, we obtain $|S| \leq \frac{(k-h)n+1}{2k-h}$.

Now we suppose that k is even. Assume for the moment that h is even. Then the left-hand side of (2.11) is even, thus

$$k|S| + (h-k)|T| \leq 0. \tag{2.12}$$

By (2.8), (2.12), and Claim 2.4, we obtain $|S| \leq \frac{(k-h)n}{2k-h}$. Assume now that h is odd. In the case where $V(G) - S - T \neq \emptyset$, we have

$$|T| \leq n - |S| - 1. \tag{2.13}$$

By (2.11), (2.13), and Claim 2.4, we obtain $|S| \leq \frac{(k-h)n}{2k-h}$. In the case where there exists $y \in T$ such that $\deg_{G-S}(y) \geq h+1$, we have

$$\sum_{y\in T}\deg_{G-S}(y) \geq h|T| + 1. \tag{2.14}$$

By (2.10) and (2.14), we have $k|S| + (h-k)|T| \leq 0$. By arguing as in the case where h is even, we obtain $|S| \leq \frac{(k-h)n}{2k-h}$. Now we may assume that $V(G) - S - T = \emptyset$ and $\deg_{G-S}(y) = h$ for any vertex $y \in T$. Thus, for any vertex $y \in T$, $\deg_T(y) = h$. Since h is odd, $|T|$ is even. Thus the left-hand side of (2.11) is even. By arguing as in the case where h is even, we obtain $|S| \leq \frac{(k-h)n}{2k-h}$. □

Claim 2.6. $h \geq 1$.

Proof. Recall that $k \geq 2$. To prove this claim we assume that $h = 0$. We set $Z = \{y \in$

$T \mid \deg_{G-S}(y) = 0\}$, then $Z \neq \emptyset$ and $N_G(V(G) - S) \cap Z = \emptyset$. Hence

$$bind(G) \leq \frac{|N_G(V(G) - S)|}{|V(G) - S|} \leq \frac{n - |Z|}{n - |S|}. \tag{2.15}$$

Since $\dfrac{(2k-1)(n-1)}{k(n-2)+1} = \dfrac{(2k-1)(n-1)}{k(n-1)-k+1} > \dfrac{2k-1}{k}$,

$$bind(G) > \frac{2k-1}{k}. \tag{2.16}$$

By (2.15) and (2.16),

$$|S| > \frac{(k-1)n + k|Z|}{2k-1}. \tag{2.17}$$

On the other hand, $\theta(S,T) \geq k|S| + (1-k)|T| - |Z|$, and hence

$$|S| \leq \frac{(k-1)n + |Z| + 1}{2k-1}$$

by (2.6) and (2.8), which contradicts (2.17). □

3. Proof of Theorems.

3.1. Proof of Theorem 1. Let k, G, n be as in Theorem 1. To give a proof by reduction to absurdity, we assume that G is not a fractional k-deleted graph, and let S, T, h be as in the paragraph preceding the statement of Claim 2.4. Since $n > 4k + 1 - 4\sqrt{k}$ and $bind(G) > \dfrac{(2k-1)(n-1)}{k(n-2)+1}$, Claims 2.4, 2.5 and 2.6 hold. By Claim 2.5,

$$|S| \leq \frac{(k-h)n + 1}{2k-h}. \tag{3.1}$$

By Lemma 2.2, $\delta(G) > \dfrac{(k-1)n+1}{2k-1} + 1$. Since $\delta(G) \leq |S| + h$,

$$|S| > \frac{(k-1)n+1}{2k-1} + 1 - h. \tag{3.2}$$

By Claims 2.4 and 2.6, $1 \leq h \leq k-1$. We assume that $h = 1$. Then $|S| > \dfrac{(k-1)n+1}{2k-1}$ by (3.2), which contradicts (3.1). Thus we may assume that $2 \leq h \leq k-1$. Applying Lemma 2.1 with $a = k$, $b = h$, $c = 1$, $x = |S|$ and $y = n$, we obtain $n \leq 4k + 1 - 4\sqrt{k-1}$, which contradicts the assumption that $n > 4k + 1 - 4\sqrt{k-1}$. □

3.2. Proof of Theorem 2. Let k, G, n be as in Theorem 2. To give a proof by reduction to absurdity, we assume that G is not a fractional k-deleted graph, and let S, T, h be as in the paragraph preceding the statement of Claim 2.4. By Lemma 2.5,

$$|S| \leq \frac{(k-h)n}{2k-h}. \tag{3.3}$$

By Lemma 2.2, $\delta(G) > \dfrac{(k-1)n}{2k-1} + 1$. Since $\delta(G) \leq |S| + h$,

$$|S| > \frac{(k-1)n}{2k-1} + 1 - h. \tag{3.4}$$

By Claims 2.4 and 2.6, $1 \leq h \leq k-1$. We assume that $h = 1$. Then $|S| > \dfrac{(k-1)n}{2k-1}$ by (3.4), which contradicts (3.3). Thus we may assume that $2 \leq h \leq k-1$. Applying Lemma 2.1 with $a = k$, $b = h$, $c = 0$, $x = |S|$ and $y = n$, we obtain $n \leq 4k + 1 - 4\sqrt{k}$, which contradicts the assumption that $n > 4k + 1 - 4\sqrt{k}$. □

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