Elliptic elements in Möbius groups

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(Communicated by Kenji Fukaya, m.j.a., Feb. 12, 2010)

Abstract: In this paper, we give some discreteness criteria for a non-elementary Möbius group by using an elliptic element as a test map.

Key words: Elliptic element; discreteness; test map.

1. Introduction. The discreteness of Möbius groups is a fundamental problem which has been extensively studied. In 1976, by using the so-called Jørgensen inequality, Jørgensen [6] established the following well-known result.

Theorem J. *A non-elementary subgroup G of $M(\overline{\mathbf{R}}^2)$ is discrete if and only if each two-generator subgroup of G is discrete.*

This important result shows that the discreteness of a non-elementary Möbius group $G \subset M(\overline{\mathbf{R}}^2)$ depends on the information of all its rank two subgroups.

Furthermore, P. Tukia and Xiantao Wang [7] obtained that

Theorem TW. *Let $G \subset M(\overline{\mathbf{R}}^2)$ be non-elementary. If G contains an elliptic element of order at least 3, then G is discrete if and only if each non-elementary subgroup generated by two elliptic elements of G is discrete.*

Theorem TW shows that if G contains elliptic elements of order at least three, then the discreteness of the subgroups generated by two elliptic elements of G is enough to secure the discreteness of G.

For a space version of Theorem TW, one has the following result as obtained in [9].

Theorem W. *Let $G \subset M(\overline{\mathbf{R}}^n)$ be non-elementary and satisfy the Parabolic Condition. Suppose G contains an elliptic element f such that f^2 is not an element of $WY(G)$. Then G is discrete if and only if $WY(G)$ is discrete and each non-elementary subgroup of G generated by two elliptic elements is discrete.*

We say that a subgroup $G \subset M(\overline{\mathbf{R}}^n)$ satisfies the *Parabolic Condition* if G contains no sequence $\{f_i\}$ such that each f_i is parabolic and $f_i \to I$ as $i \to \infty$ (cf. [9]).

Yang Shihai generalized Theorem TW to $PU(2,1)$ in [12]. Then Cao [1] obtained the generalizations of Theorems TW and W in $PU(1,n)$.

However, Chen Min [2] showed that one could even use a fixed Möbius transformation as a test map to test the discreteness of a group. Following the idea of Theorems TW and W, it is natural to ask that whether one can generalize these results by using an elliptic element as a test map. Through discussion, we obtain

Theorem 1.1. *Let $G \subset M(\overline{\mathbf{R}}^n)$ be a non-elementary group and $M(G) = \mathbf{H}^{n+1}$. Suppose that $f \in G$ is elliptic such that $f^2 \neq I$. Then G is discrete if and only if each non-elementary subgroup generated by f and an elliptic element of G is discrete.*

Theorem 1.2. *Let $G \subset M(\overline{\mathbf{R}}^n)$ be a non-elementary group. Suppose that $f \in G$ is elliptic such that $f^2 \notin WY(G)$ and the restriction of f on S is sense-preserving. Then G is discrete if and only if $WY(G)$ is discrete, and each non-elementary subgroup generated by f and an elliptic element of G is discrete.*

Theorem 1.3. *Let $G \subset PU(1,n)$ be a non-elementary group and $M(G) = \mathbf{H}^n_{\mathbf{C}}$. Suppose that $f \in G$ is elliptic with order at least 3. Then G is discrete if and only if each non-elementary group generated by f and an elliptic element of G is discrete.*

Theorem 1.4. *Let $G \subset PU(1,n)$ be a non-elementary group. Suppose that $f \in G$ is elliptic such that $f^2 \notin ker(\Phi)$ and the restriction of f on $M(G)$ is sense-preserving. Then G is discrete if and only*

2000 Mathematics Subject Classification. Primary 30F40; Secondary 20H10.

doi: 10.3792/pjaa.86.69

if ker(Φ) is discrete, and each non-elementary subgroup generated by f and an elliptic element of G is discrete.

2. Preliminaries. Throughout this paper, for a group $G \subset M(\overline{\mathbf{R}}^n)$, we will adopt the same definitions and notations as in [11] such as $\overline{\mathbf{H}}^{n+1}$, discrete group, the limit set $L(G)$, $WY(G)$, non-elementary group and so on; for a group $G \subset PU(1,n)$, we will adopt the same definitions and notations as in [1, 3] such as $\mathbf{H}^n_{\mathbf{C}}$, discrete group, the limit set $L(G)$ and so on.

We denote $M(G)$ the smallest invariant totally geodesic sub-manifold of G, $\Phi(g)$ the restriction of g to $M(G)$ for all $g \in G$, that is

$$\Phi(g) = g|_{M(G)},\ \Phi(G) = \{g|_{M(G)} :\ g \in G\}.$$

According to [3, 5], if $G \subset M(\overline{\mathbf{R}}^n)$, then by conjugation, $M(G) = \mathbf{H}^m_{\mathbf{R}}$ $(m \leq n+1)$; if $G \subset PU(1,n)$, then by conjugation,

$$M(G) = \mathbf{H}^k_{\mathbf{C}} \text{ or } \mathbf{H}^l_{\mathbf{R}},$$

where k, l are positive integers and $k, l \leq n$. It is obvious that if $G \subset PU(1,n)$ and $M(G) = \mathbf{H}^k_{\mathbf{C}}$ (resp. $\mathbf{H}^l_{\mathbf{R}}$), then for any $g \in G$, $\Phi(g)$ is an element of $PU(1,k)$ (resp. $PO(1,l)$).

For $f \in M(\overline{\mathbf{R}}^n)$, let the set of fixed points of f be

$$fix(f) = \{x \in \overline{\mathbf{H}}^{n+1} :\ f(x) = x\}.$$

For a nontrivial element $f \in M(\overline{\mathbf{R}}^n)$, f is called *loxodromic* if f has exactly two fixed points and they all lie on $\overline{\mathbf{R}}^n$, *parabolic* if f has exactly one fixed point and it lies on $\overline{\mathbf{R}}^n$, and *elliptic* if f has a fixed point in $\mathbf{H}^{n+1}$.

For $g_r = \begin{pmatrix} a_r & b_r \\ c_r & d_r \end{pmatrix} \in M(\overline{\mathbf{R}}^n)$ $(r = 1, 2)$, we define

$$\|g_1 - g_2\| = (|a_1 - a_2|^2 + |b_1 - b_2|^2 + |c_1 - c_2|^2 + |d_1 - d_2|^2)^{\frac{1}{2}}.$$

The following lemma is crucial for our investigation.

Lemma 2.1 [10]**.** *Let f, $g \in M(\overline{\mathbf{R}}^n)$. If $\langle f, g\rangle$ is a discrete and non-elementary group, then*

$$\|f - I\| \cdot \|g - I\| \geq \frac{1}{32}.$$

Let $g \in PU(1,n)$ be a nontrivial element and

$$fix(g) = \{x \in \overline{\mathbf{H}}^n_{\mathbf{C}} :\ g(x) = x\}.$$

g is called *loxodromic* if g has exactly two fixed points and they all lie on the boundary $\partial\mathbf{H}^n_{\mathbf{C}}$ of $\mathbf{H}^n_{\mathbf{C}}$, *parabolic* if f has exactly one fixed point and it lies on $\partial\mathbf{H}^n_{\mathbf{C}}$, and *elliptic* if f has a fixed point in $\mathbf{H}^n_{\mathbf{C}}$.

In order to prove the main results, we need the following lemmas.

Lemma 2.2 [1]**.** *Let G be a non-elementary subgroup of $PU(1,n)$. Then either*
(1) *G is discrete*; *or*
(2) *ker(Φ) is not discrete but Φ(G) is discrete*; *or*
(3) *$\Phi(G)$ is dense in $SU(1, M(G))$.*

Here $SU(1, M(G))$ consists of matrices in $U(1, M(G))$ with determinant 1.

Lemma 2.3 [1, 4]**.** *Suppose that two elements f and g in $PU(1,n)$ generate a discrete and non-elementary group.*
(1) *If f is parabolic or loxodromic, then we have*

$$\max\{N(f), N([f,g])\} \geq 2 - \sqrt{3},$$

where $[f,g] = fgf^{-1}g^{-1}$ is the commutator of f and g, $N(f) = \|f - I\|$ and $\|.\|$ means the Frobenius matrix norm so that $\|Q\| = [tr(QQ^*)]^{\frac{1}{2}}$ for any matrix Q.
(2) *If f is elliptic, then we have*

$$\max\{N(f), N([f,g^q]) :\ q = 1, 2, \cdots, n+1\} \geq 2 - \sqrt{3}.$$

3. The proofs of the main results. Now we first give a lemma which is important to prove Theorems 1.1 and 1.2.

Lemma 3.1. *If $f \in M(\overline{\mathbf{R}}^n)$ is elliptic, then by conjugation in $M(\overline{\mathbf{R}}^n)$, we may assume that*

$$f = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}.$$

Proof. By conjugation, we may assume that $f = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, where $\gamma \neq 0$. If $\delta = 0$, then the result follows.

Now suppose $\delta \neq 0$. Let $g = \begin{pmatrix} 1 & \gamma^{-1}\delta \\ 0 & 1 \end{pmatrix}$. Then

$$gfg^{-1} = \begin{pmatrix} \alpha + \gamma^{-1}\delta\gamma & \beta - \alpha\gamma^{-1}\delta \\ \gamma & 0 \end{pmatrix}.$$

The proof is completed. □

Proof of Theorem 1.1. The necessity is obvious. We only need to prove the sufficiency.

Since $M(G) = \mathbf{H}^{n+1}$, we know that the minimal sphere containing $L(G)$ is $\overline{\mathbf{R}}^n$. Choose $x_j \in L(G)$ and accordingly open balls U_j in $\overline{\mathbf{R}}^{n+1}$ $(j = 1, 2, \cdots, n+2)$ satisfying

(1) $x_j \in U_j$;
(2) $U_j \cap U_s = \emptyset$ whenever $j \neq s$;
(3) for any $a_j \in U_j$, there exists only one n-sphere $S(a_1, \cdots, a_{n+2})$ containing $a_1, \cdots, a_{n+2}$.

We first claim that G contains no sequence of distinct elliptic elements converging to the identity.

Suppose, on the contrary, that G contains such a sequence $\{g_i\}$ converging to the identity as $i \to \infty$. By choosing a subsequence and after relabeling U_j, $j = 1, 2, \cdots, n+2$, if necessary, we can assume that $fix(g_i^2) \cap U_1 = \emptyset$ for each large enough i.

If $fix(f^2) \cap U_1 = \emptyset$, then there exists a loxodromic element $g_1 \in G$ with $fix(g_1) \subset U_1$. Hence there is an integer t such that

$$fix(g_1^t f^2 g_1^{-t}) = g_1^t[fix(f^2)] \subset U_1.$$

If $fix(f^2) \cap U_1 \neq \emptyset$, then, since $f^2 \neq I$, there exists some U_j satisfying $fix(f^2) \cap U_j = \emptyset$, where $j \in \{2, 3, \cdots, n+2\}$. Therefore, there exist a loxodromic element $g_2 \in G$ and an integer s such that

$$fix(g_2^s f^2 g_2^{-s}) \subset U_j.$$

For $g_2^s f^2 g_2^{-s}$, there exists an integer r such that

$$fix(g_1^r g_2^s f^2 g_2^{-s} g_1^{-r}) \subset U_1.$$

So in either case, there exists an element $h \in G$ such that

$$fix(hf^2h^{-1}) \subset U_1.$$

Since $\langle hfh^{-1}, g_i\rangle = h\langle f, h^{-1}g_ih\rangle h^{-1}$, by assumption and Lemma 2.1, we know that $\langle hfh^{-1}, g_i\rangle$ is elementary for large enough i. Therefore,

$$fix(g_i^2) \cap fix(hf^2h^{-1}) \neq \emptyset,$$

which is a contradiction. We have proved the claim.

By Lemma 3.1, we may assume that $f = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$. Suppose, on the contrary, that G is not discrete. Then G is dense in $M(\overline{\mathbf{R}}^n)$ by Theorem 3.1 in [8]. Let $l_i = \begin{pmatrix} r_i & 0 \\ 0 & \frac{1}{r_i} \end{pmatrix}$, where $r_i > 1$ and $r_i \to 1$ as $i \to \infty$. Then there exists a sequence $\{h_i\} \subset G$ of distinct loxodromic elements converging to the identity such that h_i is close enough to l_i for each i. By computation we have

$$l_i f l_i^{-1} f^{-1} = \begin{pmatrix} r_i^2 & r_i^2 ba^* - ab^* \\ 0 & \frac{1}{r_i^2} \end{pmatrix} \to I.$$

It is easy to see that $l_i f l_i^{-1} f^{-1}$ is loxodromic for each i. Then $h_i f h_i^{-1} f^{-1}$ is also loxodromic.

Since $\langle f, h_i f h_i^{-1}\rangle = \langle f, h_i f h_i^{-1} f^{-1}\rangle$ and $h_i f h_i^{-1} f^{-1} \to I$, by assumption and Lemma 2.1, the subgroup $\langle f, h_i f h_i^{-1}\rangle$ is elementary for large enough i. Let $fix(h_i f h_i^{-1} f^{-1}) = \{x_i, y_i\}$. Then both f^2 and $h_i f^2 h_i^{-1}$ fix x_i and y_i. Note that f is not of order two. Then f^2 and $h_i f^2 h_i^{-1}$ are elliptic, and $\{h_i f^2 h_i^{-1} f^{-2}\}$ is a sequence of distinct elliptic elements converging to the identity, which is a contradiction.

The proof is completed. □

Proof of Theorem 1.2. We only prove the sufficiency. By conjugation, we may assume that the minimal sphere containing $L(G)$ is $S = \overline{\mathbf{R}}^k$, where $1 \leq k \leq n$. Let $g|_S$ denote the restriction of $g \in G$ to $\overline{\mathbf{R}}^k$ and

$$\psi(G) = \{g|_S : g|_S \text{ is sense-preserving and } g \in G\}.$$

Suppose, on the contrary, that G is not discrete. Since $WY(G)$ is discrete, $\psi(G)$ is dense in $M(\overline{\mathbf{R}}^k)$. By assumption, we know that $f|_S \in \psi(G)$, $f^2|_S = (f|_S)^2 \neq I$ and $\psi(G)$ is k-dimensional. By Theorem 1.1, $\psi(G)$ contains a non-elementary and non-discrete subgroup generated by $f|_S$ and an elliptic element $g|_S \in \psi(G)$. It is obvious that $g \in G$ is elliptic and $\langle f, g\rangle \subset G$ is non-elementary. The assumption implies that $\langle f, g\rangle$ is discrete, which contradicts to that $\langle f|_S, g|_S\rangle$ is non-discrete. □

Proof of Theorem 1.3. The necessity is obvious. We now prove the sufficiency. We know that $M(G) = \mathbf{H}^n_{\mathbf{C}}$. Choose $x_j \in L(G)$ and accordingly open balls U_j in $\overline{\mathbf{H}}^n_{\mathbf{C}}$ ($j = 1, 2, \cdots, 2n+1$) satisfying

(1) $x_j \in U_j$;
(2) $U_j \cap U_s = \emptyset$ whenever $j \neq s$;
(3) for any $a_j \in U_j$, there exists only one $(2n-1)$-sphere $S(a_1, \cdots, a_{2n+1})$ containing $a_1, \cdots, a_{2n+1}$.

Suppose, on the contrary, that G is not discrete. According to Corollary 4.5.2 in [3], there exists a sequence $\{g_i\} \subset G$ of distinct elliptic elements converging to the identity as $i \to \infty$. By choosing a subsequence and after relabeling U_j, if necessary, we can assume that $fix(g_i^2) \cap U_1 = \emptyset$ for each large enough i.

By similar reasoning as in the proof of Theorem 1.1, there exists a element $h \in G$ such that

$$fix(hf^2h^{-1}) \subset U_1.$$

Since $\langle hfh^{-1}, g_i\rangle = h\langle f, h^{-1}g_ih\rangle h^{-1}$, by assumption and Lemma 2.3, we know that $\langle hfh^{-1}, g_i\rangle$ is elementary for large enough i. This implies that

$$fix(g_i^2) \cap fix(hf^2h^{-1}) \neq \emptyset,$$

which is a contradiction. The proof is completed. □

Proof of Theorem 1.4. We only need to prove the sufficiency. We suppose, on the contrary, that G is not discrete. By Lemma 2.2, we know that $\Phi(G)$ is not discrete.

By conjugation, we may assume that $M(G) = \mathbf{H}_{\mathbf{C}}^k$ *or* $\mathbf{H}_{\mathbf{R}}^l$, where $1 \leq k, l \leq n$. Now we divide our proof into two cases.

Case I. $M(G) = \mathbf{H}_{\mathbf{C}}^k$.

According to Corollary 4.5.2 in [3], there exists a sequence $\{g_i\} \subset G$ of distinct elliptic elements converging to the identity as $i \to \infty$. By similar reasoning as in the proof of Theorem 1.3, we obtain a contradiction.

Case II. $M(G) = \mathbf{H}_{\mathbf{R}}^l$.

We know that $\Phi(G)$ is a subgroup of $PO(1,l)$ and all the sense-preserving elements of $\Phi(G)$ is dense in $M(\overline{\mathbf{R}}^{l-1})$. By assumption, $\Phi(f)$ is a sense-preserving elliptic element with $\Phi(f^2) = \Phi^2(f) \neq I$. Theorem 1.1 implies that there exists a non-elementary and non-discrete group generated by $\Phi(f)$ and $\Phi(g)$, where $\Phi(g)$ is a sense-preserving elliptic element. Therefore, $\langle f, g \rangle$ is a non-elementary but non-discrete subgroup of G, which is also a contradiction.

The proof is completed. □

Acknowledgments. The research was partly supported by Program for NCET (No. 40-0783), NSFs of China (No. 10771059) and Hunan (No. 05JJ10001). The author would like to thank the referee for his (her) careful reading and useful suggestions.

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