A family of integral inequalities on the circle $\mathbf{S}^1$

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Abstract: We consider the Chebychev semigroup defined on the interval $[-1,+1]$ by its Dirichlet form $\int_{-1}^{+1}(1-x^2)f'^2(x)\frac{dx}{\pi\sqrt{1-x^2}}$. We prove, via a method involving probabilistic techniques, a family of inequalities which interpolate between the Sobolev and Poincaré inequalities.

Key words: Chebychev semigroup; spectral gap; Poincaré inequality; Sobolev inequality; logarithmic Sobolev inequality.

1. Introduction. Gross' logarithmic Sobolev inequality [4] states that for all smooth functions f on $\mathbf{R}^n$,

$$\int_{\mathbf{R}^n} f^2\log f^2 d\gamma_n - \left(\int_{\mathbf{R}^n} f^2 d\gamma_n\right)\log\left(\int_{\mathbf{R}^n} f^2 d\gamma_n\right) \leq 2\int_{\mathbf{R}^n}|\nabla f|^2 d\gamma_n,$$

where $d\gamma_n$ denotes the normalized Gaussian measure on $\mathbf{R}^n$: $\gamma_n(dx) = (\sqrt{2\pi})^{-n}\exp(-|x|^2/2)dx$. In this Gaussian context, the Poincaré inequality (spectral gap inequality) is given by:

$$\int_{\mathbf{R}^n} f^2 d\gamma_n - \left(\int_{\mathbf{R}^n} f d\gamma_n\right)^2 \leq \int_{\mathbf{R}^n}|\nabla f|^2 d\gamma_n.$$

In 1989, W. Beckner [2] derived a family of generalized Poincaré inequalities that yield a sharp interpolation between the Poincaré inequality and the logarithmic Sobolev inequality:

$$\int_{\mathbf{R}^n} f^2 d\gamma_n - \int_{\mathbf{R}^n}(e^{tL}f)^2 d\gamma_n \leq (1-e^{-2t})\int_{\mathbf{R}^n}|\nabla f|^2 d\gamma_n, \quad \text{for all } t \geq 0,$$

where L is the Ornstein-Uhlenbeck operator: $L = \Delta - x\cdot\nabla$.

Similar researches on this kind of inequalities for general probability measures generated by diffusions have been done by many authors (see, for instance, [1] and [9]).

The purpose of this note is to present a family of integral inequalities on the unit circle $\mathbf{S}^1$ which provide interpolation between the Sobolev and Poincaré inequalities (see Theorem 3.1 below). These types of integral inequalities are deeply related to the aspects of the large-time behavior of parabolic PDEs (like in [7]).

2. Preliminaries. In order to keep the paper reasonably self-contained, we summarize in this section the basic notions that will be used in this work. We consider on the interval $I := [-1,+1]$ the Chebychev operator defined by

$$\mathcal{L} := (1-x^2)\frac{d^2}{dx^2} - x\frac{d}{dx} \qquad (x \in I),$$

acting on the Hilbert space $\mathbf{L}^2(I,\mu)$ with respect to the probability measure $\mu(dx) := \frac{1}{\pi\sqrt{1-x^2}}dx$. The operator $\mathcal{L}$ may be obtained as the projection of the Laplacian on the unit circle $\mathbf{S}^1$ and μ is obtained as the projection on I of the normalized Lebesgue measure on $\mathbf{S}^1$. The Chebychev polynomials $(T_n)_{n\in\mathbf{N}}$ are defined by

$$T_n(x) : = \cos(n \ \arccos x) = \frac{(-1)^n\sqrt{\pi}}{2^n\Gamma(n+\frac{1}{2})}\ \sqrt{1-x^2}\ \frac{d^n}{dx^n}\left((1-x^2)^{n-\frac{1}{2}}\right),$$

where Γ is the usual gamma function:

$$\Gamma(p) = \int_0^{+\infty} t^{p-1}e^{-t}dt.$$

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It is known that the Chebychev polynomials are eigenvectors for the operator $-\mathcal{L}$ (see, for instance, [8]):

$$-\mathcal{L}(T_n) = n^2 T_n.$$

In fact, the distribution μ is symmetrizing for $\mathcal{L}$ and the sequence $(-n^2, Vect(T_n))_{n\in\mathbf{N}}$ forms the spectral decomposition of the minimal self-adjoint extension of this operator on $\mathbf{L}^2(I,\mu)$. With the help of an integration by parts, it is easily seen that

$$\forall f,g \in \mathcal{C}^2(I), \quad \int (\mathcal{L}f)g d\mu = \int f\mathcal{L}g d\mu = -\int G(f,g)(x)d\mu(x), \tag{1}$$

where G is the positive symmetric bilinear form defined by:

$$G(f,g)(x) = (1-x^2)f'(x)g'(x).$$

An important consequence of property (1) is:

$$\forall f \in \mathcal{C}^2(I), \quad \int \mathcal{L}f d\mu = 0,$$

which expresses the invariance of the measure μ. By means of the above mentioned properties of the operator $\mathcal{L}$, essentially the one concerning the symmetry with respect to μ, we deduce the existence of a semigroup of operators $(P_t)_{t\geq 0}$ generated by $\mathcal{L}$ acting on $\mathbf{L}^2(I,\mu)$ by:

$$P_t T_n = e^{-n^2 t} T_n, \quad \forall n \in \mathbf{N}, \tag{2}$$

and such that:

1. P_t is a contraction in all spaces $\mathbf{L}^p(I,\mu)$ $(1 \leq p \leq +\infty)$;
2. P_t is symmetric: $\int (P_t f)g d\mu = \int f(P_t g)d\mu$, $\forall f$, $g \in \mathbf{L}^2(I,\mu)$;
3. P_t is positive and $P_t 1 = 1$.

This semigroup is obtained as the projection of the circular Brownian motion on a diameter. According to (2), $P_t = e^{t\mathcal{L}}$ and P_t is ergodic:

$P_t f$ tends to $\int f d\mu$ μ-almost everywhere as $t \to +\infty$.

The commutation relation between the action of the operator $\mathcal{L}$ and the derivation is given as:

$$\frac{d}{dx}\mathcal{L} = \widetilde{\mathcal{L}}\left(\frac{d}{dx}\right) - \frac{d}{dx},$$

where $\widetilde{\mathcal{L}}$ is the operator associated to the family of Chebychev polynomials of second kind:

$$\widetilde{\mathcal{L}} := (1-x^2)\frac{d^2}{dx^2} - 3x\frac{d}{dx}.$$

This commutation formula translates for the semigroup $(P_t)_{t\geq 0}$ by:

$$\frac{d}{dx}P_t = e^{-t}\widetilde{P}_t\left(\frac{d}{dx}\right), \tag{3}$$

where $\widetilde{P}_t$ designates the heat semigroup generated by $\widetilde{\mathcal{L}}$. Notice in passing that $\widetilde{P}_t$ is symmetric (and so invariant) with respect to the probability measure $\widetilde{\mu}(dx) := \frac{2}{\pi}\sqrt{1-x^2}\,dx$. The generator $\widetilde{\mathcal{L}}$ satisfies the following dissipativity formula:

$$\int (-\widetilde{\mathcal{L}}f)g d\widetilde{\mu} = \int (1-x^2)f'g'd\widetilde{\mu}, \tag{4}$$

f, g being sufficiently smooth on I. We emphasize that $\widetilde{\mathcal{L}}$ may be obtained as the projection of the Laplacian on the unit sphere $\mathbf{S}^3$ and $\widetilde{\mu}$ is obtained as the projection of the normalized Lebesgue measure on $\mathbf{S}^3$.

For $1 \leq p < +\infty$, let $\mathbf{D}_p(\mathcal{L})$ denote the domain of the generator $\mathcal{L}$ of $(P_t)_{t\geq 0}$ in $\mathbf{L}^p(I,\mu)$. In virtue of the density of $\mathcal{C}^2(I)$ in $\mathbf{D}_2(\mathcal{L})$, we may extend formula (1) to $\mathbf{D}_2(\mathcal{L})$.

3. The main result. Our objective in this section is to establish a family of integral inequalities on the unit circle $\mathbf{S}^1$ which provide interpolation between the Sobolev and the Poincaré inequalities. For $1 \leq p \leq +\infty$, we adopt the notation

$$\mathbf{L}^p_{+0}(I,\mu) = \{f \in \mathbf{L}^p(I,\mu) \mid \exists \varepsilon > 0, f \geq \varepsilon\}.$$

Let $\varphi : \mathbf{R}^+ \to \mathbf{R}$ be a strictly convex function such that $\varphi(0) = 0$. We define the φ-entropy functional $\mathbf{E}$ of $f \in \mathbf{L}^1_{+0}(I,\mu)$ by

$$\mathbf{E}(\varphi,f,t) = \int \varphi(f)d\mu - \int \varphi(P_t f)d\mu, \quad t \in [0,+\infty].$$

The quantity $\mathbf{E}(\varphi,f,t)$ is always nonnegative since P_t is invariant for the probability measure μ. By the ergodic property of the semigroup,

$$\mathbf{E}(\varphi,f,+\infty) := \mathbf{E}(\varphi,f) = \int \varphi(f)d\mu - \varphi\left(\int f d\mu\right).$$

When $\varphi(x) = x^2$, $\mathbf{E}(\varphi,f)$ coincides with the classical notion of variance,

$$\mathbf{E}(\varphi, f) = \mathrm{Var}(f) := \int f^2 d\mu - \left(\int f d\mu\right)^2,$$

and when $\varphi(x) = x \log x$,

$$\mathbf{E}(\varphi, f) = \mathbf{E}nt(f) := \int f \log f d\mu - \int f d\mu \log\left(\int f d\mu\right),$$

the usual Kullback entropy.

In the sequel, we shall restrict ourself to the following class $\mathcal{C}$ of real functions $\varphi \in \mathcal{C}^\infty(\mathbf{R}^+)$: $\varphi \in \mathcal{C}$ means that $\varphi(0) = 0$, φ'' is strictly positive on $\mathbf{R}^+$ and

$$\frac{3}{2}(\varphi''')^2 \leq \varphi'' \varphi'''' \text{ on } \mathbf{R}^+.$$

A similar class to $\mathcal{C}$ was introduced and studied with much profit by R. Latala and K. Oleszkiewicz (see [5]). These two authors have investigated a number of remarkable properties related to the functional $\mathbf{E}(\cdot, f)$ for an arbitrary probability measure.

Having in our disposal enough machinery, we are now ready to prove the following estimate of the φ-entropy functional $\mathbf{E}$:

Theorem. 3.1. *Let $\varphi \in \mathcal{C}$. Then, for all functions $f \in \mathbf{L}^\infty_{+0}(I, \mu) \cap \mathbf{D}_2(\mathcal{L})$ and $t \in [0, +\infty]$,*

$$\mathbf{E}(\varphi, f, t) \leq \frac{1}{2}(1 - e^{-2t}) \int \varphi''(f) G(f, f)\, d\mu. \tag{5}$$

Moreover, the numeric constant at the right hand side of inequality (5) is the best.

To illustrate this theorem, let us analyze some practical applications. The most important examples of the class $\mathcal{C}$ in our mind are:

$$\varphi_p(x) = \frac{-x^{\frac{2}{p}} + x}{p - 2} \quad \text{for} \quad p \in [1, +\infty[,\ p \neq 2$$

and

$$\varphi_2(x) = \frac{1}{2} x \log x,$$

which corresponds to the limiting case of φ_p as $p \to 2$. If $\varphi = \varphi_p$, inequality (5), written for $t = +\infty$, describes the Sobolev inequality: for all $p \geq 1$ ($p \neq 2$) and for all functions $f \in \mathbf{L}^\infty_{+0}(I, \mu) \cap \mathbf{D}_2(\mathcal{L})$,

$$\frac{\|f\|_p^2 - \|f\|_2^2}{p - 2} \leq \int G(f, f)\, d\mu, \tag{6}$$

where $\|f\|_p$ is the norm in $\mathbf{L}^p(I, \mu)$. Indeed,

$$\begin{aligned} \frac{\|f\|_p^2 - \|f\|_2^2}{p - 2} &= \mathbf{E}(\varphi_p, f^p, +\infty) \\ &\leq \frac{1}{2} \int \varphi_p''(f^p) G(f^p, f^p)\, d\mu \\ &= \int G(f, f)\, d\mu. \end{aligned}$$

With $\varphi = \varphi_2$ and $t = +\infty$, inequality (5) is exactly the Sobolev logarithmic inequality found by C. E. Mueller and F. B. Weissler [6] (see also [3]): replacing f (positive) by f^2, we get

$$\mathbf{E}nt(f^2) \leq 2 \int G(f, f) d\mu, \quad \forall f \in \mathbf{L}^\infty_{+0}(I, \mu) \cap \mathbf{D}_2(\mathcal{L}). \tag{7}$$

Taking into account that

$$\int G(|f|, |f|) d\mu \leq \int G(f, f)\, d\mu,$$

and using the fact that the set of bounded functions in $\mathcal{C}^2(I)$ is dense in $\mathbf{D}_2(\mathcal{L})$, we can extend inequalities (6) and (7) to $\mathbf{D}_2(\mathcal{L})$. This last inequality (7) is equivalent to the hypercontractive estimate for the semigroup $(P_t)_{t \geq 0}$: Whenever $1 < p < q < +\infty$ and $t > 0$ satisfy $e^{-t} \leq \sqrt{\frac{p-1}{q-1}}$, then, for all functions $f \in \mathbf{L}^p(I, \mu)$,

$$\|P_t f\|_q \leq \|f\|_p.$$

In other words, P_t maps $\mathbf{L}^p(I, \mu)$ in $\mathbf{L}^q(I, \mu)$ $(q > p)$ with norm one.

Proof of Theorem 3.1. By the Fubini theorem, it follows from the definition of $\mathbf{E}(\varphi, f, t)$ that for any $t > 0$,

$$\begin{aligned} \mathbf{E}(\varphi, f, t) &= -\int (\varphi(P_t f) - \varphi(P_0 f))\, d\mu \\ &= -\int_0^t \frac{d}{ds}\left[\int \varphi(P_s f) d\mu\right] ds \\ &= \int_0^t \left(\int -(\mathcal{L} P_s f) \varphi'(P_s f) d\mu\right) ds \\ &= \int_0^t \left(\int (1 - x^2)(P_s f)'^2\, \varphi''(P_s f) d\mu\right) ds \\ &= \int_0^t e^{-2s} \left(\int (1 - x^2)(\widetilde{P}_s f')^2 \varphi''(P_s f)\, d\mu\right) ds. \end{aligned}$$

The last two equalities follow from the dissipativity property (1) and the commutation formula (3), respectively. An integration by parts over the time variable s yields

$$\mathbf{E}(\varphi,f,t) = -\frac{1}{2}\,e^{-2t}\int(1-x^2)(\widetilde{P}_t f')^2\varphi''(P_t f)\,d\mu + \frac{1}{2}\int\varphi''(f)(1-x^2)f'^2 d\mu + \frac{1}{2}\int_0^t e^{-2s}\frac{d}{ds}\Big[\int(1-x^2)(\widetilde{P}_s f')^2\varphi''(P_s f)\,d\mu\Big]ds.$$

Since

$$\int_0^t \frac{d}{ds}\Big[\int(1-x^2)(\widetilde{P}_s f')^2\varphi''(P_s f)\,d\mu\Big]ds = \int(1-x^2)(\widetilde{P}_t f')^2\varphi''(P_t f)\,d\mu - \int\varphi''(f)(1-x^2)f'^2 d\mu,$$

we get

$$\mathbf{E}(\varphi,f,t) = \frac{1}{2}\,(1-e^{-2t})\int\varphi''(f)(1-x^2)f'^2 d\mu + \frac{1}{2}\int_0^t (e^{-2s}-e^{-2t})\,\frac{d}{ds} \times\Big[\int(1-x^2)(\widetilde{P}_s f')^2\varphi''(P_s f)\,d\mu\Big]ds.$$

Now,

$$e^{-2s}\frac{d}{ds}\Big[\int(1-x^2)(\widetilde{P}_s f')^2\varphi''(P_s f)\,d\mu\Big] = 2\int(1-x^2)\widetilde{\mathcal{L}}(P_s f)'\cdot\varphi''(P_s f)(P_s f)' d\mu + \int\mathcal{L}P_s f\cdot(1-x^2)\,\varphi'''(P_s f)(P_s f)'^2 d\mu.$$

Applying successively (1) and (4), the first integral in this sum is reduced to:

$$-2\int(1-x^2)^2(P_s f)''^2\varphi''(P_s f)\,d\mu - 2\int(1-x^2)^2(P_s f)''(P_s f)'^2\varphi'''(P_s f)\,d\mu,$$

while the second integral is equal to:

$$-2\int(1-x^2)^2(P_s f)'^2(P_s f)''\varphi'''(P_s f)\,d\mu - \int(1-x^2)^2(P_s f)'^4\,\varphi''''(P_s f)\,d\mu + 2\int x(1-x^2)(P_s f)'^3\varphi'''(P_s f)\,d\mu.$$

Replacing x by $\dfrac{-\widetilde{\mathcal{L}}(x)}{3}$, and invoking again the dissipativity formula (4), the last member in the preceding sum becomes:

$$2\int(1-x^2)^2(P_s f)'^2(P_s f)''\varphi'''(P_s f)\,d\mu + \frac{2}{3}\int(1-x^2)^2(P_s f)'^4\varphi''''(P_s f)\,d\mu.$$

As a consequence, after reassembling the terms, we find:

$$\frac{\mathbf{E}(\varphi,f,t)}{1-e^{-2t}} = \frac{1}{2}\int\varphi''(f)(1-x^2)f'^2 d\mu - \frac{1}{2}\int_0^t \frac{1-e^{-2(t-s)}}{1-e^{-2t}} \times\Big(\int(1-x^2)^2\xi(s,f,\varphi)d\mu\Big)ds, \qquad (8)$$

with

$$\xi(s,f,\varphi) = 2f_s''^2\varphi''(f_s) + 2f_s'^2 f_s''\varphi'''(f_s) + \frac{1}{3}\,f_s'^4\varphi''''(f_s) = \left[\sqrt{2}f_s''\sqrt{\varphi''(f_s)} + \frac{\varphi'''(f_s)}{\sqrt{2}\sqrt{\varphi''(f_s)}}f_s'^2\right]^2 + \frac{f_s'^4}{3\varphi''(f_s)}\left[\varphi''(f_s)\varphi''''(f_s) - \frac{3}{2}(\varphi'''(f_s))^2\right],$$

where we have posed $f_s = P_s f$. The positivity of $\xi(s,f,\varphi)$ then allows us to exhibit the desired inequality (5) from (8).

It remains to show that the numeric constant $\frac{1}{2}\,(1-e^{-2t})$ at the right hand side of inequality (5) is optimal. As usual, let us consider $c\in\,]0,+\infty[$ such that $\varphi''(c)>0$. If f is replaced by $c+\varepsilon f$ in (5), and we pass to the limit as ε tends to 0^+, we easily recover the Poincaré (or spectral gap) inequality with best constant:

$$\forall t\in[0,+\infty],\int f^2 d\mu - \int(P_t f)^2 d\mu \le (1-e^{-2t})\int G(f,f)\,d\mu,$$

which completes the proof. □

We close this paper by the following concluding remarks:

1. Of course, letting $t=+\infty$, inequality (5) in Theorem 3.1 gives rise to:

$$\mathbf{E}(\varphi,f) \le \frac{1}{2}\int\varphi''(f)G(f,f)\,d\mu. \qquad (9)$$

Moreover, it is a simple matter to observe that (5) provides a smooth nonincreasing interpolation for inequality (9):

$$\mathbf{E}(\varphi, f) \le \frac{\mathbf{E}(\varphi, f, t)}{1 - e^{-2t}} \le \frac{1}{2} \int \varphi''(f) G(f, f)\, d\mu.$$

2. By (8), we point out at once that, if $\frac{3}{2}(\varphi''')^2 < \varphi''\varphi''''$, the equality holds in (5) if and only if f is constant. In particular, inequalities (6) and (7) do not admit nonconstant extremal functions.

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