An existence result of the Cauchy Dirichlet problem for the Hermite heat equation

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Abstract: Using the Mehler kernel, we give an existence result of the Cauchy Dirichlet problem for the Hermite heat equation with homogeneous Dirichlet boundary conditions and continuous and bounded Cauchy data vanishing at $x = 0$.

Key words: Hermite functions; Mehler kernel; Hermite heat equation; Cauchy Dirichlet problem.

1. Introduction. The existence and uniqueness problems for the heat equation in both the half and whole line have been investigated extensively. Several examples in [1, 5, 7] and the references therein are of ample evidences. Similarly uniqueness problems for parabolic equations in a general setting have been given in [6] as described by the following theorem.

Theorem 1.1 ([6], Corollary 8.15 and Exercise 8.22)**.** *Let Ω be a domain in $\mathbf{R}^d$, let $T > 0$ and $Q = (0, T) \times \Omega$. Assume that the second-order operator*

$$L(t, x) = \sum_{i,j} a_{i,j}(t, x)\partial_{x_i}\partial_{x_j} + \sum_i b_i(t, x)\partial_{x_i} - V(t, x)$$

is uniformly elliptic with respect to x. Assume that its coefficients are real valued and satisfy the following boundedness condition:

(1.1)

$$V(t, x) \geq 0, \text{ trace } a(t, x) + x \cdot b(t, x) \leq K(1 + |x|^2)$$

for $(t, x) \in Q$. Assume that the function U is continuous and bounded on $\bar{Q}$ and satisfies the regularity conditions: *for any t the derivatives $\partial_x U(t, x)$, $\partial_x^2 U(t, x)$ exist and are continuous in Q and the derivative $\partial_t U$ exists at any point in Q. Let*

$$\partial' Q = (0, T) \times \partial\Omega \cup \{(0, x) | x \in \bar{\Omega}\}$$

denotes the parabolic boundary of Q. Then the following uniform uiqueness estimate holds:

$$\sup_Q |U| \leq T \sup_Q |\partial_t U - LU| + \sup_{\partial' Q} |U|.$$

In particular, if $U = 0$ on $\partial' Q$ and $\partial_t U - LU = 0$ in Q, then $U \equiv 0$ in Q.

With $d = 1$, $\Omega = [0, \infty)$, $(a_{i,j})$ = unit matrix of order 1, $b_i = 0$ and $V(t, x) = |x|^2$, the following uniqueness theorem for the Hermite heat equation is a particular case of Theorem 1.1.

Theorem 1.2. *Let $U(x, t)$ be a continuous function on $[0, \infty) \times [0, \infty)$ satisfying the following*

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + x^2\right) U(x, t) = 0 \text{ in } (0, \infty) \times (0, \infty),$$

for some positive constant M

$$|U(x, t)| \leq M \text{ in } [0, \infty) \times [0, \infty),$$

$U(x, 0) = 0$ for $0 \leq x < \infty$ and $U(0, t) = 0$ for $0 \leq t < \infty$. Then $U(x, t) \equiv 0$ in $[0, \infty) \times [0, \infty)$.

To prove that the equations have a solution is an even more difficult matter than that of proving uniqueness. Although a hint for finding the solution of the Caucy Dirichlet problem for the heat equation is available in [5], the case for the Hermite heat equation is different and difficult too due to the presence of the potential term x^2.

As a challenging job, the main aim of this paper is therefore to give a unique solution of the Cauchy Dirichlet problem for the Hermite heat equation in the half line with homogeneous Dirichlet boundary conditions and continuous and bounded Cauchy data vanishing at $x = 0$. While proving the main theorem, we heavily make the use of some derivations and results on the Mehler kernel.

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2. Preliminaries. As introduced in [2], we denote by $E(x,\xi,t)$ the Mehler kernel defined by

$$E(x,\xi,t)=\begin{cases}\sum_{k=0}^{\infty}e^{-(2k+1)t}h_k(x)h_k(\xi), & t>0\\ 0, & t\le 0\end{cases}$$

where h_k's are L^2 – normalized Hermite functions defined by

$$h_k(x)=\frac{(-1)^k\ e^{x^2/2}}{\sqrt{2^k k!\sqrt{\pi}}}\frac{d^k}{dx^k}e^{-x^2},\quad x\in\mathbf{R}.$$

Moreover the explicit form of $E(x,\xi,t)$ for $t>0$ is

$$E(x,\xi,t)=\frac{e^{-t}\ e^{-\frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}(x-\xi)^2-\frac{1-e^{-2t}}{1+e^{-2t}}x\xi}}{\sqrt{\pi}(1-e^{-4t})^{\frac{1}{2}}}.$$

We note that for each $\xi\in\mathbf{R}$, $E(x,\xi,t)$ satisfies the Hermite heat equation.
Since $E(x,\xi,t)=\tilde{\eta}(\xi,t)\ \tilde{E}(\xi,x,t)$ where

$$\tilde{\eta}(\xi,t)=\frac{\sqrt{2}\ e^{-t}\ e^{-\frac{1}{2}\frac{1-e^{-4t}}{1+e^{-4t}}\xi^2}}{(1+e^{-4t})^{\frac{1}{2}}}, \tag{2.1}$$

$$\tilde{E}(\xi,x,t)=\left(\frac{1+e^{-4t}}{1-e^{-4t}}\right)^{\frac{1}{2}}\frac{e^{-\frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}\left(x-\frac{2e^{-2t}}{1+e^{-4t}}\xi\right)^2}}{\sqrt{2\pi}} \tag{2.2}$$

for $x,\xi\in\mathbf{R}$ and $t>0$, we have the following lemma:

Lemma 2.1 (Lemma 3.1, [3], and Lemma 2.1, [4]). *For any $\delta>0$*

$$\int_{\mathbf{R}}\tilde{E}(\xi,x,t)dx=1, \tag{2.3}$$

$$\int_{\left|x-\frac{2e^{-2t}}{1+e^{-4t}}\xi\right|\ge\delta}\tilde{E}(\xi,x,t)dx\to 0 \tag{2.4}$$

uniformly for $\xi\in\mathbf{R}$ as $t\to 0^+$.

3. Main Result.

Theorem 3.1. *Let ϕ be a continuous and bounded function on $[0,\infty)$ with $\phi(0)=0$. Then there exists a unique solution of the following Cauchy Dirichlet problem for the Hermite heat equation*

$$\begin{cases}(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}+x^2)U(x,t)=0 & x>0,\ t>0,\\ U(x,0)=\phi(x) & x>0,\\ U(0,t)=0 & t>0.\end{cases} \tag{3.1}$$

Proof. Let

$$h(x)=\begin{cases}\phi(x), & x\ge 0\\ -\phi(-x), & x<0.\end{cases}$$

Then by hypothesis $h(x)$ is a continuous, bounded and odd function on $\mathbf{R}$. In view of the Fourier series of the odd function h on $\mathbf{R}$, it is easy to see that

$$h(x)\sim\frac{2}{\pi}\lim_{t\to 0}\int_0^L\frac{\tilde{\eta}(x,t)h(\xi)d\xi}{A_{L,x,\xi,t}}$$

where $A_{L,x,\xi,t}$ equals

$$\left(\sum_{n=0}^{\infty}\frac{\pi}{L}\sin\frac{n\pi\xi}{L}\ \sin\frac{2n\pi e^{-2t}x}{L(1+e^{-4t})}\ e^{-\frac{n^2\pi^2}{L^2}\frac{1-e^{-4t}}{2(1+e^{-4t})}}\right)^{-1}.$$

As $L\to\infty$, we have

$$h(x)\sim\frac{2}{\pi}\lim_{t\to 0}\int_0^{\infty}\frac{\tilde{\eta}(x,t)h(\xi)d\xi}{B_{x,\xi,t}}$$

where $B_{x,\xi,t}$ equals

$$\left(\int_0^{\infty}\sin\lambda\xi\ \sin\frac{2\lambda e^{-2t}x}{1+e^{-4t}}\ e^{-\frac{\lambda^2(1-e^{-4t})}{2(1+e^{-4t})}}d\lambda\right)^{-1}$$

Considering h only on $[0,\infty)$, we have

$$\phi(x)\sim\frac{2}{\pi}\lim_{t\to 0}\int_0^{\infty}\frac{\tilde{\eta}(x,t)\phi(\xi)d\xi}{B_{x,\xi,t}} \tag{3.2}$$

But

$$\frac{1}{B_{x,\xi,t}}=\frac{1}{4}(H_1-H_2) \tag{3.3}$$

where

$$H_1=\int_{-\infty}^{\infty}\cos\lambda\left(\xi-\frac{2e^{-2t}x}{1+e^{-4t}}\right)e^{-\frac{\lambda^2(1-e^{-4t})}{2(1+e^{-4t})}}d\lambda,$$

$$H_2=\int_{-\infty}^{\infty}\cos\lambda\left(\xi+\frac{2e^{-2t}x}{1+e^{-4t}}\right)\ e^{-\frac{\lambda^2(1-e^{-4t})}{2(1+e^{-4t})}}d\lambda.$$

Evaluating the integrals H_1 and H_2, we obtain

$$H_1=Re\int_{-\infty}^{\infty}e^{-\frac{\lambda^2(1-e^{-4t})}{2(1+e^{-4t})}+i\lambda\left(\xi-\frac{2e^{-2t}x}{1+e^{-4t}}\right)}d\lambda$$

$$=\frac{Re\int_{-\infty}^{\infty}e^{-\frac{2(1-e^{-4t})}{1+e^{-4t}}\left(\frac{\lambda}{2}-\frac{i}{2}\frac{1+e^{-4t}}{1-e^{-4t}}\left(\xi-\frac{2e^{-2t}x}{1+e^{-4t}}\right)\right)^2}d\lambda}{e^{\frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}\left(\xi-\frac{2e^{-2t}x}{1+e^{-4t}}\right)^2}}$$

$$=\sqrt{2\pi}\sqrt{\frac{1+e^{-4t}}{1-e^{-4t}}}e^{-\frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}\left(\xi-\frac{2e^{-2t}x}{1+e^{-4t}}\right)^2}.$$

Similarly

$$H_2=Re\int_{-\infty}^{\infty}e^{-\frac{\lambda^2(1-e^{-4t})}{2(1+e^{-4t})}+i\lambda\left(\xi+\frac{2e^{-2t}x}{1+e^{-4t}}\right)}d\lambda$$

$$=\sqrt{2\pi}\sqrt{\frac{1+e^{-4t}}{1-e^{-4t}}}e^{-\frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}\left(\xi+\frac{2e^{-2t}x}{1+e^{-4t}}\right)^2}.$$

Substituting the values of H_1 and H_2 in (3.3), we obtain from (3.2) that

$$\phi(x) \sim \lim_{t\to 0}\int_0^\infty \frac{\tilde{\eta}(x,t)\phi(\xi)d\xi}{\left\{\tilde{E}(x,\xi,t)-\tilde{E}(x,-\xi,t)\right\}^{-1}}$$

$$\phi(x)=\lim_{t\to 0}\int_0^\infty \{E(x,\xi,t)-E(x,-\xi,t)\}\phi(\xi)d\xi \tag{3.4}$$

where in view of (2.2)

$$\tilde{E}(x,\pm\xi,t)=\sqrt{\frac{1+e^{-4t}}{1-e^{-4t}}}\,\frac{e^{-\frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}\left(\xi\mp\frac{2e^{-2t}x}{1+e^{-4t}}\right)^2}}{\sqrt{2\pi}}.$$

Let

$$U(x,t):=\int_0^\infty \{E(x,\xi,t)-E(x,-\xi,t)\}\phi(\xi)d\xi.$$

Then by (3.4), we have

$$\lim_{t\to 0}U(x,t)=\phi(x).$$

Moreover, since $E(x,\xi,t)$ and $E(x,-\xi,t)$ are the Hermite heat solutions, so is $U(x,t)$. Furthermore

$$U(0,t)=0 \text{ for } t>0$$

reveals from the fact that

$$E(0,\xi,t)=E(0,-\xi,t) \text{ for all } \xi\in\mathbf{R} \text{ and all } t>0.$$

Thus $U(x,t)$ is a solution of (3.1). Next by definition of h on $\mathbf{R}$, it is easy to see that

$$\begin{aligned} U(x,t) &= \int_0^\infty \{E(x,\xi,t)-E(x,-\xi,t)\}\phi(\xi)d\xi \\ &= \int_{\mathbf{R}} E(x,\xi,t)h(\xi)d\xi. \qquad (3.5)\end{aligned}$$

Then there exists a constant $M_1>0$ such that

$$\begin{aligned} |U(x,t)| &\le \tilde{\eta}(x,t)\|h\|_\infty \int_{\mathbf{R}} \tilde{E}(x,\xi,t)d\xi \\ &\le M_1 \text{ in } [0,\infty)\times(0,\infty)\end{aligned}$$

since from (2.1) and (2.3), $\tilde{\eta}(x,t)$ is bounded and

$$\int_{\mathbf{R}} \tilde{E}(x,\xi,t)d\xi=1.$$

Then

$$|U(x,t)|\le M \text{ in } [0,\infty)\times[0,\infty) \tag{3.6}$$

where $M=M_1+\|\phi\|_\infty$. Thus in view of (3.6) and Theorem 1.2, we conclude that $U(x,t)$ is a unique solution of (3.1). □

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