# Quantum queer superalgebra and crystal bases

By Dimitar GRANTCHAROV[*)], Ji Hye JUNG[**)], Seok-Jin KANG[**)],
Masaki KASHIWARA, M.J.A.[**),***)] and Myungho KIM[**)]

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**Abstract:** In this paper, we develop the crystal basis theory for the quantum queer superalgebra $U_q(\mathfrak{q}(n))$. We define the notion of crystal bases, describe the tensor product rule, and present the existence and uniqueness of crystal bases for $U_q(\mathfrak{q}(n))$-modules in the category $\mathcal{O}_{int}^{\geq 0}$.

**Key words:** Quantum queer superalgebra; crystal bases; odd Kashiwara operators.

**1. Introduction.** The crystal bases are one of the most prominent discoveries of the modern combinatorial representation theory. Immediately after its first appearance in 1990 in [3], the crystal basis theory developed rapidly and attracted considerable mathematical attention. Many important and deep results for symmetrizable Kac-Moody algebras have been established in the last 20 years following the original works [3–5]. In particular, an explicit combinatorial realization of crystal bases for classical Lie algebras was given in [6].

In contrast to the case of Lie algebras, the crystal base theory for Lie superalgebras is still in its beginning stage. A major difficulty in the superalgebra case arises from the fact that the category of finite-dimensional representations is in general not semisimple. Nevertheless, there is an interesting category of finite-dimensional $U_q(\mathfrak{g})$-modules which is semisimple for the two super-analogues of the general linear Lie algebra $\mathfrak{gl}(n)$: $\mathfrak{g} = \mathfrak{gl}(m|n)$ and $\mathfrak{g} = \mathfrak{q}(n)$. This is the category $\mathcal{O}_{int}^{\geq 0}$ of representations that appear as subrepresentations of tensor powers $\mathbf{V}^{\otimes N}$ of the natural representation $\mathbf{V}$ of $U_q(\mathfrak{g})$. The semisimplicity of $\mathcal{O}_{int}^{\geq 0}$ is verified in [1] for $\mathfrak{g} = \mathfrak{gl}(m|n)$ and in [2] for $\mathfrak{g} = \mathfrak{q}(n)$.

The crystal basis theory of $\mathcal{O}_{int}^{\geq 0}$ for the general linear Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(m|n)$ was developed in [1]. In this case the irreducible objects in $\mathcal{O}_{int}^{\geq 0}$ are indexed by partitions having so-called $(m, n)$-hook shapes. This combinatorial description enables us to index the crystal basis of any irreducible object $V(\lambda)$ in $\mathcal{O}_{int}^{\geq 0}$ with highest weight $\lambda$ by the set $B(Y)$ of semistandard tableaux $Y$ of shape $\lambda$. In addition to the existence of the crystal basis, the decompositions of $V(\lambda) \otimes \mathbf{V}$ and $B(Y) \otimes \mathbf{B}$, where $\mathbf{B}$ is the crystal basis for $\mathbf{V}$, have been found in [1].

In this paper we focus on the second super-analogue of the general linear Lie algebra: the queer Lie superalgebra $\mathfrak{q}(n)$. It has been known since its inception that the representation theory of $\mathfrak{q}(n)$ is more complicated compared to the other classical Lie superalgebra theories. A distinguished feature of $\mathfrak{q}(n)$ is that any Cartan subsuperalgebra has a nontrivial odd part. As a result, the highest weight space of any highest weight $\mathfrak{q}(n)$-module has a structure of a Clifford module. In particular, every $\mathfrak{gl}(n)$-component of a finite-dimensional $\mathfrak{q}(n)$-module appears with multiplicity larger than one (in fact, a power of two). Important results related to the representation theory of $\mathfrak{q}(n)$ include the $\mathfrak{q}(n)$-analogue of the celebrated Schur-Weyl duality discovered by Sergeev in 1984 [8], and character formulae for all simple finite-dimensional representations found by Penkov and Serganova in 1997 [7]. The foundations of the highest weight representation theory of the quantum queer superalgebra $U_q(\mathfrak{q}(n))$ have been established in [2]. An interesting observation in [2] is that the classical limit of a simple highest weight $U_q(\mathfrak{q}(n))$-module is a simple highest weight $U(\mathfrak{q}(n))$-module or a direct sum of two highest weight $U(\mathfrak{q}(n))$-modules.

In view of the above remarks, it is clear that developing a crystal basis theory for the category $\mathcal{O}_{int}^{\geq 0}$ of $U_q(\mathfrak{q}(n))$ is a challenging problem. The purpose of this paper is to announce the results that lead to a solution of this problem. Take the base

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*) Department of Mathematics, University of Texas at Arlington, Arlington, TX76021, U.S.A.
**) Department of Mathematical Sciences and Research Institute of Mathematics, Seoul National University, Seoul 151-747, Korea.
***) Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan.

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field to be $\mathbf{C}((q))$. Our main theorem is the existence and uniqueness of the crystal bases of $U_q(\mathfrak{q}(n))$-modules in $\mathcal{O}_{int}^{\geq 0}$. The proofs will appear in full detail in a forthcoming paper. To overcome the challenges described above, we modify the notion of a crystal basis and introduce the so-called *abstract* $\mathfrak{q}(n)$*-crystal.* To do so we first define *odd Kashiwara operators* $\tilde{e}_{\bar{1}}$, $\tilde{f}_{\bar{1}}$, and $\tilde{k}_{\bar{1}}$, where $\tilde{k}_{\bar{1}}$ corresponds to an odd element in the Cartan subsuperalgebra of $\mathfrak{q}(n)$. Then, a *crystal basis* for a $U_q(\mathfrak{q}(n))$-module $M$ in the category $\mathcal{O}_{int}^{\geq 0}$ is a triple $(L, B, (l_b)_{b\in B})$, where the crystal lattice $L$ is a free $\mathbf{C}[[q]]$-submodule of $M$, $B$ is a finite $\mathfrak{gl}(n)$-crystal, $(l_b)_{b\in B}$ is a family of vector spaces such that $L/qL = \bigoplus_{b\in B} l_b$, with a set of compatibility conditions for the action of the Kashiwara operators imposed in addition. The definition of a crystal basis leads naturally to the notion of an abstract $\mathfrak{q}(n)$-crystal an example of which is the $\mathfrak{gl}(n)$-crystal $B$ in any crystal basis $(L, B, (l_b)_{b\in B})$. The modified notion of a crystal allows us to consider the multiple occurrence of $\mathfrak{gl}(n)$-crystals corresponding to a highest weight $U_q(\mathfrak{q}(n))$-module $M$ in $\mathcal{O}_{int}^{\geq 0}$ as a single $\mathfrak{q}(n)$-crystal. It is worth noting that $M$ is not necessarily a simple module and that the $\mathfrak{q}(n)$-crystal $B$ of $M$ depends only on the highest weight $\lambda$ of $M$, hence we may write $B = B(\lambda)$. In order to find the highest weight vector of $B(\lambda)$, we use the action of the Weyl group on $B(\lambda)$ and define odd Kashiwara operators $\tilde{e}_{\bar{i}}$ and $\tilde{f}_{\bar{i}}$ for $i = 2, \ldots, n-1$. Then the highest weight vector of $B(\lambda)$ is simply the unique vector annihilated by the $2n-2$ Kashiwara operators $\tilde{e}_i$ and $\tilde{e}_{\bar{i}}$. In addition to the existence and uniqueness of the crystal basis of $M$, we establish an isomorphism $\mathbf{B} \otimes B(\lambda) \simeq \bigsqcup_{\lambda+\varepsilon_j:\text{strict}} B(\lambda+\varepsilon_j)$ and explicitly describe the highest weight vectors of $\mathbf{B} \otimes B(\lambda)$ in terms of the even Kashiwara operators $\tilde{f}_i$ and the highest weight vector of $B(\lambda)$. We conjecture that the highest weight vectors of $B(\lambda) \otimes \mathbf{B}$ can be found in an analogous way with the aid of the odd Kashiwara operators $\tilde{f}_{\bar{i}}$.

**2. The quantum queer superalgebra.** For an indeterminate $q$, let $\mathbf{F} = \mathbf{C}((q))$ be the field of formal Laurent series in $q$ and let $\mathbf{A} = \mathbf{C}[[q]]$ be the subring of $\mathbf{F}$ consisting of formal power series in $q$. Let $P^\vee = \mathbf{Z}k_1 \oplus \cdots \oplus \mathbf{Z}k_n$ be a free abelian group of rank $n$ and let $\mathfrak{h} = \mathbf{C} \otimes_{\mathbf{Z}} P^\vee$. Define the linear functionals $\epsilon_i \in \mathfrak{h}^*$ by $\epsilon_i(k_j) = \delta_{ij}$ $(i, j = 1, \ldots, n)$ and set $P = \mathbf{Z}\epsilon_1 \oplus \cdots \oplus \mathbf{Z}\epsilon_n$. We denote by $\alpha_i = \epsilon_i - \epsilon_{i+1}$ the *simple roots.*

**Definition 2.1.** The *quantum queer superalgebra* $U_q(\mathfrak{q}(n))$ is the superalgebra over $\mathbf{F}$ with 1 generated by $e_i, f_i, e_{\bar{i}}, f_{\bar{i}}$ $(i = 1, \ldots, n-1)$, $q^h$ $(h \in P^\vee)$, $k_{\bar{j}}$ $(j = 1, \ldots, n)$ with the following defining relations.

$$
\begin{aligned}
&q^0 = 1, \quad q^{h_1} q^{h_2} = q^{h_1+h_2} \quad (h_1, h_2 \in P^\vee),\\
&q^h e_i q^{-h} = q^{\alpha_i(h)} e_i \quad (h \in P^\vee),\\
&q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i \quad (h \in P^\vee),\\
&q^h k_{\bar{j}} = k_{\bar{j}} q^h,\\
&e_i f_j - f_j e_i = \delta_{ij} \frac{q^{k_i - k_{i+1}} - q^{-k_i + k_{i+1}}}{q - q^{-1}},\\
&e_i e_j - e_j e_i = f_i f_j - f_j f_i = 0\\
&\qquad \text{if } |i-j| > 1,\\
&e_i^2 e_j - (q+q^{-1}) e_i e_j e_i + e_j e_i^2 =\\
&f_i^2 f_j - (q+q^{-1}) f_i f_j f_i + f_j f_i^2 = 0\\
&\qquad \text{if } |i-j| = 1,\\
&k_{\bar{i}}^2 = \frac{q^{2k_i} - q^{-2k_i}}{q^2 - q^{-2}},\\
&k_{\bar{i}} k_{\bar{j}} + k_{\bar{j}} k_{\bar{i}} = 0 \quad (i \neq j),\\
&k_{\bar{i}} e_i - q e_i k_{\bar{i}} = e_{\bar{i}} q^{-k_i},\\
&k_{\bar{i}} f_i - q f_i k_{\bar{i}} = -f_{\bar{i}} q^{k_i},\\
&e_i f_{\bar{j}} - f_{\bar{j}} e_i = \delta_{ij}(k_{\bar{i}} q^{-k_{i+1}} - k_{\overline{i+1}} q^{-k_i}),\\
&e_{\bar{i}} f_j - f_j e_{\bar{i}} = \delta_{ij}(k_{\bar{i}} q^{k_{i+1}} - k_{\overline{i+1}} q^{k_i}),\\
&e_i e_{\bar{i}} - e_{\bar{i}} e_i = f_i f_{\bar{i}} - f_{\bar{i}} f_i = 0,\\
&e_i e_{i+1} - q e_{i+1} e_i = e_{\bar{i}} e_{\overline{i+1}} + q e_{\overline{i+1}} e_{\bar{i}},\\
&q f_{i+1} f_i - f_i f_{i+1} = f_{\bar{i}} f_{\overline{i+1}} + q f_{\overline{i+1}} f_{\bar{i}},\\
&e_i^2 e_{\bar{j}} - (q+q^{-1}) e_i e_{\bar{j}} e_i + e_{\bar{j}} e_i^2 =\\
&f_i^2 f_{\bar{j}} - (q+q^{-1}) f_i f_{\bar{j}} f_i + f_{\bar{j}} f_i^2 = 0,\\
&\qquad \text{if } |i-j| = 1.
\end{aligned} \tag{1}
$$

The generators $e_i$, $f_i$ $(i = 1, \ldots, n-1)$, $q^h$ $(h \in P^\vee)$ are regarded as *even* and $e_{\bar{i}}$, $f_{\bar{i}}$ $(i = 1, \ldots, n-1)$, $k_{\bar{j}}$ $(j = 1, \ldots, n)$ are *odd*. From the defining relations, it is easy to see that the even generators together with $k_{\bar{1}}$ generate the whole algebra $U_q(\mathfrak{q}(n))$.

The superalgebra $U_q(\mathfrak{q}(n))$ is a Hopf superalgebra with the comultiplication $\Delta\colon U_q(\mathfrak{q}(n)) \to U_q(\mathfrak{q}(n)) \otimes U_q(\mathfrak{q}(n))$ defined by

$$
\begin{aligned}
&\Delta(q^h) = q^h \otimes q^h \quad \text{for } h \in P^\vee,\\
&\Delta(e_i) = e_i \otimes q^{-k_i+k_{i+1}} + 1 \otimes e_i,\\
&\Delta(f_i) = f_i \otimes 1 + q^{k_i - k_{i+1}} \otimes f_i,\\
&\Delta(k_{\bar{1}}) = k_{\bar{1}} \otimes q^{k_1} + q^{-k_1} \otimes k_{\bar{1}}.
\end{aligned} \tag{2}
$$

Let $U^+$ (resp. $U^-$) be the subalgebra of $U_q(\mathfrak{q}(n))$ generated by $e_i$, $e_{\bar{i}}$ $(i = 1, \ldots, n-1)$ (resp. $f_i$, $f_{\bar{i}}$

$(i=1,\ldots,n-1)$, and let $U^0$ be the subalgebra generated by $q^h$ $(h \in P^\vee)$ and $k_{\bar{j}}$ $(j=1,\ldots,n)$. In [2], it was shown that the algebra $U_q(\mathfrak{q}(n))$ has the *triangular decomposition*:

$$U^- \otimes U^0 \otimes U^+ \xrightarrow{\sim} U_q(\mathfrak{q}(n)). \tag{3}$$

Hereafter, a $U_q(\mathfrak{q}(n))$-module is understood as a $U_q(\mathfrak{q}(n))$-supermodule. A $U_q(\mathfrak{q}(n))$-module $M$ is called a *weight module* if $M$ has a weight space decomposition $M = \bigoplus_{\mu \in P} M_\mu$, where

$$M_\mu := \{m \in M; q^h m = q^{\mu(h)} m \quad \text{for all } h \in P^\vee\}.$$

The set of weights of $M$ is defined to be

$$\mathrm{wt}(M) = \{\mu \in P; M_\mu \neq 0\}.$$

**Definition 2.2.** A weight module $V$ is called a *highest weight module with highest weight* $\lambda \in P$ if $V$ is generated by a finite-dimensional $U^0$-module $\mathbf{v}_\lambda$ satisfying the following conditions:

(a) $e_i v = e_{\bar{i}} v = 0$ for all $v \in \mathbf{v}_\lambda$, $i=1,\ldots,n-1$,

(b) $q^h v = q^{\lambda(h)} v$ for all $v \in \mathbf{v}_\lambda$, $h \in P^\vee$.

There is a unique irreducible highest weight module with highest weight $\lambda \in P$ up to parity change. We denote it by $V(\lambda)$.

Set

$$P^{\geq 0} = \{\lambda = \lambda_1\epsilon_1 + \cdots + \lambda_n\epsilon_n \in P;\ \lambda_j \in \mathbf{Z}_{\geq 0} \quad \text{for all } j=1,\ldots,n\},$$

$$\Lambda^+ = \{\lambda = \lambda_1\epsilon_1 + \cdots + \lambda_n\epsilon_n \in P^{\geq 0};\ \lambda_i \geq \lambda_{i+1} \text{ and } \lambda_i = \lambda_{i+1} \text{ implies } \lambda_i = \lambda_{i+1} = 0 \text{ for all } i=1,\ldots,n-1\}.$$

Note that each element $\lambda \in \Lambda^+$ corresponds to a *strict partition* $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0)$. Thus we will call $\lambda \in \Lambda^+$ a strict partition.

We define $\mathcal{O}_{int}^{\geq 0}$ to be the category of finite-dimensional weight modules such that $\mathrm{wt}(M) \subset P^{\geq 0}$ and $k_{\bar{i}}|_{M_\mu} = 0$ for any $i \in \{1,\ldots,n\}$ and $\mu \in P^{\geq 0}$ satisfying $\langle k_i, \mu\rangle = 0$. The fundamental properties of the category $\mathcal{O}_{int}^{\geq 0}$ are summarized in the following proposition.

**Proposition 2.3** [2]**.**

(a) *Every* $U_q(\mathfrak{q}(n))$*-module in* $\mathcal{O}_{int}^{\geq 0}$ *is completely reducible.*

(b) *Every irreducible object in* $\mathcal{O}_{int}^{\geq 0}$ *has the form* $V(\lambda)$ *for some* $\lambda \in \Lambda^+$.

**3. Crystal bases.** Let $M$ be a $U_q(\mathfrak{q}(n))$-module in $\mathcal{O}_{int}^{\geq 0}$. For $i=1,2,\ldots,n-1$, we define the *even Kashiwara operators* on $M$ in the usual way. That is, for a weight vector $u \in M_\lambda$, consider the *i-string decomposition* of $u$:

$$u = \sum_{k \geq 0} f_i^{(k)} u_k,$$

where $e_i u_k = 0$ for all $k \geq 0$, $f_i^{(k)} = f_i^k/[k]!$, $[k] = \frac{q^k - q^{-k}}{q - q^{-1}}$, $[k]! = [k][k-1]\cdots[2][1]$, and we define the even Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$ $(i=1,\ldots,n-1)$ by

$$\tilde{e}_i u = \sum_{k \geq 1} f_i^{(k-1)} u_k, \qquad \tilde{f}_i u = \sum_{k \geq 0} f_i^{(k+1)} u_k. \tag{4}$$

On the other hand, we define the *odd Kashiwara operators* $\tilde{k}_{\bar{1}}$, $\tilde{e}_{\bar{1}}$, $\tilde{f}_{\bar{1}}$ by

$$\begin{aligned} \tilde{k}_{\bar{1}} &= q^{k_1 - 1} k_{\bar{1}}, \\ \tilde{e}_{\bar{1}} &= -(e_1 k_{\bar{1}} - q k_{\bar{1}} e_1) q^{k_1 - 1}, \\ \tilde{f}_{\bar{1}} &= -(k_{\bar{1}} f_1 - q f_1 k_{\bar{1}}) q^{k_2 - 1}. \end{aligned} \tag{5}$$

Recall that an abstract $\mathfrak{gl}(n)$-crystal is a set $B$ together with the maps $\tilde{e}_i, \tilde{f}_i\colon B \to B \sqcup \{0\}$, $\varphi_i, \varepsilon_i\colon B \to \mathbf{Z} \sqcup \{-\infty\}$ $(i=1,\ldots,n-1)$, and $\mathrm{wt}\colon B \to P$ satisfying the conditions given in [5]. In this paper, we say that an abstract $\mathfrak{gl}(n)$-crystal is a $\mathfrak{gl}(n)$*-crystal* if it is realized as a crystal basis of a finite-dimensional integrable $U_q(\mathfrak{gl}(n))$-module. In particular, we have $\varepsilon_i(b) = \max\{n \in \mathbf{Z}_{\geq 0};\ \tilde{e}_i^n b \neq 0\}$ and $\varphi_i(b) = \max\{n \in \mathbf{Z}_{\geq 0};\ \tilde{f}_i^n b \neq 0\}$ for any $b$ in a $\mathfrak{gl}(n)$-crystal $B$.

**Definition 3.1.** Let $M = \bigoplus_{\mu \in P^{\geq 0}} M_\mu$ be a $U_q(\mathfrak{q}(n))$-module in the category $\mathcal{O}_{int}^{\geq 0}$. A *crystal basis* of $M$ is a triple $(L, B, l_B = (l_b)_{b \in B})$, where

(a) $L$ is a free $\mathbf{A}$-submodule of $M$ such that
  (i) $\mathbf{F} \otimes_{\mathbf{A}} L \xrightarrow{\sim} M$,
  (ii) $L = \bigoplus_{\mu \in P^{\geq 0}} L_\mu$, where $L_\mu = L \cap M_\mu$,
  (iii) $L$ is stable under the Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$ $(i=1,\ldots,n-1)$, $\tilde{k}_{\bar{1}}$, $\tilde{e}_{\bar{1}}$, $\tilde{f}_{\bar{1}}$.

(b) $B$ is a $\mathfrak{gl}(n)$-crystal together with the maps $\tilde{e}_{\bar{1}}, \tilde{f}_{\bar{1}}\colon B \to B \sqcup \{0\}$ such that
  (i) $\mathrm{wt}(\tilde{e}_{\bar{1}} b) = \mathrm{wt}(b) + \alpha_1$, $\mathrm{wt}(\tilde{f}_{\bar{1}} b) = \mathrm{wt}(b) - \alpha_1$,
  (ii) for all $b, b' \in B$, $\tilde{f}_{\bar{1}} b = b'$ if and only if $b = \tilde{e}_{\bar{1}} b'$.

(c) $l_B = (l_b)_{b \in B}$ is a family of non-zero $\mathbf{C}$-vector spaces such that
  (i) $l_b \subset (L/qL)_\mu$ for $b \in B_\mu$,
  (ii) $L/qL = \bigoplus_{b \in B} l_b$,
  (iii) $\tilde{k}_{\bar{1}} l_b \subset l_b$,
  (iv) for $i = 1,\ldots,n-1,\bar{1}$, we have
    (1) if $\tilde{e}_i b = 0$ then $\tilde{e}_i l_b = 0$, and otherwise $\tilde{e}_i$ induces an isomorphism $l_b \xrightarrow{\sim} l_{\tilde{e}_i b}$.
    (2) if $\tilde{f}_i b = 0$ then $\tilde{f}_i l_b = 0$, and otherwise $\tilde{f}_i$ induces an isomorphism $l_b \xrightarrow{\sim} l_{\tilde{f}_i b}$.

Note that one can prove that $\tilde{e}_{\bar{1}}^2 = \tilde{f}_{\bar{1}}^2 = 0$ as endomorphisms of $L/qL$ for any crystal basis $(L, B, l_B)$.

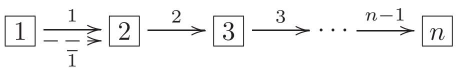
**Example 3.2.** Let

$$\mathbf{V} = \bigoplus_{j=1}^{n} \mathbf{F} v_j \oplus \bigoplus_{j=1}^{n} \mathbf{F} v_{\bar{j}}$$

be the vector representation of $U_q(\mathfrak{q}(n))$. The action of $U_q(\mathfrak{q}(n))$ on $\mathbf{V}$ is given as follows: $e_i v_j = \delta_{j,i+1} v_i$, $e_i v_{\bar{j}} = \delta_{j,i+1} v_{\bar{i}}$, $f_i v_j = \delta_{j,i} v_{i+1}$, $f_i v_{\bar{j}} = \delta_{j,i} v_{\overline{i+1}}$, $e_{\bar{i}} v_j = \delta_{j,i+1} v_{\bar{i}}$, $e_{\bar{i}} v_{\bar{j}} = \delta_{j,i+1} v_i$, $f_{\bar{i}} v_j = \delta_{j,i} v_{\overline{i+1}}$, $f_{\bar{i}} v_{\bar{j}} = \delta_{j,i} v_{i+1}$, $q^h v_j = q^{\epsilon_j(h)} v_j$, $q^h v_{\bar{j}} = q^{\epsilon_j(h)} v_{\bar{j}}$, $k_{\bar{i}} v_j = \delta_{j,i} v_{\bar{j}}$, $k_{\bar{i}} v_{\bar{j}} = \delta_{j,i} v_j$.
Set

$$\mathbf{L} = \bigoplus_{j=1}^{n} \mathbf{A} v_j \oplus \bigoplus_{j=1}^{n} \mathbf{A} v_{\bar{j}},$$

$l_j = \mathbf{C} v_j \oplus \mathbf{C} v_{\bar{j}}$, and let $\mathbf{B}$ be the crystal graph given below.




Here, the actions of $\tilde{f}_i$ $(i = 1, \ldots, n-1, \bar{1})$ are expressed by $i$-arrows. Then $(\mathbf{L}, \mathbf{B}, l_{\mathbf{B}} = (l_j)_{j=1}^n)$ is a crystal basis of $\mathbf{V}$.

**Theorem 3.3.** *Let $M_j$ be a $U_q(\mathfrak{g})$-module in $\mathcal{O}_{int}^{\geq 0}$ with crystal basis $(L_j, B_j, l_{B_j})$ $(j = 1, 2)$. Set $B_1 \otimes B_2 = B_1 \times B_2$ and*

$$l_{B_1 \otimes B_2} = (l_{b_1} \otimes l_{b_2})_{b_1 \in B_1, b_2 \in B_2}.$$

*Then*

$$(L_1 \otimes_{\mathbf{A}} L_2, B_1 \otimes B_2, l_{B_1 \otimes B_2})$$

*is a crystal basis of $M_1 \otimes_{\mathbf{F}} M_2$, where the action of the Kashiwara operators on $B_1 \otimes B_2$ are given as follows:*

$$(6) \quad \begin{aligned} \tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \\ \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases} \end{aligned}$$

$$(7) \quad \begin{aligned} \tilde{e}_{\bar{1}}(b_1 \otimes b_2) &= \begin{cases} \tilde{e}_{\bar{1}} b_1 \otimes b_2 & \text{if } \langle k_1, \mathrm{wt}\, b_2 \rangle = 0, \\ & \quad \langle k_2, \mathrm{wt}\, b_2 \rangle = 0, \\ b_1 \otimes \tilde{e}_{\bar{1}} b_2 & \text{otherwise}, \end{cases} \\ \tilde{f}_{\bar{1}}(b_1 \otimes b_2) &= \begin{cases} \tilde{f}_{\bar{1}} b_1 \otimes b_2 & \text{if } \langle k_1, \mathrm{wt}\, b_2 \rangle = 0, \\ & \quad \langle k_2, \mathrm{wt}\, b_2 \rangle = 0, \\ b_1 \otimes \tilde{f}_{\bar{1}} b_2 & \text{otherwise}. \end{cases} \end{aligned}$$

**Sketch of Proof.** Our assertion follows from the following comultiplication formulas.

$$\begin{aligned} \Delta(\tilde{k}_{\bar{1}}) &= \tilde{k}_{\bar{1}} \otimes q^{2k_1} + 1 \otimes \tilde{k}_{\bar{1}}, \\ \Delta(\tilde{e}_{\bar{1}}) &= \tilde{e}_{\bar{1}} \otimes q^{k_1+k_2} + 1 \otimes \tilde{e}_{\bar{1}} \\ &\quad - (1 - q^2)\tilde{k}_{\bar{1}} \otimes e_1 q^{2k_1}, \\ \Delta(\tilde{f}_{\bar{1}}) &= \tilde{f}_{\bar{1}} \otimes q^{k_1+k_2} + 1 \otimes \tilde{f}_{\bar{1}} \\ &\quad - (1 - q^2)\tilde{k}_{\bar{1}} \otimes f_1 q^{k_1+k_2-1}. \end{aligned}$$

□

Motivated by the properties of crystal bases, we introduce the notion of abstract crystals.

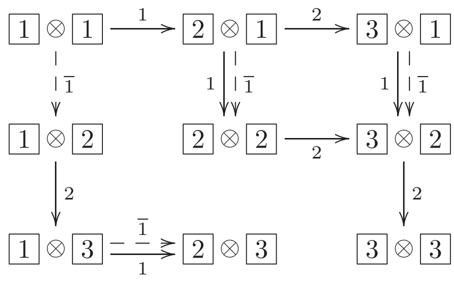
**Definition 3.4.** An *abstract $\mathfrak{q}(n)$-crystal* is a $\mathfrak{gl}(n)$-crystal together with the maps $\tilde{e}_{\bar{1}}, \tilde{f}_{\bar{1}} : B \to B \sqcup \{0\}$ satisfying the following conditions:

(a) $\mathrm{wt}(B) \subset P^{\geq 0}$,
(b) $\mathrm{wt}(\tilde{e}_{\bar{1}} b) = \mathrm{wt}(b) + \alpha_1$, $\mathrm{wt}(\tilde{f}_{\bar{1}} b) = \mathrm{wt}(b) - \alpha_1$,
(c) for all $b, b' \in B$, $\tilde{f}_{\bar{1}} b = b'$ if and only if $b = \tilde{e}_{\bar{1}} b'$.

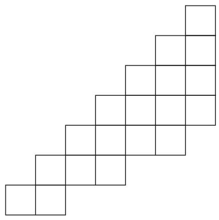
Let $B_1$ and $B_2$ be abstract $\mathfrak{q}(n)$-crystals. The *tensor product* $B_1 \otimes B_2$ of $B_1$ and $B_2$ is defined to be the $\mathfrak{gl}(n)$-crystal $B_1 \otimes B_2$ together with the maps $\tilde{e}_{\bar{1}}$, $\tilde{f}_{\bar{1}}$ defined by (7). Then it is an abstract $\mathfrak{q}(n)$-crystal. Note that $\otimes$ satisfies the associative axiom.

**Example 3.5.**

(a) If $(L, B, l_B)$ is a crystal basis of a $U_q(\mathfrak{q}(n))$-module $M$ in the category $\mathcal{O}_{int}^{\geq 0}$, then $B$ is an abstract $\mathfrak{q}(n)$-crystal.
(b) The crystal graph $\mathbf{B}$ is an abstract $\mathfrak{q}(n)$-crystal.
(c) By the tensor product rule, $\mathbf{B}^{\otimes N}$ is an abstract $\mathfrak{q}(n)$-crystal. When $n = 3$, the $\mathfrak{q}(n)$-crystal structure of $\mathbf{B} \otimes \mathbf{B}$ is given below.




(d) For a strict partition $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0)$, let $Y_\lambda$ be the skew Young diagram having $\lambda_1$ many boxes in the first diagonal, $\lambda_2$ many boxes in the second diagonal, etc. For example, if $\lambda$ is given by $(7 > 6 > 4 > 2 > 0)$, then we have

$Y_\lambda =$  .

Let $\mathbf{B}(Y_\lambda)$ be the set of all semistandard tableaux of shape $Y_\lambda$ with entries from $1, 2, \ldots, n$. Then by an *admissible reading* introduced in [1], $\mathbf{B}(Y_\lambda)$ is embedded in $\mathbf{B}^{\otimes|\lambda|}$ and it is stable under $\tilde{e}_i, \tilde{f}_i$, $\tilde{e}_{\bar{1}}, \tilde{f}_{\bar{1}}$. Hence it becomes an abstract $\mathfrak{q}(n)$-crystal. Moreover, the $\mathfrak{q}(n)$-crystal structure thus obtained does not depend on the choice of admissible readings.

Let $B$ be an abstract $\mathfrak{q}(n)$-crystal. For $i = 2, \ldots, n-1$, let $w$ be an element of the Weyl group $W$ with shortest length such that $w(\alpha_i) = \alpha_1$. Such an element is unique and we may choose $w = s_2 \cdots s_i s_1 \cdots s_{i-1}$. We define the *odd Kashiwara operators* $\tilde{e}_{\bar{i}}$, $\tilde{f}_{\bar{i}}$ $(i = 2, \ldots, n-1)$ by

$$\tilde{e}_{\bar{i}} = S_{w^{-1}} \tilde{e}_{\bar{1}} S_w, \quad \tilde{f}_{\bar{i}} = S_{w^{-1}} \tilde{f}_{\bar{1}} S_w.$$

Here $S_w$ is the Weyl group action on the $\mathfrak{gl}(n)$-crystal. The operators $\tilde{e}_{\bar{i}}$, $\tilde{f}_{\bar{i}}$ do not depend on the choice of reduced expressions of $w$. We say that $b \in B$ is a *highest weight vector* if $\tilde{e}_i b = \tilde{e}_{\bar{i}} b = 0$ for all $i = 1, \ldots, n-1$.

**4. Existence and uniqueness.** In this section, we present the main result of our paper.

**Theorem 4.1.**

(a) *Let $\lambda \in \Lambda^+$ be a strict partition and let $M$ be a highest weight $U_q(\mathfrak{q}(n))$-module in the category $\mathcal{O}_{int}^{\geq 0}$ with highest weight $\lambda$. If $(L, B, l_B)$ is a crystal basis of $M$, then $L_\lambda$ is invariant under $\tilde{k}_{\bar{i}} := q^{k_i - 1} k_{\bar{i}}$ for all $i = 1, \ldots, n$. Conversely, if $M_\lambda$ is generated by a free $\mathbf{A}$-submodule $L_\lambda^0$ invariant under $\tilde{k}_{\bar{i}}$ $(i = 1, \ldots, n)$, then there exists a unique crystal basis $(L, B, l_B)$ of $M$ such that*
   (i) $L_\lambda = L_\lambda^0$,
   (ii) $B_\lambda = \{b_\lambda\}$,
   (iii) $L_\lambda^0 / q L_\lambda^0 = l_{b_\lambda}$,
   (iv) *$B$ is connected.*

   *Moreover, as an abstract $\mathfrak{q}(n)$-crystal, $B$ depends only on $\lambda$. Hence we may write $B = B(\lambda)$.*

(b) *The $\mathfrak{q}(n)$-crystal $B(\lambda)$ has a unique highest weight vector $b_\lambda$.*

(c) *If $b \in \mathbf{B} \otimes B(\lambda)$ is a highest weight vector, then we have*

$$b = 1 \otimes \tilde{f}_1 \cdots \tilde{f}_{j-1} b_\lambda$$

   *for some $j$ such that $\lambda + \epsilon_j$ is a strict partition.*

(d) *Let $M$ be a $U_q(\mathfrak{q}(n))$-module in $\mathcal{O}_{int}^{\geq 0}$, and let $(L, B, l_B)$ be a crystal basis of $M$. Then there exist decompositions $M = \bigoplus_{a \in A} M_a$ as a $U_q(\mathfrak{q}(n))$-module, $L = \bigoplus_{a \in A} L_a$ as an $\mathbf{A}$-module, $B = \bigsqcup_{a \in A} B_a$ as a $\mathfrak{q}(n)$-crystal, parametrized by a set $A$ such that for any $a \in A$ the following conditions hold*:
   (i) *$M_a$ is a highest weight module with highest weight $\lambda_a$ and $B_a \simeq B(\lambda_a)$ for some strict partition $\lambda_a$,*
   (ii) $L_a = L \cap M_a$, $L_a / q L_a = \bigoplus_{b \in B_a} l_b$,
   (iii) *$(L_a, B_a, l_{B_a})$ is a crystal basis of $M_a$.*

(e) *Let $M$ be a highest weight $U_q(\mathfrak{q}(n))$-module in the category $\mathcal{O}_{int}^{\geq 0}$ with highest weight $\lambda$. Assume that $M$ has a crystal basis $(L, B(\lambda), l_{B(\lambda)})$ such that $L_\lambda / q L_\lambda = l_{b_\lambda}$. Then we have*
   (i) *$\mathbf{V} \otimes M = \bigoplus_{\lambda + \epsilon_j : strict} M_j$, where $M_j$ is a highest weight $U_q(\mathfrak{q}(n))$-module with highest weight $\lambda + \epsilon_j$ and $\dim(M_j)_{\lambda + \epsilon_j} = 2 \dim M_\lambda$,*
   (ii) $L_j = (\mathbf{L} \otimes L) \cap M_j$,
   (iii) *$\mathbf{B} \otimes B(\lambda) \simeq \coprod_{\lambda + \epsilon_j : strict} B_j$, where*

$$B_j \simeq B(\lambda + \epsilon_j), \quad L_j / q L_j = \bigoplus_{b \in B_j} l_b.$$

We will prove all of our assertions at once by induction on the length of $\lambda$. The proof is involved because our theorem consists of several interlocking statements. The key ingredient is a combinatorial proof of (c).

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