The discrete mean square of Dirichlet L-function at integral arguments

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Abstract: In this paper we shall make complete structural elucidation of the explicit formula for the (discrete) mean square of Dirichlet L-function at integral arguments, save for the case $s = 1$, this being completely settled in [1] recently. We shall treat the cases of negative and positive integers arguments separately, the former case being a preliminary and inclusive in the second. It will turn out that in respective cases the characteristic difference properties of Bernoulli polynomials and of the Hurwitz zeta-function are essential and telescoping the resulting difference equations, we obtain the results, revealing the underlying simple structure (known before 1905 at least).

Key words: Discrete mean square; characteristic difference equations; Dirichlet L-function; Bernoulli polynomials; Hurwitz zeta-function.

1. Introduction. The discrete mean value of the special values of the Dirichlet L-function $L(s,\chi)$–especially that of $L(1,\chi)$ in view of its relevance to the class number of the associated number fields–has been the subject of many researches. One can consult an excellent survey of Matsumoto [2] for the reference to [1], where the $s = 1$ case has been completely to structurally settled (also cf. [15]). The discrete mean values square of at positive integers has been also considered extensively by several authors. Katsurada and Matsumoto [3] were the first who obtained the result by their beta-function integral method. Louboutin [4–6] considered the same problem by different methods. Liu and Zhang [7] considered the cases of the product of two Dirichlet L-functions, which include all the above cases. Their result has been fully generalized by [8].

However in none of these papers (save for [1]), attention is paid on the reason why the formula is to hold, i.e. the underlying structure that forces the formula to hold has never been studied and only ad-hoc methods have been adopted.

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In this paper we shall concentrate on structural side of the problem and make a methodological taking over of all the previous results. We shall show that the underlying principle (which should be known to [14] at least before 1905) is exactly the same as in [9] if we use another basis (Hurwitz zeta-function) for relevant periodic functions, cf. (1.4), as elucidated by [10]. Then secondly we shall show that the characteristic difference properties of the Hurwitz zeta-function will show its essential effect and just telescoping gives the result, as in infinitesimal calculus–differentiation and integration! However, to show some historically interesting feature of the problem, we treat the case of negative integer case separately in terms of Bernoulli polynomials (using a 1923 result of Nielsen) although this case could be included in the positive integer case in terms of the Hurwitz zeta-function through (1.2). We get an interesting convolution identity as a bonus (Proposition 1).

Notation. Let

$$\zeta(s,x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}, \quad \mathrm{Re}\, s = \sigma > 1. \tag{1.1}$$

be the Hurwitz zeta-function, $0 < x \le 1$, whose special case with $x = 1$ is the Riemann zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \sigma > 1.$$

These zeta-functions have meromorphic continuations over the whole s-plane with a unique simple pole at $s = 1$ with residue 1. For this and other results on those functions which we use in this paper, we refer e.g. to [11].

For a non-negative integer k, we introduce the kth Bernoulli polynomial by

$$\zeta(1-k,x) = -\frac{1}{k}B_k(x). \tag{1.2}$$

Then it follows that

$$B_k(x) = \sum_{r=0}^{k}\binom{k}{r}B_{k-r}x^r. \tag{1.3}$$

$B_k = B_k(1)$ is the kth Bernoulli number ($k \geq 2$) with $B_1 = -\frac{1}{2}$. $\overline{B}_k(x)$ is the period Bernoulli polynomial defined by

$$\overline{B}_k(x) = B_k(x-[x])$$

with $[x]$ designating the integral part of x. (cf. [11, Chapter 4]).

For a Dirichlet character χ to the modulus q, let the Dirichlet L-function $L(s,\chi)$ associated with χ be defined by

$$L(s,\chi) = q^{-s}\sum_{a=1}^{q-1}\chi(a)\zeta\left(s,\frac{a}{q}\right) \tag{1.4}$$

for $\sigma > 1$ in the first instance. It has the Dirichlet series expansion

$$L(s,\chi) = \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}, \quad \sigma = \operatorname{Re} s > 1 \tag{1.5}$$

where the series on the right-hand side is uniformly convergent in s, (for χ non-trivial) so that it is analytic in $\sigma > 0$.

In the remaining region, the functional equation for the Dirichlet L-function gives its analytic continuation over the whole plane ([11, (8.17), p. 171]), and we can speak of their special values at integer points $n \leq 1$. We assume throughout that $q \geq 3$.

$$J_k(q) = \sum_{d|q}\mu\left(\frac{q}{d}\right)d^k \tag{1.6}$$

is the Jordan totient function, where the summation is extended over all positive divisors of q and μ is the Möbius function. Note that

$$J_1(q) = \phi(q) = \sum{}^{*}1 \tag{1.7}$$

is the Euler function, where $*$ on the summation sign means that it is extended over those natural numbers relatively prime to q.

We are in a position to state our results.

Theorem 1. *For a non-negative integer n, we have*

$$\begin{aligned}\sum_{\chi \bmod q}|L(-n,\chi)|^2 &= \phi(q)q^{2n}\frac{B_{2n+2}}{(n+1)^2}J_{-2n-1}(q)\\ &+ \frac{2\phi(q)q^{2n}}{n+1}\sum_{r=0}^{n-1}\binom{n+1}{r}\frac{B_{n-r+1}B_{n+r+1}}{n+r+1}J_{-n-r}(q)\\ &+ (-1)^n\phi^2(q)q^{2n}\frac{1}{(n+1)^2}\frac{B_{2n+2}}{\binom{2n+2}{n+2}}.\end{aligned} \tag{1.8}$$

Substituting (1.2), we may express (1.8) as

$$\begin{aligned}\sum_{\chi \bmod q}|L(-n,\chi)|^2 &= \frac{2}{n+1}\phi(q)q^{2n}\zeta(-2n-1)J_{-2n-1}(q) + \phi(q)\\ &\times q^{2n}\sum_{r=0}^{n-1}\binom{n}{r}\zeta(-n+r)\zeta(-n-r)J_{-n-r}(q)\\ &+ \frac{2(-1)^n}{n+1}\phi^2(q)q^{2n}\frac{1}{\binom{2n+2}{n+1}}\zeta(-2n-1),\end{aligned} \tag{1.9}$$

which is Theorem 6 of Katsurada and Matsumoto [3].

We now turn to the positive integer case $s = k > 1$ treated by Katsurada and Motsumoto [3], Louboutin [5], Liu-Zhang [7] and Kanemitsu-Ma-Zhang [8]. The following result amounts to the Katsurada-Matsumoto Theorem in the spirit of [3].

Theorem 2. *For integers $N, k > 1$ we have*

$$\begin{aligned}\sum_{\chi \bmod q}|L(k,\chi)|^2 &= \frac{\phi(q)}{q^{2k}}J_{2k}(q)\zeta(2k)\\ &+ \frac{2\phi(q)}{q^{2k}}\sum_{\substack{r=0\\ r\neq k-1}}^{N}\binom{k+r-1}{r-1}\zeta(k+r)\zeta(k-r)J_{k-r}(q)\\ &+ \frac{2(-1)^{k-1}}{q^{2k}}\binom{2k-2}{k-1}\zeta(2k-1)\phi(q)^2\\ &\times\left(\log q + \sum_{p|q}\frac{\log p}{p-1} - \frac{\zeta'}{\zeta}(2k-1) + \gamma - \sum_{n=k}^{2k-2}\frac{1}{n}\right)\\ &+ \frac{\phi(q)}{q^{2k}}R_N(q),\end{aligned}$$

where γ is the Euler constant and $R_N(q)$ is defined by

$$\begin{aligned}R_N(q) &= \sum_{d|q}\mu\left(\frac{q}{d}\right)\\ &\times\sum_{r=0}^{k-1}\binom{k+r-1}{r}\binom{r-k}{N-k+1}\frac{1}{d^{N-k}}\end{aligned} \tag{1.10}$$

$$\sum_{m=1}^{\infty} \frac{1}{m^{k+N}} \int_1^{\infty} \overline{B}_{N-k+1}(dmz) z^{r-N-1}\,\mathrm{d}z.$$

2. Proof of Theorem 1.

Lemma 1. *For $s \neq 1$,*

$$\sum_{\chi} |L(s,\chi)|^2 = \frac{\phi(q)}{q^{2\sigma}} \sum_{d|q} \mu\left(\frac{q}{d}\right) \sum_{a=1}^{d} \left|\zeta\left(s, \frac{a}{d}\right)\right|^2. \tag{2.1}$$

Proof easily follows from (1.4).

Lemma 2 (Nielsen [13, p. 76, (10)]). *For each $n \geq 1$,*

$$B_n(x)^2 = B_{2n}(x) + \frac{(-1)^{n-1} B_{2n}}{\binom{2n}{n}} + 2n \sum_{k=0}^{n-2} \binom{n}{k} B_{n-k} \frac{B_{n+k}(x)}{n+k}. \tag{2.2}$$

Proof. We give a proof independent of Nielsen's, which helps to understand the characteristic difference property of the Bernoulli polynomials. We apply the method of undetermined coefficients. First we have the Bernoulli polynomial expansion

$$B_n(x)^2 = \sum_{k=1}^{2n} a_k B_k(x) + a_0, \tag{2.3}$$

where a_k's are to be determined ($a_k = a_k(n)$). To this end we compare the two expressions for the difference

$$\Delta B_n^2(x) = B_n(x+1)^2 - B_n(x)^2.$$

On one hand, by the characteristic difference equation

$$B_n(x+1) - B_n(x) = nx^{n-1}, \tag{2.4}$$

we obtain

$$\Delta B_n^2(x) = nx^{n-1}\bigl(nx^{n-1} + 2B_n(x)\bigr). \tag{2.5}$$

Hence by (1.3)

$$\Delta B_n^2(x) = n^2 x^{2n-2} + 2n \sum_{r=0}^{n} \binom{n}{r} B_{n-r} x^{n+r-1}. \tag{2.6}$$

On the other hand, by (2.4) again,

$$\Delta B_n^2(x) = \sum_{k=1}^{n-1} k a_k x^{k-1} + \sum_{r=0}^{n} (n+r) a_{n+r} x^{n+r-1}. \tag{2.7}$$

Comparing the coefficients in (2.6) and (2.7), we establish (2.2) up to the value of $a_0 = a_0(n)$. Integrating (2.2) and using the orthogonality, we obtain

$$a_0 = a_0(n) = \int_0^1 B_n(x)^2 \mathrm{d}x$$

(cf. e.g. [12] or [13]), and the value of the last integral is known to be

$$\int_0^1 B_n(x)^2 \mathrm{d}x = (-1)^{n-1} \frac{B_{2n}}{\binom{2n}{n}}, \quad (n = 1, 2, \cdots)$$

whence (2.2) follows. □

As a bonus (2.2) with $x = 1$ deserves mentioning.

Proposition 1. *For $n \geq 2$, we have*

$$B_n^2 - B_{2n} - 2n \sum_{r=0}^{n-2} \binom{n}{r} B_{n-r} \frac{B_{n+r}}{n+r} = (-1)^{n-1} \frac{B_{2n}}{\binom{2n}{n}}.$$

Proof of Theorem 1.

By (1.2),

$$S(d) := \sum_{a=1}^{d} \left|\zeta\left(-n, \frac{a}{d}\right)\right|^2 = \frac{1}{(n+1)^2} \sum_{a=1}^{d} B_{n+1}\left(\frac{a}{d}\right)^2.$$

Substituting (2.2) and using the Kubert relation for the Bernoulli polynomial (cf. [11, p. 4 (1.8)])

$$B_s(dx) = d^{s-1} \sum_{a=0}^{d-1} B_s\left(x + \frac{a}{d}\right), \tag{2.8}$$

we obtain

$$S(d) = \frac{1}{(n+1)^2} B_{2n+2} d^{-2n-1} + \frac{d}{(n+1)^2} (-1)^n \frac{B_{2n+2}}{\binom{2n+2}{n+1}} + \frac{2}{n+1} \sum_{r=0}^{n-1} \binom{n+1}{r} B_{n-r+1} \times \frac{B_{n+r+1}}{n+r+1} d^{-n-r}. \tag{2.9}$$

Now substituting (2.9) in the formula of Lemma 1, we complete the proof of Theorem 1.

3. Proof of Theorem 2. The main ingredient in the proof is the following Lemma 4 in the proof of which we need the integral representation and estimate for sums of powers $L_u(N,x) = \sum_{n=0}^{N} (x+n)^u$. We recall the following from [11].

Lemma 3. *For any $l \in \mathbf{N}$ with $l > \operatorname{Re} u + 1$ and for any $x \geq 0$, we have the integral representation*

$$L_u(N,x) = \sum_{r=1}^{l} \frac{\Gamma(u+1)}{\Gamma(u+2-r)} \frac{(-1)^r \overline{B}_r(N)}{r!} (N+x)^{u-r+1} + \frac{(-1)^l}{l!} \frac{\Gamma(u+1)}{\Gamma(u+1-l)} \int_N^\infty \overline{B}_l(t)\,(t+x)^{u-l}\,\mathrm{d}t + \begin{cases} \dfrac{1}{u+1}(N+x)^{u+1} + \zeta(-u,x), & u \neq -1 \\ \log(N+x) - \psi(x), & u = -1, \end{cases}$$

where $\psi(x)$ *is the Euler digamma function.*

We can now prove a lemma corresponding to Lemma 2.

Lemma 4. *For any integer* $k \geq 2$, *we have*

$$\zeta(k,x)^2 = \zeta(2k,x) + 2\sum_{r=0}^{k-2} \binom{k+r-1}{r} \times ((-1)^k + (-1)^r)\zeta(k+r)\zeta(k-r,x) + 2(-1)^{k-1} \sum_{r=0}^{k-1} \binom{k+r-1}{r} \times \sum_{m=1}^{\infty} \sum_{n \leq m} \frac{1}{m^{k+r}(x+n-1)^{k-r}}. \tag{3.1}$$

Proof. We consider the difference equation for $f_k(x) = \zeta(k,x)^2$ corresponding to (2.5):

$$\Delta f_k = f_k(x+1) - f_k(x) := g_k(x), \tag{3.2}$$

where

$$g_k(x) = -\frac{2}{x^k}\zeta(k,x+1) - \frac{1}{x^{2k}}. \tag{3.3}$$

This follows from the difference equation

$$\zeta(k,x+1) - \zeta(k,x) = -\frac{1}{x^k}$$

corresponding to (2.4).

In the following we shall show that we do can telescope (3.2).

We recall the following 1905 result from [14, p. 48, (9)]

$$\frac{1}{x^k}\zeta(k,x+1) = (-1)^k \sum_{r=0}^{k-1} \binom{k+r-1}{r} \sum_{m=1}^{\infty} \frac{1}{m^{k+r}(x+m)^{k-r}} + \sum_{r=0}^{k-1} \binom{k+r-1}{r} (-1)^r \zeta(k+r) \frac{1}{x^{k-r}}. \tag{3.4}$$

First we telescope first N terms in (3.2) to get:

$$-\left(\zeta(k,x+N+1)^2 - \zeta(k,x)^2\right) = -\sum_{n=0}^{N} \Delta f_k(x+n) = -\sum_{n=0}^{N} g_k(x+n) := S_N,$$

say. To compute the sum S_N we substitute (3.4) in (3.3), substitute $x+n, 0 \leq n \leq N$ for x and sum over $n, 0 \leq n \leq N$. We then obtain

$$S_N = 2(-1)^k \sum_{r=0}^{k-1} \binom{k+r-1}{r} \sum_{m=1}^{\infty} \frac{1}{m^{k+r}} \times (L_{r-k}(N+m,x) - L_{r-k}(m-1,x)) + 2\sum_{r=0}^{k-1} \binom{k+r-1}{r} (-1)^r \zeta(k+r) L_{r-k}(N,x) + L_{-2k}(N,x). \tag{3.5}$$

Those terms in S_N with $r = k-1$ contribute:

$$2(-1)^k \binom{2k-2}{k-1} \sum_{m=1}^{\infty} \frac{1}{m^{2k-1}} \times (L_{-1}(N+m,x) - L_{-1}(N,x)) + 2(-1)^{k-1} \binom{2k-2}{k-1} \times \sum_{m=1}^{\infty} \frac{1}{m^{2k-1}} L_{-1}(m-1,x). \tag{3.6}$$

By Lemma 3 we see that the first term in (3.6) $\to 0$ as $N \to \infty$, so that (3.6) tends to

$$2(-1)^{k-1} \binom{2k-2}{k-1} \sum_{m=1}^{\infty} \frac{1}{m^{2k-1}} L_{-1}(m-1,x),$$

which is the term with $r = k-1$ (without the term $L_{r-k}(N+m,x)$) in the first sum in (3.5). By Lemma 3, other terms $L_{r-k}(N,x)$ tend to $\zeta(k-r)$ as $N \to \infty$, whence taking the limits

$$-\sum_{n=0}^{\infty} \Delta f_k(x+n) = -\sum_{n=0}^{\infty} g_k(x+n) = \lim_{N\to\infty} S_N,$$

we conclude (3.1), completing the proof. □

We further need the following combinational identities which are interesting in their own right.

Lemma 5.

$$\sum_{r=0}^{k-1} \binom{k-1+r}{r} = \frac{1}{2}\binom{2k}{k}. \tag{3.7}$$

$$\sum_{\nu=0}^{k-1} \binom{r+\nu-1}{\nu} \frac{1}{k-\nu} \tag{3.8}$$

$$= \binom{r+k-1}{k}(\psi(r+k)-\psi(r)).$$

(3.9)
$$\sum_{r=0}^{k-1}\binom{k+r-1}{r}\binom{r-k}{j}(-1)^j = \binom{2k+j-1}{k-1}.$$

Proof follows by generating functionology and is omitted.

Now we may turn to

Completion of Proof of Theorem 2. We use Lemma 1 with $s=k$,

$$\sum_{\chi}|L(k,\chi)|^2 = \frac{\phi(q)}{q^{2k}}\sum_{d|q}\mu\left(\frac{q}{d}\right)\sum_{a=1}^{d}\zeta\left(k,\frac{a}{d}\right)^2.$$

Noting that the sum over $1\le a\le d$ of the third term on the right of (3.1) with $x=\frac{a}{d}$, S_3, say, is

$$2(-1)^{k-1}\sum_{r=0}^{k-1}\binom{k+r-1}{r}d^{k-r}\sum_{m=1}^{\infty}\frac{1}{m^{k+r}}L_{r-k}(dm),$$

we substitute Lemma 3 to deduce that

$$\begin{aligned}\frac{(-1)^{k-1}}{2}S_3 &= \sum_{r=0}^{k-2}\binom{k+r-1}{r}\\ &\times\left\{-\frac{1}{k-r-1}\zeta(2k-1)d+\zeta(k+r)\zeta(k-r)d^{k-r}\right\}\\ &+\binom{2k-2}{k-1}\\ &\times(d\log d\zeta(2k-1)-d\zeta'(2k-1)+d\gamma\zeta(2k-1))\\ &+\sum_{r=0}^{k-1}\binom{k+r-1}{r}\sum_{j=1}^{l}(-1)^{j-1}\binom{r-k}{j-1}\\ &\times\zeta(1-j)\zeta(2k+j-1)d^{1-j}\\ &+(-1)^l\sum_{r=0}^{k-1}\binom{k+r-1}{r}\binom{r-k}{l}d^{k-r}\\ &\times\sum_{m=1}^{\infty}\frac{1}{m^{k+r}}\int_{dm}^{\infty}\bar{B}_l(t)t^{r-k-l}\,\mathrm{d}t,\end{aligned}$$

on separating the case $r=k-1$. Substituting this in (3.1) completes the proof. □

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