Lyapunov inequality for elliptic equations involving limiting nonlinearities

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Abstract: This note deals with a generalization of the famous Lyapunov inequality giving a necessary condition for the existence of solutions to a boundary value problem for an ordinary differential equation. The problem we consider is closely related to a well-known problem on an asymptotic behavior of positive solutions of a class of semilinear elliptic equations of nearly critical Sobolev growth.

Key words: Elliptic equations; critical exponents; Lyapunov inequality.

Introduction. Motivated by the study of nonlinear boundary value problems at resonance, in [4], the authors considered the following linear boundary value problem

$$(1)\qquad \begin{aligned} &u''(x) + a(x)u(x) = 0, \quad x \in (L_1, L_2),\\ &u(L_1) = u(L_2) = 0, \end{aligned}$$

where $a(x) \in \Lambda_0$ and Λ_0 is defined by

$$\Lambda_0 = \{a \in C[L_1, L_2] \setminus \{0\} : \text{Problem (1) has a nontrivial solution}\}.$$

Note that the well-known Lyapunov inequality [10] states that if $a(x) \in \Lambda_0$, then necessarily

$$\int_{L_1}^{L_2} |a(x)|\,dx > \frac{4}{L_2 - L_1}.$$

This inequality is sharp in the sense that the constant on the right cannot be replaced by a larger number. Thus,

$$\beta_1 \equiv \inf_{a\in\Lambda_0} \|a\|_1 = \frac{4}{L_2 - L_1},$$

and the value of β_1 is not attained. A. Cañada, J. A. Montero and S. Villegas generalized this result by considering the quantity

$$\beta_p \equiv \inf_{a\in\Lambda_0} \|a\|_p,$$

for all p, $1 \le p \le \infty$, and obtaining an explicit expression for β_p in terms of p, L_1 and L_2. In their further work [5], they treated an analogous problem for partial differential equations. More precisely, the following problem was considered

$$\begin{cases} -\Delta u(x) = a(x)\,u(x) & x \in \Omega\\ u(x) = 0 & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbf{R}^N$ $(N \ge 2)$ is a smooth bounded domain, $a \in L^q(\Omega)$, for some $q \ge 1$, and the qualitative study of the quantity

$$\beta_p \equiv \inf_{a\in\Lambda\cap L^p(\Omega)} \|a\|_p, \quad 1 \le p \le \infty,$$

where Λ is defined similarly to Λ_0, was made (see also [5, Remark 5]). The dimension of the problem plays an important rôle in this instance. In particular, A. Cañada, J. A. Montero and S. Villegas showed that when $N = 2$, the constant $\beta_p > 0$ if, and only if, $1 < p \le \infty$. If $N \ge 3$, then $\beta_p > 0$ if, and only if, $\frac{N}{2} \le p \le \infty$. Moreover, if $N \ge 2$ and $\frac{N}{2} < p \le \infty$, then β_p is attained. Note that a complete study of the critical case corresponding to the value of $p = \frac{N}{2}$ is left open in [5]. In the present paper we provide a detailed treatment, when $N \ge 3$, of this critical case.

In conclusion, mention may be made of the fact that our result is a generalization of some results from [8] and [12] on an asymptotic behavior of positive solutions to a well-known class of semilinear elliptic equations with nearly critical nonlinearity.

The main result. Let Ω be a smooth bounded domain in $\mathbf{R}^N$, $N \ge 3$. Consider the following Dirichlet boundary value problem

$$(2)\qquad \begin{cases} -\Delta u(x) = a(x)\,u(x) & x \in \Omega\\ u(x) = 0 & x \in \partial\Omega, \end{cases}$$

where the function $a : \Omega \to \mathbf{R}$ belongs to the set

$$\Lambda = \{a \in L^{N/2}(\Omega) \setminus \{0\} : \text{Problem (2) has a nontrivial solution}\}.$$

The eigenvalues of the eigenvalue problem

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$$\begin{cases} -\Delta u(x) = \lambda\, u(x) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega \end{cases}$$

belong to the set Λ. Hence the quantity

$$\beta_{\frac{N}{2}} = \inf_{a\in\Lambda} \|a\|_{\frac{N}{2}}$$

is well defined.

Theorem 1. *The value of* $\beta_{\frac{N}{2}}$ *is given by*

$$\beta_{\frac{N}{2}} = S_N,$$

where S_N *is the best Sobolev constant in* $\mathbf{R}^N$:

$$S_N = \pi N(N-2)\left[\frac{\Gamma(N/2)}{\Gamma(N)}\right]^{2/N},$$

and $\beta_{\frac{N}{2}}$ *is not attained.*

Proof. Let $a \in \Lambda$, and $u \in H_0^1(\Omega)$ be a corresponding nontrivial solution of Problem (2). Multiplying the equation in (2) by u, and integrating by parts using the boundary condition, we obtain

$$\int_\Omega |\nabla u|^2 = \int_\Omega a\, u^2. \tag{3}$$

It follows from the Hölder inequality that

$$\int_\Omega |\nabla u|^2 \le \ \|a\|_{\frac{N}{2}} \|u^2\|_{\frac{N}{N-2}}.$$

Note that the exponent $2N/(N-2)$ is critical for the embedding of the Sobolev space $H_0^1(\Omega)$ into Lebesgue spaces.

From the last inequality we have

$$\|a\|_{\frac{N}{2}} \ge \frac{\int_\Omega |\nabla u|^2}{\|u\|^2_{\frac{2N}{N-2}}} \ge \inf_{v\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla v|^2}{\|v\|^2_{\frac{2N}{N-2}}} = S_N. \tag{4}$$

Therefore,

$$\beta_{\frac{N}{2}} = \inf_{a\in\Lambda} \|a\|_{\frac{N}{2}} \ge S_N. \tag{5}$$

Consider now the problem

$$\begin{cases} -\Delta u(x) = N(N-2)\, u^{p-\varepsilon}(x) & x \in \Omega \\ u(x) > 0 & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega, \end{cases} \tag{6}$$

where $p = (N+2)/(N-2)$ and $\varepsilon \ge 0$. It is well known that when $\varepsilon > 0$ Problem (6) has a solution u_ε. Hence for any $\varepsilon > 0$ the functions $a_\varepsilon(x) := N(N-2)\ u_\varepsilon^{p-1-\varepsilon}(x)$ belong to the set Λ. Note that if $\varepsilon = 0$ the existence of solutions of Problem (6) depends on the topological properties of the domain Ω. In particular, when Ω is starshaped it is proved in [11] that (6) does not have any solution. The asymptotic behavior of solutions of Problem (6) as ε goes to zero was studied in the papers [8,12,13] (see also [1,2] for the case of spherical domains).

Let u_ε be a solution of Problem (6), and assume that $\{u_\varepsilon\}_{\varepsilon>0}$ is a minimizing sequence for the Sobolev inequality, i.e.

$$\lim_{\varepsilon\to 0} \frac{\int_\Omega |\nabla u_\varepsilon|^2}{\|u_\varepsilon\|^2_{p+1-\varepsilon}} = S_N. \tag{7}$$

Multiplying (6) by u_ε and integrating by parts, we obtain

$$\int_\Omega |\nabla u_\varepsilon|^2 = N(N-2)\int_\Omega u_\varepsilon^{p+1-\varepsilon}.$$

Then, from the assumption (7) we have

$$\begin{aligned} &\lim_{\varepsilon\to 0} N(N-2)\|u_\varepsilon\|^{p-1-\varepsilon}_{p+1-\varepsilon} \\ &\quad = \lim_{\varepsilon\to 0}\left\|N(N-2)u_\varepsilon^{p-1-\varepsilon}\right\|_{\frac{p+1-\varepsilon}{p-1-\varepsilon}} = S_N. \end{aligned}$$

Hence

$$\begin{aligned} \|a_\varepsilon\|_{\frac{N}{2}} &= \|a_\varepsilon\|_{\frac{p+1}{p-1}} = \left\|N(N-2)u_\varepsilon^{p-1-\varepsilon}\right\|_{\frac{p+1}{p-1}} \\ &= S_N + o(1) \quad \text{as} \quad \varepsilon \to 0. \end{aligned}$$

Therefore

$$\beta_{\frac{N}{2}} = \inf_{a\in\Lambda}\|a\|_{\frac{N}{2}} \le \lim_{\varepsilon\to 0}\|a_\varepsilon\|_{\frac{N}{2}} = S_N,$$

which together with (5) gives

$$\beta_{\frac{N}{2}} = S_N. \tag{8}$$

Now, let $\{a_n\}_{n\in\mathbf{N}}$ be an arbitrary minimizing sequence for $\beta_{\frac{N}{2}}$, i.e.

$$\lim_{n\to\infty}\|a_n\|_{\frac{N}{2}} = \beta_{\frac{N}{2}} = \inf_{a\in\Lambda}\|a\|_{\frac{N}{2}}.$$

For any $n \in \mathbf{N}$, denote by u_n a nontrivial solution of (2) corresponding to the function a_n. Then from (4) and (8) we have

$$\lim_{n\to\infty}\|a_n\|_{\frac{N}{2}} = \lim_{n\to\infty}\frac{\int_\Omega |\nabla u_n|^2}{\|u_n\|^2_{\frac{2N}{N-2}}} = S_N.$$

Thus $\{u_n\}_{n\in\mathbf{N}}$ is a minimizing sequence for the Sobolev inequality. It is well known that the best Sobolev constant S_N is never achieved on a bounded domain (see, e.g., [15,16]). Hence we deduce from (4) that $\beta_{\frac{N}{2}}$ is not attained. □

Theorem 2. *For any point* $x_0 \in \overline{\Omega}$ *there exists a minimizing sequence* $\{a_n\}_{n\in\mathbf{N}}$ *for* $\beta_{\frac{N}{2}}$ *such that* $\{|a_n|^{N/2}\}_{n\in\mathbf{N}}$ *converges in the sense of measures* *to* $S_N^{N/2}\delta_{x_0}$, *where* δ_{x_0} *denotes the Dirac mass concentrated at the point* x_0.

Proof. Let x_0 be an arbitrary point of $\overline{\Omega}$. We choose Q to be a $C(\overline{\Omega})\cap C^3(\Omega)$ non-negative function which has x_0 as its unique (non-degenerate) maximum point in $\overline{\Omega}$. Let

$$Q_M := \max_{x\in\overline{\Omega}} Q(x) = Q(x_0).$$

Recall that an asymptotic behavior of solutions of the following boundary value problem was investigated in [6],

$$(9)\quad \begin{cases} -\Delta u = Q(x)|u|^{p-1}u + \varepsilon|u|^{\sigma-1}u & x\in\Omega \\ u = 0 & x\in\partial\Omega, \end{cases}$$

where $p=(N+2)/(N-2)$, $\sigma\in[1,p)$, $\varepsilon>0$, and the function Q can be taken as above. The existence of at least one positive solution of (9) was established in [7] for $\sigma=1$ and ε less than the first eigenvalue of $-\Delta$ with zero Dirichlet boundary condition.

We take now $\sigma=1$ and note that for ε small enough the functions $a_\varepsilon(x)=Q(x)|u_\varepsilon(x)|^{p-1}+\varepsilon$, where u_ε is a least energy solution of Problem (9), belong to the set Λ. Therefore, using the Minkowski inequality and the fact that $N/2=(p+1)/(p-1)$ we have

$$\begin{aligned} S_N &\le \lim_{\varepsilon\to 0}\|a_\varepsilon\|_{\frac{N}{2}} = \lim_{\varepsilon\to 0}\left\|Q(x)|u_\varepsilon(x)|^{p-1}+\varepsilon\right\|_{\frac{N}{2}} \\ &\le \lim_{\varepsilon\to 0}\left[\int_\Omega Q(x)^{\frac{N}{2}}|u_\varepsilon(x)|^{p+1}\right]^{2/N} \\ &\le \lim_{\varepsilon\to 0}\left[Q_M^{\frac{N}{2}-1}\int_\Omega Q(x)|u_\varepsilon(x)|^{p+1}\right]^{2/N} \\ &= \left[Q_M^{(N-2)/2}\frac{S_N^{N/2}}{Q_M^{(N-2)/2}}\right]^{2/N} = S_N. \end{aligned}$$

The value of the last limit is calculated in [6, (2.8)]. Thus

$$(10)\qquad \lim_{\varepsilon\to 0}\|a_\varepsilon\|_{\frac{N}{2}} = S_N.$$

In particular, we see that $a_\varepsilon(x)=Q(x)|u_\varepsilon(x)|^{p-1}+\varepsilon$, $\varepsilon>0$, is a minimizing sequence for $\beta_{\frac{N}{2}}$.

Under the assumed conditions on the function Q, Theorem 1.1 in [6] asserts that (after passing to a subsequence)

$$|u_\varepsilon|^{p+1} \rightharpoonup Q_M^{-(N/2)}S_N^{N/2}\delta_{x_0} \quad \text{as} \quad \varepsilon\to 0$$

in the sense of measures, where we recall that x_0 is the unique maximum point of the function Q. This fact, together with (10), implies that (after passing to a subsequence)

$$|a_\varepsilon|^{N/2} \rightharpoonup S_N^{N/2}\delta_{x_0} \quad \text{as} \quad \varepsilon\to 0$$

in the sense of measures, and the theorem follows. □

Remark. In a forthcoming paper [3] we give a different, more constructive proof of the last theorem, revealing the nature of blow-up behavior of minimizing sequences for $\beta_{\frac{N}{2}}$. Employing the knowledge on the minimizing sequences for the best Sobolev constant S_N from [9,14], we also prove that any minimizing sequence for $\beta_{\frac{N}{2}}$ converges in the sense of measures to a multiple of the Dirac mass centered at some point $x_0\in\overline{\Omega}$. In addition, when $N=1$ we show that the blowing-up occurs only at one point of the domain, the center of the interval, pointing out a deep difference with respect to the multidimensional case.

In the two-dimensional case, $N=2$, the L_1-norm in the expression of $\beta_{\frac{N}{2}}$ is not natural from the viewpoint of the limiting cases of the Sobolev embedding theorem. We observe a rather degenerate behavior of minimizing sequences here, in the sense that the concentration may occur at any finite number of points of the domain. Consequently, we redefine the constant $\beta_{\frac{N}{2}}$ by changing the L_1-norm by a suitable Orlicz norm $\|\cdot\|_A$ stemming from the Moser-Trudinger inequality, the latter giving the critical growth in the two-dimensional situation. The norm $\|\cdot\|_A$ is defined by means of a Young function $A(t)=e^{4\pi t}-4\pi t-1$, $t\ge 0$. We show that this new quantity β_A is bounded away from zero, and the following estimate is valid

$$\beta_A \ge \frac{1}{2\nu^{\frac{1}{2}}} > 0,$$

where

$$\nu := \sup_{u\in H_0^1(\Omega),\,\|\nabla u\|_2=1}\int_\Omega\left(e^{4\pi u^2}-4\pi u^2-1\right).$$

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