Smooth projective toric varieties whose nontrivial nef line bundles are big

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Abstract: For any $n \geq 3$, we explicitly construct smooth projective toric n-folds of Picard number ≥ 5, where any nontrivial nef line bundles are big.

Key words: Toric variety; Mori theory; nef cone; pseudo-effective cone.

1. Introduction. The following question is our main motivation of this note.

Question 1.1. *Are there any smooth projective toric varieties $X \not\simeq \mathbf{P}^n$ such that*

$$\partial \mathrm{Nef}(X) \cap \partial \mathrm{PE}(X) = \{0\} \ ?$$

Here, $\mathrm{Nef}(X)$ *is the nef cone of X and* $\mathrm{PE}(X)$ *is the pseudo-effective cone of X.*

By definition, the nef cone $\mathrm{Nef}(X)$ is included in the pseudo-effective cone $\mathrm{PE}(X)$. We note that $\partial \mathrm{Nef}(X) \cap \partial \mathrm{PE}(X) = \{0\}$ is equivalent to the condition that any nontrivial nef line bundles on X are big.

In this note, we explicitly construct smooth *projective* toric threefolds of Picard number ≥ 5 on which any nontrivial nef line bundles are big. The main parts of this note are nontrivial examples given in Section 4. See Examples 4.2 and 4.3. In general, it seems to be hard to find those examples. Therefore, it must be valuable to describe them explicitly here. This short note is a continuation and a supplement of the papers: [F2] and [FP].

Let us see the contents of this note. Section 2 is a supplement to the toric Mori theory. We introduce the notion of 'general' complete toric varieties. By the definition of 'general' projective toric varieties, it is obvious that the final step of the MMP for a $\mathbf{Q}$-factorial 'general' projective toric variety is a $\mathbf{Q}$-factorial projective toric variety of Picard number one. It is almost obvious if we understand Reid's combinatorial description of toric extremal contraction morphisms. Moreover, it is easy to check that any nontrivial nef line bundles on a 'general' complete toric variety are always big. In Section 3, we recall the basic definitions and properties of *primitive collections* and *primitive relations* after Batyrev. By the result of Batyrev, any *smooth* projective toric variety is 'general' if and only if it is isomorphic to the projective space. So, the results obtained in Section 2 can not be used to construct examples in Section 4. The first author first considered that there are plenty of 'general' smooth projective toric varieties. So, he thought that the examples in Section 4 is worthless. Section 4 is the main part of this note. We give smooth projective toric threefolds of Picard number ≥ 5, where any nontrivial nef line bundles are always big. We note that this phenomenon does not occur for smooth projective toric surfaces. Let X be a smooth projective toric surface. Then we can easily see that there exists a morphism $f : X \to \mathbf{P}^1$ if X is not isomorphic to $\mathbf{P}^2$. So, the line bundle $f^*\mathcal{O}_{\mathbf{P}^1}(1)$ on X is nef but not big. Let X be a smooth projective toric variety and let Δ be the corresponding fan. If Δ is sufficiently complicated combinatorially in some sense, then any nontrivial nef line bundles are big. However, we do not know how to define 'complicated' fans suitably. Therefore, the explicit examples in Section 4 seem to be useful. We note that it is difficult to calculate nef cones or pseudo-effective cones for projective (not necessarily toric) varieties. In the final section: Section 5, we collect miscellaneous results. We explain how to generalize examples in [FP] and in Section 4 into dimension $n \geq 4$. We also treat $\mathbf{Q}$-factorial projective toric varieties with $\mathrm{Nef}(X) = \mathrm{PE}(X)$.

Let us fix the notation used in this note. For the details, see [R] or [FS]. For the basic results on the toric geometry, see the standard text books: [MO], [O], or [Fl].

Notation. We will work over some fixed field k throughout this note. Let X be a complete toric variety; a 1-cycle of X is a formal sum $\sum a_i C_i$ with

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complete curves C_i on X, and $a_i \in \mathbf{Z}$. We put

$$Z_1(X) := \{\text{1-cycles of } X\},$$

and

$$Z_1(X)_{\mathbf{R}} := Z_1(X) \otimes \mathbf{R}.$$

There is a pairing

$$\mathrm{Pic}(X) \times Z_1(X)_{\mathbf{R}} \to \mathbf{R}$$

defined by $(\mathcal{L}, C) \mapsto \deg_C \mathcal{L}$, extended by bilinearity. Define

$$N^1(X) := (\mathrm{Pic}(X) \otimes \mathbf{R})/\equiv$$

and

$$N_1(X) := Z_1(X)_{\mathbf{R}}/\equiv,$$

where the *numerical equivalence* $\equiv$ is by definition the smallest equivalence relation which makes N^1 and N_1 into dual spaces.

Inside $N_1(X)$ there is a distinguished cone of effective 1-cycles,

$$\mathrm{NE}(X) = \{Z | Z \equiv \sum a_i C_i \text{ with } a_i \in \mathbf{R}_{\geq 0}\} \subset N_1(X).$$

It is known that $\mathrm{NE}(X)$ is a rational polyhedral cone. A subcone $F \subset \mathrm{NE}(X)$ is said to be *extremal* if $u, v \in \mathrm{NE}(X)$, $u + v \in F$ imply $u, v \in F$. The cone F is also called an *extremal face* of $\mathrm{NE}(X)$. A one-dimensional extremal face is called an *extremal ray*.

We define the *Picard number* $\rho(X)$ by

$$\rho(X) := \dim_{\mathbf{R}} N^1(X) < \infty.$$

An element $D \in N^1(X)$ is called *nef* if $D \geq 0$ on $\mathrm{NE}(X)$.

We define the *nef cone* $\mathrm{Nef}(X)$, the *ample cone* $\mathrm{Amp}(X)$, and the *pseudo-effective cone* $\mathrm{PE}(X)$ in $N^1(X)$ as follows:

$$\mathrm{Nef}(X) = \{D \,|\, D \text{ is nef}\},$$

$$\mathrm{Amp}(X) = \{D \,|\, D \text{ is ample}\}$$

and

$$\mathrm{PE}(X) = \{D \equiv \sum a_i D_i \,|\, D_i \text{ is an effective Weil divisor and } a_i \in \mathbf{R}_{\geq 0}\}.$$

It is not difficult to see that $\mathrm{PE}(X)$ is a rational polyhedral cone in $N^1(X)$ since X is toric. For the usual definition of $\mathrm{PE}(X)$, see, for example, [L, Definition 2.2.25]. It is easy to see that $\mathrm{Amp}(X) \subset \mathrm{Nef}(X) \subset \mathrm{PE}(X)$.

From now on, we assume that X is projective. Let D be an $\mathbf{R}$-Cartier divisor on X. Then D is called *big* if $D \equiv A + E$ for an ample $\mathbf{R}$-divisor A and an effective $\mathbf{R}$-divisor E. For the original definition of a big divisor, see, for example, [L, 2.2 Big Line Bundles and Divisors]. We define the *big cone* $\mathrm{Big}(X)$ in $N^1(X)$ as follows:

$$\mathrm{Big}(X) = \{D \,|\, D \text{ is big}\}.$$

It is well known that the big cone is the interior of the pseudo-effective cone and the pseudo-effective cone is the closure of the big cone. See, for example, [L, Theorem 2.2.26].

In [F2] and [FP], we mainly treated *non-projective* toric varieties. In this note, we are interested in *projective* toric varieties.

2. Supplements to the toric Mori theory. We introduce the following new notion. It will not be useful when we construct various examples of *smooth* projective toric varieties in Section 4. However, we include it here for the future usage. By the simple observations in this section, we know that the great mass of complete toric varieties have no nontrivial non-big nef line bundles.

Definition 2.1. Let X be a complete toric variety with $\dim X = n$. Let Δ be the fan corresponding to X. Let $G(\Delta) = \{v_1, \cdots, v_m\}$ be the set of all primitive vectors spanning one dimensional cones in Δ. If there exists a relation

$$a_{i_1} v_{i_1} + \cdots + a_{i_k} v_{i_k} = 0$$

such that $\{i_1, \cdots, i_k\} \subset \{1, \cdots, m\}$, $a_{i_j} \in \mathbf{Z}_{>0}$ for any $1 \leq j \leq k$ with $k \leq n$, then X is called *'special'*. If X is not 'special', then X is called *'general'*.

Example 2.2. The projective space $\mathbf{P}^n$ is 'general' in the sense of Definition 2.1.

Let us introduce the following easy but useful lemmas for the toric Mori theory. The proofs are obvious. So, we omit them.

Lemma 2.3. *Let X be a complete toric variety and let $\pi : \widetilde{X} \to X$ be a small projective toric $\mathbf{Q}$-factorialization* (*cf.* [F1, Corollary 5.9]). *Assume that X is 'general'* (*resp. 'special'*). *Then $\widetilde{X}$ is also 'general'* (*resp. 'special'*).

More generally, we have the following lemma.

Lemma 2.4. *Let X and X' be complete toric varieties and let $\varphi : X \dashrightarrow X'$ be a proper birational toric map. Assume that φ is an isomorphism in codimension one. Then X is 'general' if and only if so is X'.*

Lemma 2.5. *Let X and Z be a complete toric varieties and let $\pi : X \to Z$ be a birational toric morphism. Assume that X is 'general'. Then Z is 'general'. We note that Z is not necessarily 'special' even if X is 'special'.*

We have two elementary properties.

Proposition 2.6. *Let X be a complete toric variety and let $f : X \to Y$ be a proper surjective toric morphism onto Y. Assume that X is 'general' and that* $\dim Y < \dim X$. *Then Y is a point.*

Proof. It is obvious. □

Corollary 2.7. *Let X be a complete toric variety. Assume that X is 'general'. Let D be a nef Cartier divisor on X such that $D \not\sim 0$. Then D is big.*

Proof. Since D is nef, the linear system $|D|$ defines a proper surjective toric morphism $\Phi_{|D|} : X \to Z$. Apply Proposition 2.6 to $\Phi_{|D|} : X \to Z$. Then we obtain $\dim Z = \dim X$. Therefore, D is big. □

The next proposition is also obvious. We include it for the reader's convenience because it has not been stated explicitly in the literature. For the details of the toric Mori theory, see [F1, Section 5] and [FS].

Proposition 2.8 (MMP for 'general' projective toric varieties)**.** *Let X be a $\mathbf{Q}$-factorial projective toric variety and let B be a Cartier divisor on X such that B is not pseudo-effecitve. Assume that X is 'general'. We run the MMP with respect to B. Then we obtain a sequence of B-negative divisorial contractions and B-flips*:

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_i$$
$$\dashrightarrow X_{i+1} \dashrightarrow \cdots \dashrightarrow X_l,$$

where X_l is a $\mathbf{Q}$-factorial projective toric variety with $\rho(X_l) = 1$.

Proof. Run the MMP with respect to B, where B is not pseudo-effective, for example, $B = K_X$. Since B is not pseudo-effective, the final step is a Fano contraction $X_l \to Z$. Since X is 'general', X_l is also 'general' by Lemmas 2.4 and 2.5. Therefore, Z must be a point by Corollary 2.6. This means that X_l is a $\mathbf{Q}$-factorial projective toric variety with $\rho(X_l) = 1$. □

We will see that any smooth projective toric variety X, which is not isomorphic to the projective space, is 'special' by [B]. See Proposition 3.5 below. So, the results in this section can not be applied to *smooth* projective toric varieties.

2. Primitive collections and relations. Let us recall the notion of primitive collections and primitive relations introduced by Batyrev (cf. [B]). It is very useful to compute some explicit examples of toric varieties. Note that this section is not indispensable for understanding the examples in Section 4.

Let Δ be a complete non-singular n-dimensional fan and let $G(\Delta)$ be the set of all primitive generators of Δ.

Definition 3.1 (Primitive collection)**.** A non-empty subset $\mathcal{P} = \{v_1, \cdots, v_k\} \subset G(\Delta)$ is called a *primitive collection* if, for each element $v_i \in \mathcal{P}$, the set $\mathcal{P} \setminus \{v_i\}$ generates a $(k-1)$-dimensional cone in Δ, while $\mathcal{P}$ does not generate any k-dimensional cone in Δ.

Definition 3.2 (Focus)**.** Let $\mathcal{P} = \{v_1, \cdots, v_k\}$ be a primitive collection in $G(\Delta)$. Let $S(\mathcal{P})$ denote $v_1 + \cdots + v_k$. The *focus* $\sigma(\mathcal{P})$ of $\mathcal{P}$ is the cone in Δ of the smallest dimension containing $S(\mathcal{P})$.

Definition 3.3 (Primitive relation)**.** Let $\mathcal{P} = \{v_1, \cdots, v_k\}$ be a primitive collection in $G(\Delta)$ and $\sigma(\mathcal{P})$ its focus. Let $w_1, \cdots, w_m$ be the primitive generators of $\sigma(\mathcal{P})$. Then there exists a unique linear combination $a_1w_1 + \cdots + a_mw_m$ with positive integer coefficients a_i which is equal to $v_1 + \cdots + v_k$. Then the linear relation $v_1 + \cdots + v_k - a_1w_1 - \cdots - a_mw_m = 0$ is called the *primitive relation associated with* $\mathcal{P}$.

Then we have the description of $\mathrm{NE}(X)$ by primitive relations.

Theorem 3.4 (cf. [B, 2.15 Theorem])**.** *Let Δ be a projective non-singular fan and $X = X(\Delta)$ the corresponding toric variety. Then the Kleiman-Mori cone* $\mathrm{NE}(X)$ *is generated by all primitive relations. The primitive relation which spans an extremal ray of* $\mathrm{NE}(X)$ *is said to be extremal.*

Let Δ be a *projective* non-singular n-dimensional fan. Then, Batyrev obtained the following important result.

Proposition 3.5 (cf. [B, 3.2 Proposition])**.** *There exists a primitive collection $\mathcal{P} = \{v_1, \cdots, v_k\}$ in $G(\Delta)$ such that the associated primitive relation is of the form*

$$v_1 + \cdots + v_k = 0.$$

In other words, the focus $\sigma(\mathcal{P}) = \{0\}$.

We close this section with an elementary remark.

Remark 3.6. If $k = n+1$ in Proposition 3.5, then $X(\Delta) \simeq \mathbf{P}^n$.

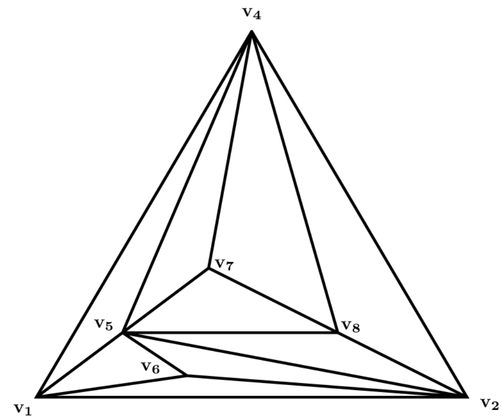
Therefore, a smooth projective toric variety X is 'general' if and only if X is isomorphic to the projective space. By this reason, it is not so easy to construct smooth projective toric varieties on which any nontrivial nef line bundles are big.

4. Examples. First, let us recall the following example, which is not a toric variety. For the details, see [MM] and [M, p. 67].

Example 4.1 (MM, no. 30 Table 2)**.** Let X be the blowing-up of $\mathbf{P}^3_{\mathbf{C}}$ along a smooth conic. Then X is a smooth Fano threefold with $\rho(X)=2$. It is known that X has two extremal divisorial contractions. One contraction is the inverse of the blowing-up $X\to\mathbf{P}^3$. Another one is a contraction of $\mathbf{P}^2$ on X into a smooth point. Therefore, it is not difficult to see that every nef Cartier divisor $D\not\equiv 0$ is big.

The next example is the main theme of this short note. It is hard for the non-experts to find it. Therefore, we think it is worthwhile to describe it explicitly here.

Example 4.2. We put $v_1=(1,0,0), v_2=(0,1,0), v_3=(0,0,1)$, and $v_4=(-1,-1,-1)$. We consider the standard fan of $\mathbf{P}^3$ generated by v_1,v_2,v_3, and v_4. We subdivide the cone $\langle v_1,v_2,v_4\rangle$ as follows: Take a blow-up $X_1\to\mathbf{P}^3$ along the vector $v_5=(1,-1,-2)=3v_1+v_2+2v_4$. We take a blow-up $X_2\to X_1$ along the vector $v_6=(1,0,-1)=\frac{1}{2}(v_1+v_2+v_5)$ and a blow-up $X_3\to X_2$ along $v_7=(0,-1,-2)=\frac{1}{3}(v_2+2v_4+2v_5)$. Finally, we take a blow-up X_3 along the vector $v_8=(0,0,-1)=\frac{1}{2}(v_2+v_7)$ and obtain X. Then, it is obvious that X is projective and $\rho(X)=5$. It is easy to see that X is smooth. In this case, $\mathrm{NE}(X)$ is spanned by the following five extremal primitive relations, $v_1+v_2+v_5-2v_6=0$, $v_4+v_5+v_8-2v_7=0$, $v_2+v_7-2v_8=0$, $v_6+v_8-v_2-v_5=0$, and $v_3+v_5-2v_1-v_4=0$. This toric variety X is nothing but the one labeled as [8-10] in [MO, Theorem 9.6]. The picture below helps us understand the combinatorial data of X.



Claim. *There are no projective surjective toric morphism* $f:X\to Y$ *with* $\dim Y=1$ *or* 2.

Proof. The variety X is obtained by successive blowing-ups of $\mathbf{P}^3$ inside the cone $\langle v_1,v_2,v_4\rangle$. So, X does not admit to a morphism to a curve. Thus, we have to consider the case when Y is a surface. By considering primitive relations, $f:X\to Y$ must be induced by the projection $\mathbf{Z}^3\to\mathbf{Z}^2:(x,y,z)\mapsto(x,y)$ because $v_3+v_8=0$. The image of the cone $\langle v_2,v_5,v_8\rangle$ is the cone spanned by $(0,1)$ and $(1,-1)$. On the other hand, the image of the cone $\langle v_1,v_4,v_5\rangle$ is the cone spanned by $(1,0)$ and $(-1,-1)$. Therefore, there are no surjective morphisms $f:X\to Y$ with $\dim Y=2$. □

Thus, every nef divisor $D\not\sim 0$ is big, that is, $\partial\mathrm{Nef}(X)\cap\partial\mathrm{PE}(X)=\{0\}$.

By the following example, the reader understands the advantage of using the toric geometry to construct examples. We do not know what happens if we take blow-ups of X in Example 4.1.

Example 4.3. By taking blowing-ups inside the cone $\langle v_5,v_7,v_8\rangle$ in Example 4.2, we obtain a smooth projective toric threefold X_k for any $k\geq 6$ such that $\rho(X_k)=k$ and $\partial\mathrm{Nef}(X_k)\cap\partial\mathrm{PE}(X_k)=\{0\}$, that is, every nef divisor $D\not\sim 0$ on X_k is big. More explicitly, for example, X_6 is the blow-up of X along $u_6=v_5+v_7+v_8$ and X_{k+1} is the blow-up of X_k along $u_{k+1}=v_5+v_7+u_k$ for $k\geq 6$.

We can easily check that any smooth projective toric threefolds of Picard number $2\leq\rho\leq 4$ have some nontrivial non-big nef line bundles by the classification table in [MO, Theorem 9.6]. For smooth non-projective toric variety, the following example will help the reader. It is the most famous example of smooth complete non-projective toric threefold.

Example 4.4. Let Δ be the fan whose rays are spanned by $v_1=(1,0,0)$, $v_2=(0,1,0)$, $v_3=(0,0,1)$, $v_4=(-1,-1,-1)$, $v_5=(0,-1,-1)$, $v_6=(-1,0,-1)$, $v_7=(-1,-1,0)$, and whose maximal cones are $\langle v_1,v_2,v_3\rangle$, $\langle v_4,v_5,v_6\rangle$, $\langle v_4,v_6,v_7\rangle$, $\langle v_4,v_5,v_7\rangle$, $\langle v_1,v_2,v_5\rangle$, $\langle v_2,v_5,v_6\rangle$, $\langle v_2,v_3,v_6\rangle$, $\langle v_3,v_6,v_7\rangle$, $\langle v_1,v_3,v_7\rangle$, $\langle v_1,v_5,v_7\rangle$. Then $X=X(\Delta)$ is the most famous non-projective smooth toric threefold with $\rho(X)=4$ obtained by Miyake and Oda. By removing three two-dimensional walls $\langle v_1,v_7\rangle$, $\langle v_2,v_5\rangle$, and $\langle v_3,v_6\rangle$ from Δ, we obtain a flopping contraction $f:X\to Y$. It is easy to see that Y is a projective toric threefold with $\rho(Y)=2$ and three ordinary double points. We can check that every nef divisor D can be written as $D=f^*D'$ for some nef divisor D' on Y. On the other hand, $\mathrm{Nef}(Y)$ is a two dimensional cone and every nef divisor on Y is big.

Therefore, $\mathrm{Nef}(X)$ is also two-dimensional and all the nef divisors on X are big. We note that $\mathrm{Nef}(X)$ is thin in $N^1(X)$ by Kleiman's ampleness criterion since X is a smooth complete non-projective variety.

The reader can find many smooth complete non-projective toric threefolds X with $\mathrm{Nef}(X) = \{0\}$ in [FP].

5. Miscellaneous comments. In this final section, we collect miscellaneous results. First, we explain how to generalize Examples 4.2 and 4.3 in dimension ≥ 4.

5.1. We put $v_1 = (1, 0, \cdots, 0)$, $v_2 = (0, 1, 0, \cdots, 0)$, $v_3 = (0, 0, 1, 0, \cdots, 0)$, $v_4 = (-1, -1, \cdots, -1) \in N = \mathbf{Z}^n$. We consider $w_1 = (0, 0, 0, 1, 0, \cdots, 0)$, $w_2 = (0, 0, 0, 0, 1, 0, \cdots, 0), \cdots$, $w_{n-3} = (0, \cdots, 0, 1) \in N$. By these vectors, we can construct a fan corresponding to $\mathbf{P}^n$ as usual. We take $v_5 = 3v_1 + v_2 + 2v_4 = (1, -1, -2, \cdots, -2)$, $v_6 = \frac{1}{2}(v_1 + v_2 + v_5) = (1, 0, -1, \cdots, -1)$, $v_7 = \frac{1}{3}(v_2 + 2v_4 + 2v_5) = (0, -1, -2, \cdots, -2)$, and $v_8 = \frac{1}{2}(v_2 + v_7) = (0, 0, -1, \cdots, -1)$. We take a sequence of blow-ups

$$X \to X_3 \to X_2 \to X_1 \to \mathbf{P}^n$$

as in Examples 4.2. In this case, the center of each blow-up is $(n-3)$-dimensional. We can easily check that X is a smooth projective toric n-fold. We note that $v_3 + w_1 + \cdots + w_{n-3} + v_8 = 0$.

Claim. *If $f : X \to Y$ is a proper surjective toric morphism and Y is not a point, then* $\dim Y = n$.

Proof of Claim. By considering linear relations among $v_1, v_2, \cdots, v_8, w_1, \cdots, w_{n-3}$ as in Definition 2.1, f should be induced by the projection $\mathbf{Z}^n \to \mathbf{Z}^2 : (x_1, x_2, \cdots, x_n) \mapsto (x_1, x_2)$ if $\dim Y < n$. By the same arguments as in the proof of Claim in Example 4.2, it can not happen. Therefore, we obtain $\dim Y = n$. □

Thus, any nontrivial nef line bundles on X are big.

So, for any (n, ρ), where $n \geq 4$ and $\rho \geq 5$, we can construct a smooth projective toric n-fold X with $\rho(X) = \rho$ on which any nontrivial nef line bundles are big (cf. Example 4.3). We leave the details for the reader's exercise. The next one is a higher dimensional analogue of [FP].

5.2 (Smooth complete toric varieties without nontrivial nef line bundles)**.** Let X be a smooth complete toric variety without nontrivial nef line bundles. We put $\mathcal{E} = \mathcal{O}_X^{\oplus k} \oplus \mathcal{L}$ for $k \geq 1$, where $\mathcal{L}$ is a nontrivial line bundle on X. We consider the $\mathbf{P}^k$-bundle $\pi : Y = \mathbf{P}_X(\mathcal{E}) \to X$. Then Y is a $(\dim X + k)$-dimensional complete toric variety. It is easy to see that Y has no nontrivial nef line bundle. So, for $n \geq 4$, we can construct many n-dimensional smooth complete toric varieties of Picard number ≥ 6 without nontrivial nef line bundles by [FP].

Finally, we close this note with an easy result. We treat the other extreme case: $\mathrm{Nef}(X) = \mathrm{PE}(X)$.

Proposition 5.3. *Let X be a $\mathbf{Q}$-factorial projective toric variety with $\rho(X) = \rho$. Assume that $\mathrm{Nef}(X) = \mathrm{PE}(X)$, that is, every effective divisor is nef. Then there is a finite toric morphism $\mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_\rho} \to X$ with $n_1 + \cdots + n_\rho = \dim X$. When X is smooth, $X \simeq \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_\rho}$ with $n_1 + \cdots + n_\rho = \dim X$.*

Proof. The condition $\mathrm{Nef}(X) = \mathrm{PE}(X)$ implies that every extremal ray of $\mathrm{NE}(X)$ is a Fano type.

First, we assume that X is smooth. We obtain a Fano contraction $f : X \to Y$ with $\rho(Y) = \rho(X) - 1$, where Y is a smooth projective toric variety and $\mathrm{Nef}(Y) = \mathrm{PE}(Y)$. It is well known that X is a projective space bundle over Y. By the induction, we obtain $Y \simeq \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_{\rho-1}}$. Therefore, we can easily check that $X \simeq \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_\rho}$ and f is the projection. Lemma 5.4 may help the reader check it.

Next, we just assume that X is a $\mathbf{Q}$-factorial projective toric variety with $\mathrm{Nef}(X) = \mathrm{PE}(X)$. As above, we have a Fano contraction $f : X \to Y$ with $\rho(Y) = \rho(X) - 1$. In this case, Y is a $\mathbf{Q}$-factorial projective toric variety with $\mathrm{Nef}(Y) = \mathrm{PE}(Y)$. By applying the induction, we have a finite toric surjective morphism $g : W' = \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_{\rho-1}} \to Y$. If we need, we take a higher model $W = \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_{\rho-1}} \to W' \to Y$ and can assume that $V \to W$ is a fiber bundle, where V is the normalization of $W \times_Y X$. We note that $\mathrm{Nef}(V) = \mathrm{PE}(V)$. Let F be a general fiber of $V \to W$, Δ_F the associated fan in the sub-lattice $N_F \subset N$ and $G(\Delta_F) = \{v_1, \ldots, v_{\rho+1}\}$. We remark that $\{v_1, \ldots, v_\rho\}$ is a basis for $N_F \otimes \mathbf{R}$. Fix a basis B for $N_{\mathbf{R}}$ such that $\{v_1, \ldots, v_\rho\} \subset B \subset G(\Delta_V)$. For any $1 \leq i \leq \rho$, put U_i be the irreducible torus invariant closed subvariety on V associated to the cone $\sum_{v \in G(\Delta_F) \setminus \{v_i, v_{\rho+1}\}} \mathbf{R}_{\geq 0} v$. Then, $\dim U_i = \dim W + 1$ and $U_i \to W$ is surjective. We can see that U_i is a $\mathbf{P}^1$-bundle over W and $\mathrm{Nef}(U_i) = \mathrm{PE}(U_i)$. Therefore, $U_i \simeq W \times \mathbf{P}^1$ and $U_i \to W$ is the first projection by the previous step. This means for any $v \in G(\Delta_V) \setminus G(\Delta_F)$, v is a linear combination of elements in $B \setminus \{v_1, \ldots, v_\rho\}$. By these observations, we can see that $V \simeq W \times F$, where F is a $\mathbf{Q}$-factorial projective toric variety with $\rho(F) = 1$.

Thus, we obtain a desired finite toric morphism $\mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_\rho} \to X$. □

The following property is a key lemma.

Lemma 5.4. *Let X be a $\mathbf{Q}$-factorial projective toric variety with* $\mathrm{Nef}(X) = \mathrm{PE}(X)$. *Let Z be any irreducible torus invariant closed subvariety of X. Then Z is a $\mathbf{Q}$-factorial projective toric variety with* $\mathrm{Nef}(Z) = \mathrm{PE}(Z)$.

Proof. It is obvious. □

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References

[B] V. V. Batyrev, On the classification of smooth projective toric varieties, Tohoku Math. J. (2) **43** (1991), no. 4, 569–585.

[F1] O. Fujino, Notes on toric varieties from Mori theoretic viewpoint, Tohoku Math. J. (2) **55** (2003), no. 4, 551–564.

[F2] O. Fujino, On the Kleiman-Mori cone, Proc. Japan Acad. Ser. A Math. Sci. **81** (2005), no. 5, 80–84.

[FP] O. Fujino and S. Payne, Smooth complete toric threefolds with no nontrivial nef line bundles, Proc. Japan Acad. Ser. A Math. Sci. **81** (2005), no. 10, 174–179.

[FS] O. Fujino and H. Sato, Introduction to the toric Mori theory, Michigan Math. J. **52** (2004), no. 3, 649–665.

[Fl] W. Fulton, *Introduction to toric varieties*, Ann. of Math. Stud., 131, Princeton Univ. Press, Princeton, NJ, 1993.

[L] R. Lazarsfeld, *Positivity in algebraic geometry.* I. *Classical setting: line bundles and linear series*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 48, Springer-Verlag, Berlin, 2004.

[M] K. Matsuki, Weyl groups and birational transformations among minimal models, Mem. Amer. Math. Soc. **116** (1995), no. 557, vi+133 pp.

[MM] S. Mori and S. Mukai, Classification of Fano 3-folds with $B_2 \geq 2$, Manuscripta Math. **36** (1981/82), no. 2, 147–162.

[MO] T. Oda, *Torus embeddings and applications*, Based on joint work with Katsuya Miyake. Tata Institute of Fundamental Research Lectures on Mathematics and Physics, 57, Tata Institute of Fundamental Research, Bombay, Springer-Verlag, Berlin-New York, 1978.

[O] T. Oda, *Convex bodies and algebraic geometry*, An introduction to the theory of toric varieties, Translated from the Japanese, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)] 15, Springer-Verlag, Berlin, 1988.

[R] M. Reid, Decomposition of toric morphisms, in *Arithmetic and geometry, Vol.* II, Progr. Math., 36, Birkhäuser Boston, Boston, MA., pp. 395–418, 1983.