# Growth functions for Artin monoids

By Kyoji SAITO

Institute for the Physics and Mathematics of the Universe, The University of Tokyo
5-1-5 Kashiwanoha, Kashiwa, 277-8568, Japan

(Communicated by Masaki KASHIWARA, M.J.A., June 12, 2009)

**Abstract:** In [S1], we showed that the growth function $P_M(t)$ for an Artin monoid associated with a Coxeter matrix $M$ of finite type is a rational function of the form $1/(1-t)N_M(t)$, where $N_M(t)$ is a polynomial determined by the Coxeter-Dynkin graph for $M$, and is called the denominator polynomial of type $M$. We formulated three conjectures on the zeros of the denominator polynomial. In the present note, we prove that the same denominator formula holds for an arbitrary Artin monoid, and formulate slightly modified conjectures on the zeros of the denominator polynomials of affine types. The new conjectures are verified for types $\tilde{A}_2, \cdots, \tilde{A}_8, \tilde{C}_2, \cdots, \tilde{C}_8, \tilde{D}_4, \tilde{E}_7, \tilde{E}_8, \tilde{F}_4, \tilde{G}_2$ among others. In Appendix, we define the elliptic denominator polynomials by formally applying the denominator polynomial formula to the elliptic diagrams for elliptic root systems [S2]. Then, the new conjectures are verified also for elliptic denominator polynomials of types $A_2^{(1,1)}, \cdots, A_7^{(1,1)}, D_4^{(1,1)}, E_6^{(1,1)}, E_7^{(1,1)}, E_8^{(1,1)}$ and $G_2^{(1,1)}$.

**Key words:** Artin monoid; growth function; denominator polynominal; irreducible polynomial.

**1. Growth function for an Artin monoid.** In the present section we recall the definition (1.3) of the *spherical growth function* for an Artin monoid, and show that it is the quotient of 1 divided by the *denominator polynomial* given by a formula (1.6).

Let $M = (m_{ij})_{i,j \in I}$ be a Coxeter matrix [B]. The Artin monoid $G_M^+$ [B-S, §1.2] associated with $M$ (or, of type $M$) is a monoid generated by the letters $a_i$, $i \in I$ which are subordinate to the relation generated by

$$(1.1) \qquad a_i a_j a_i \cdots = a_j a_i a_j \cdots \quad i, j \in I,$$

where both hand sides of (1.1) are words of alternating sequences of letters $a_i$ and $a_j$ of the same length $m_{ij} = m_{ji}$ with the initials $a_i$ and $a_j$, respectively. More precisely, $G_M^+$ is the quotient of the free monoid generated by the letters $a_i$ ($i \in I$) by the equivalence relation: two words $U$ and $V$ in the letters are equivalent, if there exists a sequence $U_0 := U, U_1, \cdots, U_m := V$ such that the word $U_k$ ($k = 1, \cdots, m$) is obtained by replacing a phrase in $U_{k-1}$ of the form on left hand side of (1.1) by right hand side of (1.1) for some $i, j \in I$. We write by $U \underset{\bullet}{=} V$ if $U$ and $V$ are equivalent. The equivalence class (i.e. an element of $G_M^+$) of a word $W$ is denoted by the same notation $W$. By the definition, equivalent words have the same length. Hence, we define the degree homomorphism:

$$(1.2) \qquad \deg : \ G_M^+ \to \mathbf{Z}_{\geq 0}$$

by assigning to each equivalence class of words the length of the words.

The growth function $P_{G_M^+, I}(t)$ for the Artin monoid $G_M^+$ is defined by

$$(1.3) \quad P_{G_M^+, I}(t) := \sum_{n \in \mathbf{Z}_{\geq 0}} \#\{W \in G_M^+ \mid \deg(W) \leq n\} t^n.$$

The *spherical growth function of the monoid* $G_M^+$ of type $M$ is defined by

$$(1.4) \qquad \dot{P}_{G_M^+, I}(t) \ := \sum_{n \in \mathbf{Z}_{\geq 0}} \#(\deg^{-1}(n)) t^n,$$

so that one has the obvious relation: $P_{G_M^+, I}(t) = \dot{P}_{G_M^+, I}(t)/(1-t)$.

**Theorem.** *Let $G_M^+$ be the Artin monoid of any type $M$. Then the spherical growth function of the monoid is given by the Taylor expansion of the rational function of the form*

$$(1.5) \qquad \dot{P}_{G_M^+, I}(t) = \frac{1}{N_M(t)}.$$

2000 Mathematics Subject Classification. Primary 16G10; Secondary 16G20, 16G21.

doi: 10.3792/pjaa.85.84

*Here, $N_M(t)$ is called the* denominator polynomial *and is given by*

$$(1.6) \qquad N_M(t) := \sum_{J\subset I} (-1)^{\#(J)} t^{\deg(\Delta_J)},$$

*where the summation index $J$ runs over subsets of $I$ such that the restricted Coxeter matrix $M|_J$ is of finite type,*[*)] *and $\Delta_J$ is the fundamental element in $G_M^+$ associated with $J$* ([B-S, §5 Definition]. *See also* Lemma-Definition 2 *and* Remark 1.2 *of the present note*).

*Proof.* The proof is achieved by the recursion formula (1.12) on the coefficients of the growth function. For the proof of the formula, we use the method used to solve the word problem for the Artin monoid [B-S, §6.1], which we recall below. We first recall the fact that an Artin monoid satisfies the cancellation condition in the following sense [B-S, Prop. 2.3].

**Lemma 1.1.** *Let $A, B, X, Y \in G_M^+$. If $AXB \doteq AYB$. Then $X \doteq Y$.*

A word $U$ is said to be divisible (from the left) by a word $V$, and denoted by $V|U$, if there exists a word $W$ such that $U \doteq VW$. Since $U \doteq V'$, $U \doteq U'$ and $V|U$ implies $V'|U'$, we use the notation "$|$" of divisibility also between elements of the monoid $G_M^+$. We have the following basic concepts [B-S, §5 Definition and §6.1].

**Lemma-Definition. (a)** *Let $M = (m_{ij})_{i,j\in I}$ be any Coxeter matrix, and let $J \subset I$ be a subset of $I$ such that $M|_J$ is of finite type (which may not necessarily be indecomposable). Then, there exists a unique element $\Delta_J \in G_M^+$, called the* fundamental element, *such that i) $a_i|\Delta_J$ for all $i \in J$, and ii) if $W \in G_M^+$ and $a_i|W$ for all $i \in J$, then $\Delta_J|W$.*

**(b)** *To an element $W \in G_M^+$, we associate the subset of $I$:*

$$(1.7) \qquad I(W) := \{i \in I \mid a_i|W\}.$$

*The restricted Coxeter matrix $M|_{I(W)}$ is of finite type for any $W \in G_M^+$.*

*Proof.* **(a)** and **(b)** These follow from the fact that the existence of $\Delta_J$ is achieved under a weaker assumption than $M_J$ is of finite type, rather that there exists a common multiple of $a_j$ for $j \in J$ in $G_M^+$ (see [B-S, Prop. (4.1)]). □

[*)] For a Coxeter matrix $M = (m_{ij})_{i,j\in I}$ and a subset $J$ of $I$, we define the restricted Coxeter matrix by $M|_J := (m_{ij})_{i,j\in J}$, which, obviously, is again a Coxeter matrix.

By the definition (1.7), one has $\Delta_{I(W)}|W$, and $\Delta_J|W$ implies $J \subset I(W)$.

We return to the Proof of Theorem.

For $n \in \mathbf{Z}_{\geq 0}$ and for any subset $J \subset I$, put

$$(1.8) \qquad G_n^+ := \{W \in G_M^+ \mid \deg(W) = n\}$$

$$(1.9) \qquad G_{n,J}^+ := \{W \in G_n^+ \mid I(W) = J\}.$$

We note that $G_{n,J}^+ = \emptyset$ if $M|_J$ is not of finite type. By the definition, we have the disjoint decomposition:

$$(1.10) \qquad G_n^+ = \amalg_{J\subset I} G_{n,J}^+,$$

where $J$ runs over all subsets of $I$. Note that $G_{n,\emptyset}^+ = \emptyset$ if $n > 0$ but $G_{0,\emptyset}^+ = \{\emptyset\} \neq \emptyset$. For any subset $J$ of $I$, the union $\amalg_{J\subset K\subset I} G_{n,K}^+$, where the index $K$ runs over all subsets of $I$ containing $J$, is equal to the subset of $G_n^+$ consisting of elements divisible by $a_j$ for $j \in J$. That is, one has

$$\amalg_{J\subset K\subset I} G_{n,K}^+ = \begin{cases} \Delta_J \cdot G_{n-\deg(\Delta_J)}^+ & \text{if } M|_J \text{ is of finite type,} \\ \emptyset & \text{if } M|_J \text{ is not of finite type.} \end{cases}$$

Thus, if $M|_J$ is of finite type, due to the cancellation condition Lemma 1.1, the multiplication map of $\Delta_J$ is injective and we obtain a bijection: $G_{n-\deg(\Delta_J)}^+ \simeq \amalg_{J\subset K\subset I} G_{n,K}^+$. This implies a numerical relation:

$$(1.11) \qquad \#(G_{n-\deg(\Delta_J)}^+) = \textstyle\sum_{J\subset K\subset I} \#(G_{n,K}^+).$$

If $M|_J$ is not of finite type, still the formula (1.11) holds formally, by putting $\deg(\Delta_J) := \infty$ and $G_{-\infty}^+ := \emptyset$, i.e. $\#(G_{n-\deg(\Delta_J)}^+) := 0$. Then, for $n > 0$, using (1.11), we get the recursion relation:

$$(1.12) \qquad \sum_{J\subset I} (-1)^{\#(J)} \#(G_{n-\deg(\Delta_J)}^+) = 0,$$

where the index $J$ may run either over all subsets of $I$, or, over only subset $J$ such that the restricted Coxeter matrix $M|_J$ is of finite type. Together with $\#(G_0^+) = 1$ for $n = 0$, the recursion formula is equivalent to the formula:

$$(1.13) \qquad \dot{P}_{G_M^+,I}(t) N_M(t) = 1.$$

This completes the Proof of Theorem. □

**Remark 1.2.** Recently, Albenque and Nadeau [A-N, (1.2)] have shown a generalization of present Theorem that the growth function of a cancellative monoid is a quotient of 1 divided by a polynomial if a subset of atomic generators has common multiple then it admits a least common multiple. Actually, an Artin monoid has the required properties [B-S, 2.3, §5].

**Remark 1.3.** We have the equality [B-S, §5.7]: $\deg(\Delta_J) = \#\{\text{reflections in } \overline{G}_{M|_J}\}$ = the length of the longest element of $\overline{G}_{M|_J}$, where $\overline{G}_M$ is the Coxeter group associated with the Coxeter matrix $M$.

By the definition (1.6) of the denominator polynomial, one has

$$N_M(1) = \sum_{\substack{J\subset I,\, M|_J \text{ is} \\ \text{of finite type}}} (-1)^{\#J}.$$

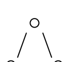
This, in particular, implies

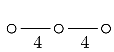
i) $N_M(t)$ has the factor $1-t$ if the graph of $M$ contains a component of finite type, and

ii) $N_M(1) = (-1)^l$ if $M$ is of indecomposable affine type of rank $l$ (i.e. $M$ is indecomposable and affine such that $\#(I) = l+1$).**)

We refer to [S1] for examples of the denominator polynomials of finite type. Here, we give a few examples of affine type.

**Example.** There are three types of indecomposable affine Coxeter matrices of rank 2. In the following, for each type, we associate the Coxeter diagram $\Gamma_M$ and the denominator polynomial $N_M(t)$.

1. $\tilde{A}_2$ $\Gamma_{\tilde{A}_2} =$  $N_{\tilde{A}_2}(t) = 1 - 3t + 3t^3$,

2. $\tilde{C}_2$ $\Gamma_{\tilde{C}_2} =$  $N_{\tilde{C}_2}(t) = 1 - 3t + t^2 + 2t^4$,

3. $\tilde{G}_2$ $\Gamma_{\tilde{G}_2} =$  $N_{\tilde{G}_2}(t) = 1 - 3t + t^2 + t^3 + t^6$.

**2. A bound on the zeros of the denominator polynomial $N_M(t)$ of affine type.** Motivated by a study of the author on certain limit partition functions associated with finitely generated monoids or groups (see [S4, §11 and 12]), we are interested in the distribution of the zero-loci of the denominator polynomials. The following lemma gives a numerical bound on the zeros of the denominator polynomials for indecomposable affine type.

**Lemma 2.1.** *Let $M$ be a Coxeter matrix of indecomposable affine type of rank $l$. Then, all the roots of $N_M(t) = 0$ are contained in the open disc of radius $r$ centered at the origin, where $r$ is give by*

$$(2.1) \qquad r := \Big(\frac{2^{l+1} - s - 1}{s}\Big)^{1/(\deg(\Delta_{M|_{I\setminus\{v\}}}) - d)},$$

*where $\deg(\Delta_{M|_{I\setminus\{v\}}})$, $d$, $s$ are invariants of $M$ explained in the proof.*

*Proof.* In the affine Coxeter graph $\Gamma_M$ (whose vertex set is identified with $I$, and hence $\#(\Gamma_M) = \#(I) = l+1$), there is a vertex $v$, called *special* [B, p. 87] such that $\Gamma_M \setminus \{v\}$ is the Coxeter graph of the finite Coxeter group isomorphic to the radical quotient of the affine Coxeter group $\overline{G}_M$. Let $s$ be the number of special vertexes in $\Gamma_M$. For types $\tilde{A}_l$, $\tilde{B}_l$, $\tilde{C}_l$, $\tilde{D}_l$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, $\tilde{F}_4$, $\tilde{G}_2$, the number $s$ is given by $l+1$, 2, 2, 4, 3, 2, 1, 1, 1, respectively.

Noting the fact that the type of $\Gamma_{M\setminus\{v\}}$ (and, hence, $\deg(\Delta_{M|_{I\setminus\{v\}}})$) does not depend on the choice of a special vertex $v$, we see that the monomial $N(t) := (-1)^l s \cdot t^{\deg(\Delta_{M|_{I\setminus\{v\}}})}$ ($v$ a special vertex) is the leading term of $N_M(t)$. One has $|N_M(t) - N(t)| \le (2^{l+1} - s - 1)|t|^d$ for $t \in \mathbf{C}$ with $|t| > 1$ (strict inequality holds except for the type $\tilde{A}_1$), where we put

$$d := \max\{\deg(\Delta_J) \mid J \subset I \text{ such that } I\setminus J \text{ is not a single special vertex}\}.$$

Hence

$$|N_M(t) - N(t)|/|N(t)| \le \frac{2^{l+1} - s - 1}{s} |t|^{d - \deg(\Delta_{M|_{I\setminus\{v\}}})}.$$

If $r \in \mathbf{R}_{>1}$ satisfies an inequality

$$\frac{2^{l+1} - s - 1}{s} r^{d - \deg(\Delta_{M|_{I\setminus\{v\}}})} \le 1,$$

then, due to Rouche's theorem, the number of zeros of $N_M(t) = 0$ in the disc of radius $r$ is equal to that of $N(t) = 0$, which has zeros only at 0 of multiplicity $\deg(N(t)) = \deg(N_M(t))$. That is, all roots of $N_M(t) = 0$ are in the disc $\{|t| < r\}$ for $r$ given in (2.1). □

**3. Conjectures on the zeros of the denominator polynomial $N_M(t)$ of affine type.** Some discussions and examples at the end of §1 lead us to the following three conjectures on the distribution of

**) The discrepancy between the rank $l$ and the number $\#(I) = l+1$ for a Coxeter matrix $M$ of indecomposable affine type comes from the fact that the associated affine Coxeter group acts on a positive semi-definite $\mathbf{R}$-vector space of with corank 1.

the zeros of the denominator polynomial $N_M(t)$ of indecomposable finite or affine type.***)

**Conjecture 1.** i) The polynomial $\tilde{N}_M(t) := N_M(t)/(1-t)$ is irreducible over $\mathbf{Z}$ for any indecomposable finite type $M$. ii) The polynomial $N_M(t)$ is irreducible over $\mathbf{Z}$ for any indecomposable affine type $M$.

**Conjecture 2.** There are $l$ mutually distinct roots of $N_M(t) = 0$ on the interval $(0, 1]$ where $l$ is the number of positive eigenvalues of $B_M$.

**Conjecture 3.** Let $r_M$ be the smallest among the roots on the interval $(0, 1]$. Then, the absolute values of the other roots of $N_M(t) = 0$ are strictly larger than $r_W$.

Conjectures on the denominator polynomials of finite type were already stated in [S1] and verified by computer calculations for the types $A_l, B_l, C_l, D_l$ ($l \leq 30$), $E_6$, $E_7$, $E_8$, $F_4$, $G_2$, $H_3$, $H_4$ and $I_2(p)$ ($p \in \mathbf{Z}_{\geq 3}$) by M. Fuchiwaki, S. Tsuchioka and others. Some theoretical approach on the conjectures is in progress by S. Yasuda.

Conjectures on the affine denominator polynomial are positively confirmed directly for the three types $\tilde{A}_2$, $\tilde{C}_2$ and $\tilde{G}_2$ of rank 2 from the explicit expressions in §2 Example. S. Tsuchioka confirmed the conjectures for further cases, including $\tilde{A}_3, \cdots, \tilde{A}_8$, $\tilde{C}_3, \cdots, \tilde{C}_8, \tilde{D}_4$, $\tilde{E}_7$, $\tilde{E}_8$ and $\tilde{F}_4$, by use of computer.

**Remark 3.1.** As we conjectured, a denominator polynomial $N_M(t)$ of finite type has zeros of order 1 at $t = 1$, and that of affine type does not vanish at $t = 1$. In Appendix, we observe that a denominator polynomial of elliptic type does not vanish there either. Among 14 denominator polynomial of hyperbolic type (see Remark 3.3.), types (2,3,7), (2,4,5), (3,3,4), (2,3,8), (3,3,5), (2,5,5), (2,3,9), (2,4,7), (2,5,6), (3,4,5) or (4,4,4) has zeros at $t = 1$ but types (2,4,6), (3,3,6) or (3,4,4) does not. It is interesting to find a formula of the order $d$ of zeros at $t = 1$ and to ask precise question than Conjecture 1: whether or when is $N_M(t)/(1-t)^d$ irreducible (see [S4, §12, Problem 3. iii)])?

***) As we shall observe in Appendix, these conjectures are (formally) valid also for elliptic root systems [S2]. After a suitable modification, the conjectures seem to be valid also for some Artin monoids of hyperbolic type (see Remark 3.1). It is interesting to clarify how far the conjectures are valid, and to develop a unified understanding of them (hopefully, in connection with the original motivation to study the limit functions associated with monoids).

****) Associated with an elliptic root system, there are concepts of an elliptic Weyl group, elliptic Lie alebra and group, elliptic Hecke algebra, ... etc. However, at present, there is no clear definition of elliptic Artin monoid (since they are not associated with Coxeter matrices).

**Remark 3.2.** In Conjecture 3, the fact that $r_M$ is less than or equal to the absolute values of any other roots of $N_M(t) = 0$ is trivially true, since $r_M$ is equal to the radius of convergence of the power series $P_M(t)$ of non-negative real coefficients due to Pringsheim Theorem (see [H, Theore 5.7.1.]). Therefore, the true question here is that there are no other roots of $N_M(t) = 0$ whose absolute value is equal to $r_W$. This question is motivated from a study of the author on certain limit functions associated with the monoid $G_M^+$ (see [S1, §5] and [S4, §11]).

**Appendix.** Pursuing formal analogy (i.e. without an explicit relation with the growth functions of any monoid****)), let us introduce the denominator polynomial $N_X(t)$ of elliptic type: let $(R, G)$ be an irreducible marked elliptic root system of type $X$ and let $\Gamma_X := \Gamma(R, G)$ be the associated elliptic Dynkin diagram [S2, I, §8]. Then, similar to (1.6), we define the *elliptic denominator polynomial of type* $X$ by

$$N_X(t) := \sum_{J \subset \Gamma_X} (-1)^{\#(J)} t^{\deg(\Delta_J)}, \tag{3.1}$$

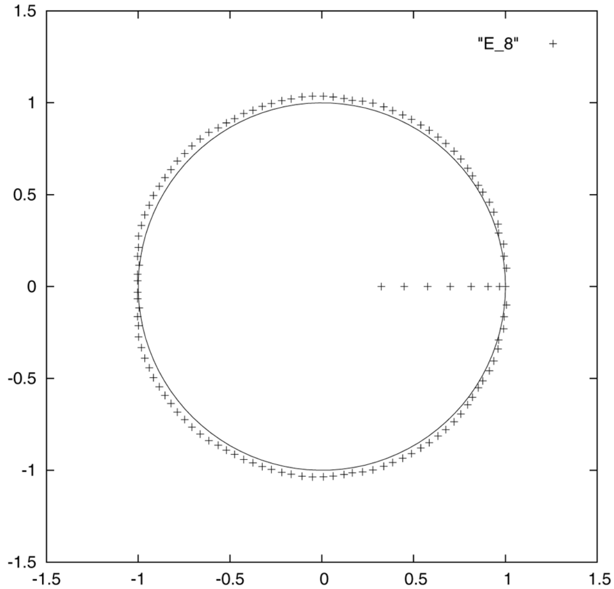
where the summation index $J$ runs over all subdiagrams of $\Gamma_X$ (not necessarily connected) which is of finite type. For these polynomials, we ask again:

**Conjecture 4.** Conjecture 1.ii) (replacing the phrase "indecomposable affine type" by the phrase "irreducible marked elliptic type"), Conjecture 2. (replacing $B_M$ by the Killing form of an elliptic root system) and Conjecture 3. in section 3 hold also for elliptic denominator polynomials.

Using computer, S. Tsuchioka has verified that Conjecture 4. hold for the types $A_2^{(1,1)}, \cdots, A_7^{(1,1)}$, $D_4^{(1,1)}, E_6^{(1,1)}, E_7^{(1,1)}, E_8^{(1,1)}$ and $G_2^{(1,1)}$ among others.

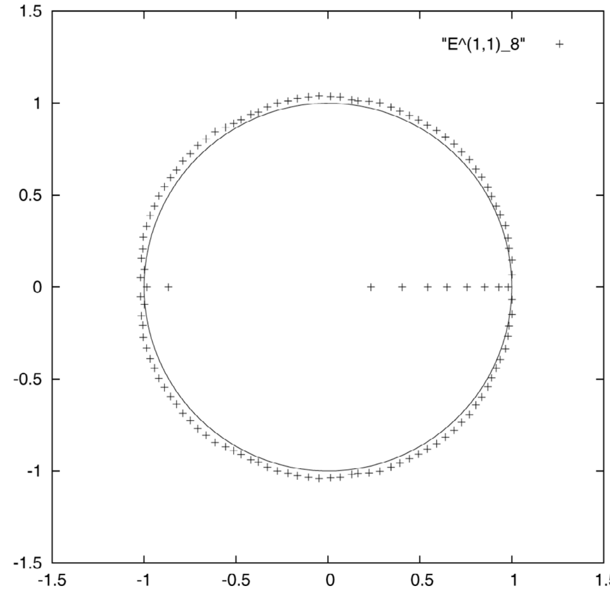
**Remark 3.3.** Recall that there are 14 regular systems of weights with $\varepsilon = -1$, which are associated with the 14 exceptional singularities by Arnold, and that two diagrams are associated with each of them, one: the basis of vanishing cycles, called the Gabrielov diagram, and the other: the basis of Picard lattice of the K3 surface of the Pinkham compactification of the Milnor fibers [S3, §13 and §18]. It is interesting to define, formally similar to the formula (3.1), the denominator polynomials associated with the two diagrams and to compare them.

**Example** (S. Tsuchioka). We illustrate the zero loci of the denominator polynomials of finite type $E_8$, affine type $\tilde{E}_8$ and elliptic type $E_8^{(1,1)}$. In the following figures, zero-loci are indicated by crosses "+".



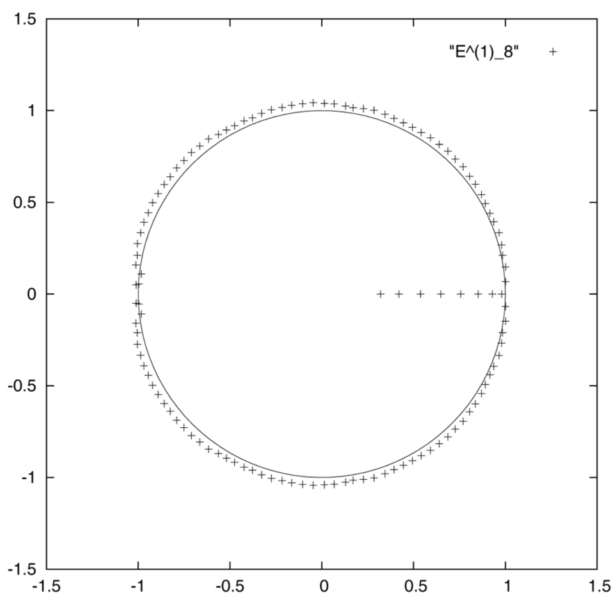

**Type $\boldsymbol{E_8}$**

$N_{E_8}(t) = 1 - 8t + 21t^2 - 14t^3 - 21t^4 + 28t^5 - 7t^6 + 12t^7 - 8t^8 - 10t^9 + 10t^{10} - 12t^{11} + 7t^{12} + 2t^{13} - t^{14} - 3t^{15} + 2t^{16} - 2t^{20} + 6t^{21} - t^{22} - t^{23} - t^{28} + t^{30} + t^{36} - t^{37} - t^{42} - t^{63} + t^{120}.$




**Type $\boldsymbol{E_8^{(1,1)}}$**

$N_{E_8^{(1,1)}}(t) = 1 - 10t + 33t^2 - 32t^3 - 35t^4 + 73t^5 - 23t^6 + 21t^7 - 30t^8 - 28t^9 + 36t^{10} - 38t^{11} + 34t^{12} + 12t^{13} - 8t^{14} - 5t^{15} + 5t^{16} - 4t^{17} - 5t^{18} + t^{19} - 2t^{20} + 18t^{21} - 8t^{22} - 6t^{23} + 2t^{26} - 6t^{28} + 2t^{29} + 2t^{30} - 2t^{31} + 4t^{36} - 4t^{37} + 2t^{39} - 2t^{42} + 2t^{56} - 2t^{63} + 2t^{64} + 2t^{120}.$




**Type $\boldsymbol{\tilde{E}_8}$**

$N_{\tilde{E}_8}(t) = 1 - 9t + 28t^2 - 28t^3 - 22t^4 + 54t^5 - 20t^6 + 10t^7 - 17t^8 - 13t^9 + 21t^{10} - 23t^{11} + 19t^{12} + 7t^{13} - 5t^{14} - 3t^{15} + 4t^{16} - 3t^{17} - 3t^{18} + t^{19} - t^{20} + 9t^{21} - 4t^{22} - 3t^{23} + t^{26} - 3t^{28} + t^{29} + t^{30} - t^{31} + 2t^{36} - 2t^{37} + t^{39} - t^{42} + t^{56} - t^{63} + t^{64} + t^{120}.$

**Acknowledgements.** The author expresses his gratitude to Shunsuke Tsuchioka for help calculating examples as well as drawing figures by computer, and also for informing him of [A-N]. He is also grateful to Mikael Pichot for his interest and discussions on the subjects, and to Ken Shackleton for carefully reading an earlier version of this note.

## References

[A-N] M. Albenque and P. Nadeau, Growth function for a class of monoids, *FPSAC (Hagenberg, 2009)*, DMTCS.

[ B ] N. Bourbaki, Groups et algébres de Lie, Chapitres 4, 5 et 6. Éléments de Mathèmatique XXXIV. Hermann, Paris, 1968.

[B-S] E. Brieskorn and K. Saito, Artin-Gruppen und Coxeter-Gruppen, Invent. Math. **17** (1972), 245–271.

[ H ] E. Hille, *Analytic function theory. Vol.* 1, Ginn and Company, Boston, 1959.

[ S1 ] K. Saito, Growth functions associated with Artin monoids of finite type, Proc. Japan Acad. Ser. A Math. Sci. **84** (2008), no. 10, 179–183.

[ S2 ] K. Saito, Extended affine root systems I (Coxeter transformations), III (Elliptic Weyl groups), Publ. RIMS, Kyoto-Univ. **21** (1985), 75–179, *ibid.* **33** (1997), 301-329.

[ S3 ] K. Saito, Towards a categorical construction of Lie algebras, in *Algebraic geometry in East Asia—Hanoi 2005*, 101–175, Adv. Stud. Pure Math., 50, Math. Soc. Japan, Tokyo.

[ S4 ] K. Saito, Limit elements in the configuration algebra for discrete group, RIMS-1593. (Preprint).