# The squaring operation on $\mathcal{A}$-generators of the Dickson algebra

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**Abstract:** We study the squaring operation $Sq^0$ on the dual of the minimal $\mathcal{A}$-generators of the Dickson algebra. We show that this squaring operation is isomorphic on its image. We also give vanishing results for this operation in some cases. As a consequence, we prove that the Lannes-Zarati homomorphism vanishes (1) on every element in any finite $Sq^0$-family in $Ext^*_{\mathcal{A}}(\mathbf{F}_2, \mathbf{F}_2)$ except possibly the family initial element, and (2) on almost all known elements in the Ext group. This verifies a part of the algebraic version of the classical conjecture on spherical classes.

**Key words:** Modular representations; invariant theory; cohomology of the Steenrod algebra; spherical classes; Lannes-Zarati homomorphism.

**1. Statement of results.** Throughout the paper, the coefficient ring for homology and cohomology is always $\mathbf{F}_2$, the field of two elements. Let $\mathbf{V}_s$ be an $s$-dimensional $\mathbf{F}_2$-vector space. The general linear group $GL_s := GL(\mathbf{V}_s)$ acts regularly on $\mathbf{V}_s$ and therefore on $H_*(B\mathbf{V}_s)$. Let $P(\mathbf{F}_2 \underset{GL_s}{\otimes} H_*(B\mathbf{V}_s))$ be the submodule of $\mathbf{F}_2 \underset{GL_s}{\otimes} H_*(B\mathbf{V}_s)$ consisting of all elements, which are annihilated by every positive-degree operation in the mod 2 Steenrod algebra, $\mathcal{A}$.

The subject of the present paper is the squaring operation

$$Sq^0 : P(\mathbf{F}_2 \underset{GL_s}{\otimes} H_*(B\mathbf{V}_s))_\delta \to P(\mathbf{F}_2 \underset{GL_s}{\otimes} H_*(B\mathbf{V}_s))_{s+2\delta},$$

which is defined by the first named author in [11] as an analogue of the classical squaring operation on the cohomology of the Steenrod algebra, $Ext^*_{\mathcal{A}}(\mathbf{F}_2, \mathbf{F}_2)$.

The most important property of the squaring operation is that it commutes with the classical squaring operation $Sq^0$ on $Ext^*_{\mathcal{A}}(\mathbf{F}_2, \mathbf{F}_2)$ through the Lannes-Zarati homomorphism

$$\varphi_s : Ext^{s,s+\delta}_{\mathcal{A}}(\mathbf{F}_2, \mathbf{F}_2) \to P(\mathbf{F}_2 \underset{GL_s}{\otimes} H_*(B\mathbf{V}_s))_\delta,$$

for any $s$ (see [12]). Therefore the investigation of the squaring operation is useful to the study of the Lannes-Zarati homomorphism.

The Lannes-Zarati homomorphism, defined in [18], is the one corresponding to an associated graded of the Hurewicz map $H : \pi^s_*(S^0) \cong \pi_*(Q_0S^0) \to H_*(Q_0S^0)$. So, the following is an algebraic version of the conjecture on spherical classes.

**Conjecture 1.1** [11]. *$\varphi_s = 0$ in any positive stem for $s > 2$.*

That the conjecture is no longer valid for s = 1 and 2 is respectively an exposition of the existence of Hopf invariant one and Kervaire invariant one classes. (See Adams [1], Browder [3], Curtis [6] for a discussion on spherical classes; and see Lannes-Zarati [18], Goerss [9], Hưng [11, 12] for a discussion on the homomorphism.)

The squaring operation on $P(\mathbf{F}_2 \underset{GL_s}{\otimes} H_*(B\mathbf{V}_s))$ is derived from the Kameko squaring operation on $\mathbf{F}_2 \underset{GL_s}{\otimes} PH_*(B\mathbf{V}_s)$ in such a way that these two squaring operations commute with each other through the canonical homomorphism

$$j^*_s : \mathbf{F}_2 \underset{GL_s}{\otimes} PH_*(B\mathbf{V}_s) \to P(\mathbf{F}_2 \underset{GL_s}{\otimes} H_*(B\mathbf{V}_s))$$

induced by the identity map on $\mathbf{V}_s$ (see [11]). The first named author also showed in [11] that $j^*_s = \varphi_s \circ Tr_s$. Here $Tr_s$ is the algebraic transfer, which was defined by Singer [21] and was shown to be highly nontrivial by Singer [21], Boardman [2], Bruner-Hà-Hưng [5], Hưng [13], Hà [10], Nam [20], and the authors [17]. Further, Hưng and Nam proved in [14] that $j^*_s = 0$ in positive degree for $s > 2$, or equivalently that the Lannes-Zarati homomorphism vanishes on the positive stem part of the algebraic transfer's image for the homological degree $s > 2$.

A basis of the $\mathbf{F}_2$-vector space $P(\mathbf{F}_2 \underset{GL_s}{\otimes} H_*(B\mathbf{V}_s))$ was determined by Singer [21] for $s = 1, 2$, by Hưng

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and Peterson [15] for $s = 3, 4$, and by Giambalvo and Peterson [8] for $s = 5$. It is still unknown for $s > 5$. The squaring operation on $P(\mathbf{F}_2 \underset{GL_s}{\otimes} H_*(B\mathbf{V}_s))$ is explicitly computed in [11] for $s \leq 4$. This result shows that $Sq^0$ is an isomorphism for $s = 1, 2$ and is no longer an isomorphism for $s = 3, 4$.

The Dickson algebra of all $GL_s$-invariants was determined in [7] as follows:

$$\begin{aligned} D_s := H^*(B\mathbf{V}_s)^{GL_s} &\cong \mathbf{F}_2[x_1, \cdots, x_s]^{GL_s} \\ &= \mathbf{F}_2[Q_{s,0}, Q_{s,1}, \cdots, Q_{s,s-1}], \end{aligned}$$

where $Q_{s,i}$ denotes the Dickson invariant of degree $2^s - 2^i$. Let $d(i_0, i_1, \ldots, i_{s-1}) \in \mathbf{F}_2 \underset{GL_s}{\otimes} H_*(B\mathbf{V}_s)$ be the element that is dual to $Q_{s,0}^{i_0} \ldots Q_{s,s-1}^{i_{s-1}}$ with respect to the basis of $D_s$ consisting of all monomials in the Dickson invariants.

The following theorem, which claims that the squaring operation is ''eventually isomorphic'' on $P(\mathbf{F}_2 \underset{GL_s}{\otimes} H_*(B\mathbf{V}_s))$, is the first main result of this paper.

**Theorem 1.2.** *The squaring operation*

$$Sq^0 : P(\mathbf{F}_2 \underset{GL_s}{\otimes} H_*(B\mathbf{V}_s)) \to P(\mathbf{F}_2 \underset{GL_s}{\otimes} H_*(B\mathbf{V}_s))$$

*is an isomorphism on its image $Im(Sq^0)$. Further, if $d(i_0, \ldots, i_{s-1})$ is an element in $P(\mathbf{F}_2 \underset{GL_s}{\otimes} H_*(B\mathbf{V}_s))$, then*

$$Sq^0 d(i_0, \ldots, i_{s-1}) = \begin{cases} d(s-2, 2i_1 + 1, \ldots, 2i_{s-1} + 1), & i_0 = s-2, \\ 0, & \text{otherwise.} \end{cases}$$

Evidently, this theorem could be applied to investigate the structure of the space $P(\mathbf{F}_2 \underset{GL_s}{\otimes} H_*(B\mathbf{V}_s))$ or of its dual space $\mathbf{F}_2 \underset{\mathcal{A}}{\otimes} D_s$. The theorem is an analogue of the result by the first name author [13, Theorem 1.1] stating that $Sq^0 : \mathbf{F}_2 \underset{GL_s}{\otimes} PH_*(B\mathbf{V}_s) \to \mathbf{F}_2 \underset{GL_s}{\otimes} PH_*(B\mathbf{V}_s)$ is an isomorphism on the image of $(Sq^0)^{s-2}$.

A sequence $\{a_i |\, i \geq 0\}$ of elements in $P(\mathbf{F}_2 \underset{GL_s}{\otimes} H_*(B\mathbf{V}_s))$ (or in $Ext^s_{\mathcal{A}}(\mathbf{F}_2, \mathbf{F}_2)$) is called an $Sq^0$-family if $a_i = Sq^0(a_{i-1})$ for every $i > 0$. It is called finite with length $s$ if it has exactly $s$ non-zero elements. Otherwise, it is called infinite.

The following is an immediate consequence of the above theorem.

**Corollary 1.3.** *Any $Sq^0$-family in $P(\mathbf{F}_2 \underset{GL_s}{\otimes} H_*(B\mathbf{V}_s))$ is either infinite or finite with length* 1.

This is an analogue of the result by the first name author [13, Corollary 1.7] stating that any $Sq^0$-family in $\mathbf{F}_2 \underset{GL_s}{\otimes} PH_*(B\mathbf{V}_s)$ is either infinite or finite with length at most $s - 2$.

Let $\alpha(\delta)$ be the number of ones in the dyadic expansion of $\delta$, and $\nu(\delta)$ the exponent of the highest power of 2 dividing $\delta$, with convention $2^{\nu(0)} = 0$.

Following Giambalvo and Peterson [8], the function $\kappa_s$ is defined by setting $\kappa_s(r) = r + 2^{\nu(s-2-r)}$. For convenience, set $\kappa_s^0(r) = r$. Finally, let $\kappa_s^\ell(r) = \kappa_s(\kappa_s^{\ell-1}(r))$ for $\ell \geq 1$. Actually, Giambalvo and Peterson denoted the function $\kappa_s$ by $x_s$. However, the letter $x_s$ will be used in this paper to name an another object, so we denote it by $\kappa_s$. A discussion on an earlier version of this function defined by Hưng and Peterson [15] will be given in an associate detailed paper.

The following is the second main result of the paper.

**Theorem 1.4.** *The squaring operation $Sq^0$ on $P(\mathbf{F}_2 \underset{GL_s}{\otimes} H_*(B\mathbf{V}_s))$ vanishes in any degree $\delta$, which*

(i) *either satisfies $\nu(\delta + s) \leq [\log_2(s-2)] + 1$ for $s \geq 3$,*

(ii) *or is not of the form $\delta_s$ defined inductively for $s \geq 3$ as follows*:

$$\delta_s = \delta_{s-1} - 1 + 2^{s-1}[\kappa_1^{j_{s-1}} \kappa_2^{j_{s-2}} \ldots \kappa_{s-1}^{j_1}(s-2) + 1],$$

*for arbitrary non-decreasing sequence $[\log_2(s-2)] < j_1 \leq j_2 \leq \ldots \leq j_{s-1}$, where $\delta_2 = 2^{j_1+1} - 2$.*

The theorem does not seem to be possibly improved in the meaning that, $Sq^0$ acts non-trivially in every degree $\delta_s$ given in the theorem at least for $s = 3, 4$ and 5. Unfortunately, the vector space $P(\mathbf{F}_2 \underset{GL_s}{\otimes} H_*(B\mathbf{V}_s))$ is unknown for $s > 5$ so far.

By means of the formula in Theorem 1.4 (ii), we find explicitly the list of all the degrees $\delta_s$ for $5 \leq s \leq 7$ in Lemmas 2.2–2.4. In principle, this procedure of computing can inductively be extended for any bigger value of $s$. In particular, the following is an immediate consequence of the above theorem.

**Corollary 1.5.** *$Sq^0$ on $P(\mathbf{F}_2 \underset{GL_s}{\otimes} H_*(B\mathbf{V}_s))$ vanishes in any degree $\delta$, which satisfies one of the two conditions:*

(i) *$\nu(\delta + s) \leq [\log_2(s-2)] + 1$ for $s \geq 3$,*

(ii) *$\delta + s$ is not of the forms listed respectively in Lemmas* 2.2–2.4 *for* $5 \leq s \leq 7$.

The remaining part of this paper deals with some applications of the above results to the study of Conjecture 1.1.

The group $Ext_{\mathcal{A}}^s(\mathbf{F}_2, \mathbf{F}_2)$ was determined for $s = 1, 2$ by Adams [1], for $s = 3$ by Wang [23], and for $s = 4$ by Lin (see [19]). It is unknown for $s > 4$. Based on these results, Conjecture 1.1 was proved by the first named author in [11, 12] for $s = 3, 4$.

Hư̛ng and Peterson showed in [16] that $\varphi = \oplus\varphi_s$ is a homomorphism of algebras and it vanishes on decomposable elements. So, in order to prove Conjecture 1.1, it suffices to study the Lannes-Zarati homomorphism on indecomposable elements.

Our first result on the Lannes-Zarati homomorphism is the following consequence of Theorem 1.2.

**Corollary 1.6.** *If $\{a_i | i \geq 0\}$ is a finite $Sq^0$-family in $Ext_{\mathcal{A}}^s(\mathbf{F}_2, \mathbf{F}_2)$, then*

$$\varphi_s(a_i) = 0 \quad for \quad i > 0.$$

Our second result on the Lannes-Zarati homomorphism is the following application of Theorem 1.4 and Corollary 1.5.

**Proposition 1.7.** *Let $\{a_i | i \geq 0\}$ be an $Sq^0$-family in $Ext_{\mathcal{A}}^s(\mathbf{F}_2, \mathbf{F}_2)$. Suppose $\delta = Stem(a_0)$ satisfies one of the following conditions*

(i) $\nu(\delta + s) \leq [\log_2(s-2)] + 1$ *for* $s \geq 3$;

(ii) *$\delta$ is not of the form $\delta_s$ given in Theorem 1.4. In particular, $\delta + s$ is not of the forms listed respectively in Lemmas 2.2–2.5 for $5 \leq s \leq 7$.*

*Then $\varphi_s(a_i) = 0$ for any $i > 0$.*

We note that every $Sq^0$-family listed in the paper by Tangora [22] as well as in that by Bruner [4] satisfies either the hypothesis of Corollary 1.6 or the one of Proposition 1.7. Therefore, if $\{a_i | i \geq 0\}$ denotes such a family in $Ext_{\mathcal{A}}^s(\mathbf{F}_2, \mathbf{F}_2)$, then $\varphi_s(a_i) = 0$ for any $i > 0$. It should be noted that the above results do not conclude whether the Lannes-Zarati homomorphism vanishes on the initial element $a_0$ of the $Sq^0$-family in question. The following proposition gives an answer to this problem in the case where $Stem(a_0)$ is rather small.

**Proposition 1.8.** *If $\{a_i | i \geq 0\}$ is an $Sq^0$-family in $Ext_{\mathcal{A}}^s(\mathbf{F}_2, \mathbf{F}_2)$ with $Stem(a_0) < 2^{s-1}$, then $\varphi_s(a_i) = 0$ for any $i \geq 0$.*

## 2. Vanishing degrees of the squaring operation in small ranks.

An element in $D_s$ is called decomposable if it is in $\bar{\mathcal{A}}D_s$, where $\bar{\mathcal{A}}$ denotes the augmentation ideal of the Steenrod algebra $\mathcal{A}$. Otherwise, it is called indecomposable.

**Definition 2.1** [8]. *A monomial $Q(I) = Q_{s,0}^{i_0} Q_{s,1}^{i_1} \ldots Q_{s,s-1}^{i_{s-1}}$ of $D_s$ is called reducible if there exists an s-tuple $J = [j_0, j_1, \ldots, j_{s-1}]$ of non-negative integers such that*

$$i_0 = \kappa_s^{j_0}(0),$$
$$i_k = \kappa_{s-k}^{j_k}(i_0 + i_1 + \cdots + i_{k-1}) - (i_0 + i_1 + \cdots + i_{k-1}),$$

*for $1 \leq k \leq s-1$. Then $J = [j_0, j_1, \ldots, j_{s-1}]$ is called the reduced form of $I$. The reduced form $J = [j_0, j_1, \ldots, j_{s-1}]$ is said to have non-decreasing terms if $j_\ell \leq j_{\ell+1}$ for all $\ell$.*

Using the formula in Theorem 1.4 (ii), we give in the following lemmas some degrees, in which the squaring operation would not vanish for $5 \leq s \leq 7$.

**Lemma 2.2.** *Let $Q(I) = Q(3, i_1, i_2, i_3, i_4)$ be an indecomposable monomial in degree $\delta$ of $D_5$ with the non-decreasing reduced form $[j_1, j_2, j_3, j_4]$. Then, $j_1 \geq 2$ and*

$$\delta + 5 = \begin{cases} 2^{j_1+1} + 2^{j_2+3} + 2^{j_3+4} + 2^{j_4+5}, & j_1 \leq j_2 < j_3 < j_4, \\ 2^{j_1+1} + 2^{j_2+3} + 2^{j_4+6}, & j_1 \leq j_2 < j_3 = j_4, \\ 2^{j_1+1} + 2^{j_3+5} + 2^{j_4+6}, & j_1 \leq j_2 = j_3 \leq j_4. \end{cases}$$

**Lemma 2.3.** *Let $Q(I) = Q(4, i_1, i_2, i_3, i_4, i_5)$ be an indecomposable monomial in degree $\delta$ of $D_6$ with the non-decreasing reduced form $[j_1, j_2, j_3, j_4, j_5]$. Then, $j_1 \geq 3$ and*

$$\delta + 6 =$$
$$\begin{cases} 2^{j_1+1} + 2^{j_2+3} + 2^{j_3+4} + 2^{j_4+5} + 2^{j_5+6}, & j_1 \leq j_2 < j_3 < j_4 < j_5, \\ 2^{j_1+1} + 2^{j_2+3} + 2^{j_3+4} + 2^{j_5+7}, & j_1 \leq j_2 < j_3 < j_4 = j_5, \\ 2^{j_1+1} + 2^{j_2+3} + 2^{j_4+6} + 2^{j_5+7}, & j_1 \leq j_2 < j_3 = j_4 \leq j_5, \\ 2^{j_1+1} + 2^{j_3+5} + 2^{j_5+7}, & j_1 \leq j_2 = j_3 \leq j_4 = j_5 - 1, \\ 2^{j_1+1} + 2^{j_3+5} + 2^{j_4+6} + 2^{j_5+6}, & j_1 \leq j_2 = j_3 \leq j_4 < j_5 - 1, \\ 2^{j_1+1} + 2^{j_3+5} + 2^{j_4+5} + 2^{j_4+6} + 2^{j_4+7}, & j_1 \leq j_2 = j_3 < j_4 = j_5, \\ 2^{j_1+1} + 2^{j_2+8}, & j_1 \leq j_2 = j_3 = j_4 = j_5. \end{cases}$$

**Lemma 2.4.** *Let $Q(I) = Q(5, i_1, i_2, i_3, i_4, i_5, i_6)$ be an indecomposable monomial in degree $\delta$ of $D_7$ with the non-decreasing reduced form $[j_1, j_2, j_3, j_4, j_5, j_6]$. Then, $j_1 \geq 3$ and*

$$\delta + 7 =$$
$$\begin{cases} 2^{j_1+1} + 2^{j_2+3} + 2^{j_3+4} + 2^{j_4+5} + 2^{j_5+6} + 2^{j_6+7}, & j_2 < j_3 < j_4 < j_5 < j_6, \\ 2^{j_1+1} + 2^{j_2+3} + 2^{j_3+4} + 2^{j_4+5} + 2^{j_5+8}, & j_2 < j_3 < j_4 < j_5 = j_6, \\ 2^{j_1+1} + 2^{j_2+3} + 2^{j_3+4} + 2^{j_5+7} + 2^{j_6+8}, & j_2 < j_3 < j_4 = j_5 \leq j_6, \\ 2^{j_1+1} + 2^{j_2+3} + 2^{j_3+9}, & j_2 < j_3 = j_4 = j_5 = j_6, \\ 2^{j_1+1} + 2^{j_2+3} + 2^{j_3+6} + 2^{j_5+6} + 2^{j_5+7} + 2^{j_5+8}, & j_2 < j_3 = j_4 < j_5 = j_6, \\ 2^{j_1+1} + 2^{j_2+3} + 2^{j_3+6} + 2^{j_5+9}, & j_2 < j_3 = j_4 \leq j_5 = j_6 - 1, \\ 2^{j_1+1} + 2^{j_2+3} + 2^{j_3+6} + 2^{j_5+7} + 2^{j_6+7}, & j_2 < j_3 = j_4 \leq j_5 < j_6 - 1, \\ 2^{j_1+1} + 2^{j_3+5} + 2^{j_5+7} + 2^{j_6+8}, & j_2 = j_3 \leq j_4 = j_5 - 1 \leq j_6 - 1, \\ 2^{j_1+1} + 2^{j_3+5} + 2^{j_4+6} + 2^{j_5+8}, & j_2 = j_3 \leq j_4 < j_5 - 1 = j_6 - 1, \\ 2^{j_1+1} + 2^{j_3+5} + 2^{j_4+6} + 2^{j_5+6} + 2^{j_6+7}, & j_2 = j_3 \leq j_4 < j_5 - 1 < j_6 - 1, \\ 2^{j_1+1} + 2^{j_3+5} + 2^{j_4+5} + 2^{j_4+6} + 2^{j_4+9}, & j_2 = j_3 < j_4 = j_5 = j_6, \\ 2^{j_1+1} + 2^{j_3+5} + 2^{j_4+5} + 2^{j_4+6} + 2^{j_4+7} + 2^{j_6+8}, & j_2 = j_3 < j_4 = j_5 \leq j_6 - 1, \\ 2^{j_1+1} + 2^{j_2+7} + 2^{j_2+9}, & j_2 = j_3 = j_4 = j_5 = j_6, \\ 2^{j_1+1} + 2^{j_2+8} + 2^{j_6+8}, & j_2 = j_3 = j_4 = j_5 < j_6. \end{cases}$$

The contains of this note will be published in detail elsewhere.

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