On observability of parallel-flow diffusive heat exchanger equations with boundary output

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Abstract: This paper is concerned with the observability problem of a parallel-flow two-fluid heat exchanger equation with diffusion term. First, the case where two fluid temperatures are measured at the outlet is considered. It is shown that the observed system with the measurements becomes observable on any interval of time through a concrete series expression of the solution. Next, the two cases where each one of two fluid temperatures is measured at the outlet are considered. It is also shown that the observed system with the only one measurement becomes observable on any interval of time except for the special cases of physical constants appearing in the equation. For the exceptional cases the unobservable subspace is finite dimensional and is characterized by using the eigenfunctions of heat equation with fluid transfer term.

Key words: Parallel-flow heat exchanger equation; boundary output; observability; C_0-semigroup.

1. Introduction. In this paper, we shall consider the following type of parallel-flow two-fluid heat exchange process with diffusion terms

$$\begin{aligned} \frac{\partial z_1}{\partial t}(t,x) &= D\frac{\partial^2 z_1}{\partial x^2}(t,x) - \alpha\frac{\partial z_1}{\partial x}(t,x) \\ &\quad + h_1(z_2(t,x) - z_1(t,x)), \\ \frac{\partial z_2}{\partial t}(t,x) &= D\frac{\partial^2 z_2}{\partial x^2}(t,x) - \alpha\frac{\partial z_2}{\partial x}(t,x) \\ &\quad + h_2(z_1(t,x) - z_2(t,x)), \\ &\quad (t,x) \in (0,\infty)\times(0,1), \end{aligned} \tag{1}$$

$$D\frac{\partial z_1}{\partial x}(t,0) - \alpha z_1(t,0) = 0,$$

$$D\frac{\partial z_2}{\partial x}(t,0) - \alpha z_2(t,0) = 0,$$

$$\frac{\partial z_1}{\partial x}(t,1) = 0, \quad \frac{\partial z_2}{\partial x}(t,1) = 0, \quad t \in (0,\infty),$$

$$z_1(0,x) = \varphi_1(x), \quad z_2(0,x) = \varphi_2(x), \quad x \in [0,1]$$

with the output equation

$$y(t) = [y_1(t), y_2(t)]^T = [z_1(t,1), z_2(t,1)]^T. \tag{2}$$

In this, $z_1(t,x)$, $z_2(t,x) \in \mathbf{R}$ are the temperature variations at time t and at the point $x \in [0,1]$ with respect to an equilibrium point, $y_1(t)$, $y_2(t) \in \mathbf{R}$ are the measurement outputs, and D, α, h_1, h_2 are positive physical constants.

In [2] a counter-flow heat exchange process with diffusion terms, in which the effect of the temperatures of the tubes was taken into consideration, was treated and a robust controller was constructed for it. However, the dynamical analysis such as observability/reachability was not done in that paper.

In this paper, we discuss on observability of a parallel-flow two-fluid heat exchange process (1) with the output equation (2). It is proved that the parallel-flow heat exchange process (1) with the measurements (2) is observable on any interval of time. In addition, we consider the two cases where only one temperature is measured at the outlet, that is, the system (1) with the output equation

$$y(t) = y_1(t) = z_1(t,1),$$

and the system (1) with the output equation

$$y(t) = y_2(t) = z_2(t,1).$$

It is also proved that the observed system with the only one measurement is observable on any interval of time except for the special cases of constants D, α, h_1, h_2. For the exceptional cases the unobservable subspace is shown to be finite dimensional and is characterized by using the eigenfunctions of heat equation with fluid transfer term.

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Remark 1.1. *As for a heat exchange process without diffusion terms (i.e. the case of $D = 0$), the transfer function approach was adopted to analyze the process in* [7]*, and the exact transient solution was concretely given in* [1]*. Recently, in the case of $D = 0$, the dynamical analysis for the system with the approximated output equation is studied in* [3]*, and the dynamical analysis for the system with the original output equation is carried out in* [4, 5]*.*

2. Preliminaries. In this section we state the results on the operators appearing in system (1) and associated C_0-semigroups. Let $L^2(0,1)$ be the Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let β be a nonnegative constant and define the unbounded operator $A_\beta : D(A_\beta) \subset L^2(0,1) \to L^2(0,1)$ by

$$(A_\beta \varphi)(x) = -D\frac{d^2\varphi(x)}{dx^2} + \alpha\frac{d\varphi(x)}{dx} + \beta\varphi(x),$$
$$D(A_\beta) = \{\varphi \in H^2(0,1)\,; \quad D\frac{d\varphi}{dx}(0) - \alpha\varphi(0) = 0,\ \frac{d\varphi}{dx}(1) = 0\}.$$

Then the operator $-A_\beta$ becomes a Riesz-spectral operator in $L^2(0,1)$. The spectrum of $-A_\beta$ consists only of isolated eigenvalues with multiplicity one. More specifically, the spectrum of $-A_\beta$ is given by

$$\sigma(-A_\beta) = \{-\lambda_n^\beta :\ n \ge 1\} \subset (-\infty, 0),$$

where the eigenvalues λ_n^β, $n \ge 1$ of A_β are positive and are given by

$$\lambda_n^\beta = \frac{s_n^2 + \alpha^2}{4D} + \beta \equiv \lambda_n^0 + \beta > 0, \quad \forall n \ge 1,$$

where $\{s_n :\ n \ge 1\}$ is the set of all solutions to the following transcendental equation

$$\tan\Big(\frac{s}{2D}\Big) = \frac{2\alpha s}{s^2 - \alpha^2}, \quad s > 0$$

satisfying $0 < s_n < s_{n+1}$, $\forall n \ge 1$. The roots $\{s_n :\ n \ge 1\}$ are independent of β and have the following asymptotics

$$s_n = 2(n-1)D\pi + o(1). \tag{3}$$

So that we have

$$0 < \lambda_n^\beta < \lambda_{n+1}^\beta \to \infty, \quad \lambda_{n+1}^\beta - \lambda_n^\beta \to \infty.$$

Furthermore, we have the concrete representations of eigenfunctions $\phi_n(x)$ corresponding to λ_n^β which are independent of β:

$$\phi_n(x) = K_n e^{\frac{\alpha}{2D}x}\Big[\cos\Big(\frac{s_n}{2D}x\Big) + \frac{\alpha}{s_n}\sin\Big(\frac{s_n}{2D}x\Big)\Big],\ \forall n \ge 1,$$

where the constants K_n are those for the normalization in $L^2(0,1)$, which are nonzero and uniformly bounded in n. In what follows, for notational brevity, we use the symbols $' = \frac{d}{dx}$, $'' = \frac{d^2}{dx^2}$. The adjoint operator of A_β is given by

$$(A_\beta^*\varphi)(x) = -D\varphi''(x) - \alpha\varphi'(x) + \beta\varphi(x),$$
$$D(A_\beta^*) = \big\{\varphi \in H^2(0,1)\,; \quad \varphi'(0) = 0,\ D\varphi'(1) + \alpha\varphi(1) = 0\big\}.$$

The operator $-A_\beta^*$ is also a Riesz-spectral operator. It is verified that $\sigma(-A_\beta^*) = \sigma(-A_\beta) = \{-\lambda_n^\beta :\ n \ge 1\}$, and the eigenfunctions $\{\psi_n :\ n \ge 1\}$ corresponding to λ_n^β of A_β^* are given by

$$\psi_n(x) = M_n\phi_n(1-x), \quad x \in [0,1],$$

where the constants M_n are chosen such as the systems $\{\phi_n\}$ and $\{\psi_n\}$ are biorthonormal systems. We note here that $\psi_n(x)$ are independent of β and M_n are uniformly bounded in n. Hence we have the following eigenfunction expansions

$$\varphi = \sum_{n=1}^\infty \langle \varphi, \psi_n\rangle \phi_n, \quad \forall\varphi \in L^2(0,1).$$

Lemma 2.1. *The operator $-A_\beta$ generates an exponentially stable C_0-semigroup e^{-tA_β} in $L^2(0,1)$, i.e. there exists an $M_1 > 0$ such that*

$$\|e^{-tA_\beta}\|_{\mathcal{L}(L^2)} \le M_1 \exp\Big(-\Big(\frac{\alpha^2}{4D} + \beta\Big)t\Big), \quad \forall t \ge 0.$$

Moreover, the semigroup e^{-tA_β} is given for any initial state $\varphi \in L^2(0,1)$ and for all $t > 0$, by

$$\Big(e^{-tA_\beta}\varphi\Big)(x) = \sum_{n=1}^\infty e^{-\lambda_n^\beta t}\langle\varphi, \psi_n\rangle\phi_n(x), \quad x \in [0,1].$$

Especially, we define the two operators A_1, A_2 by using A_β as follows:

$$A_1 := A_{h_1+h_2}, \quad A_2 := A_0.$$

Now, we give the state space setting for (1). For that purpose, we introduce a Hilbert space $X := [L^2(0,1)]^2$ with inner product

$$\langle\varphi, \psi\rangle_X := \int_0^1 \{\varphi_1(x)\psi_1(x) + \varphi_2(x)\psi_2(x)\}dx,$$
$$\varphi = [\varphi_1, \varphi_2]^T \in X,\ \psi = [\psi_1, \psi_2]^T \in X.$$

Define the unbounded operator $A : D(A) \subset X \to X$ by

$$A\varphi = \begin{bmatrix} D\varphi_1'' - \alpha\varphi_1' + h_1(\varphi_2 - \varphi_1) \\ D\varphi_2'' - \alpha\varphi_2' + h_2(\varphi_1 - \varphi_2)\end{bmatrix}, \tag{4}$$
$$\varphi = [\varphi_1, \varphi_2]^T \in D(A),$$

$$D(A) = \{\varphi = [\varphi_1, \varphi_2]^T \in [H^2(0,1)]^2;\ D\varphi_1'(0) - \alpha\varphi_1(0) = D\varphi_2'(0) - \alpha\varphi_2(0) = 0,\ \varphi_1'(1) = \varphi_2'(1) = 0\}.$$

Since

$$A = \begin{bmatrix} -A_2 & 0 \\ 0 & -A_2 \end{bmatrix} + \begin{bmatrix} -h_1 & h_1 \\ h_2 & -h_2 \end{bmatrix}$$

and the operator $\begin{bmatrix} -A_2 & 0 \\ 0 & -A_2 \end{bmatrix}$ generates a C_0- semigroup $\begin{bmatrix} e^{-tA_2} & 0 \\ 0 & e^{-tA_2} \end{bmatrix}$ on X and $\begin{bmatrix} -h_1 & h_1 \\ h_2 & -h_2 \end{bmatrix}$ is a bounded operator on X, we have by the pertubation theorem for C_0-semigroups [8] that A generates a C_0-semigroup $e^{tA} =: T(t)$ on X. The (mild) solution $z(t) := [z_1(t,\cdot), z_2(t,\cdot)]^T \in X$ of (1) is given by

$$z(t) = T(t)\varphi = T(t)[\varphi_1, \varphi_2]^T, \quad t > 0.$$

In order to give the concrete representation of $T(t)\varphi$, we utilize the following linear transformations.

First, for system (1), defining

$$f(t,x) := z_1(t,x) - z_2(t,x) \tag{5}$$

and then using Lemma 2.1, we have

$$f(t,\cdot) = e^{-tA_1}(\varphi_1 - \varphi_2). \tag{6}$$

Similarly, for system (1), defining

$$g(t,x) := h_2 z_1(t,x) + h_1 z_2(t,x) \tag{7}$$

and using Lemma 2.1 again, we have

$$g(t,\cdot) = e^{-tA_2}(h_2\varphi_1 + h_1\varphi_2). \tag{8}$$

Therefore, it follows from (5)–(8) that

$$\begin{aligned} z_1(t,\cdot) &= h_0^{-1}(h_1 f(t,\cdot) + g(t,\cdot)) \\ &= h_0^{-1} e^{-tA_1}\{h_1(\varphi_1 - \varphi_2)\} \\ &\quad + h_0^{-1} e^{-tA_2}(h_2\varphi_1 + h_1\varphi_2), \end{aligned} \tag{9}$$

$$\begin{aligned} z_2(t,\cdot) &= h_0^{-1}(-h_2 f(t,\cdot) + g(t,\cdot)) \\ &= h_0^{-1} e^{-tA_1}\{-h_2(\varphi_1 - \varphi_2)\} \\ &\quad + h_0^{-1} e^{-tA_2}(h_2\varphi_1 + h_1\varphi_2), \end{aligned} \tag{10}$$

where $h_0 = h_1 + h_2$. Applying Lemma 2.1 to equations (9) and (10), we can verify the following lemma.

Lemma 2.2. *The operator A generates an exponentially stable C_0-semigroup $e^{tA} =: T(t)$ on $X = [L^2(0,1)]^2$, i.e., there exists an $M > 0$ such that*

$$\|T(t)\|_{\mathcal{L}(X)} \le M \exp\Big(-\frac{\alpha^2}{4D}t\Big), \quad \forall t \ge 0.$$

Moreover, the semigroup $T(t)$ is given by the following Fourier series expansions for any initial state $\varphi = [\varphi_1, \varphi_2]^T \in X$ and for all $t \ge 0$:

$$\Big(T(t)\varphi\Big)(x) = [z_1(\varphi;t,x), z_2(\varphi;t,x)]^T,$$

where

$$\begin{aligned} &z_1(\varphi;t,x) \\ &= h_0^{-1}\Big\{\sum_{n=1}^{\infty} e^{-(\lambda_n^0 + h_0)t}\langle h_1(\varphi_1 - \varphi_2), \psi_n\rangle \phi_n(x) \\ &\qquad + \sum_{n=1}^{\infty} e^{-\lambda_n^0 t}\langle h_2\varphi_1 + h_1\varphi_2, \psi_n\rangle \phi_n(x)\Big\}, \end{aligned} \tag{11}$$

$$\begin{aligned} &z_2(\varphi;t,x) \\ &= h_0^{-1}\Big\{\sum_{n=1}^{\infty} e^{-(\lambda_n^0 + h_0)t}\langle -h_2(\varphi_1 - \varphi_2), \psi_n\rangle \phi_n(x) \\ &\qquad + \sum_{n=1}^{\infty} e^{-\lambda_n^0 t}\langle h_2\varphi_1 + h_1\varphi_2, \psi_n\rangle \phi_n(x)\Big\}, \\ &\qquad\qquad x \in [0,1]. \end{aligned} \tag{12}$$

Remark 2.1. *In general, the operator A is not a Riesz-spectral operator on X.*

3. Observability. In this section, we study the observability problem for the system (1) with the output equation (2). This observed system can be formulated as follows:

$$\frac{dz(t)}{dt} = Az(t), \quad z(0) = \varphi_0 = [\varphi_1, \varphi_2]^T \in X, \tag{13}$$

$$y(t) = Cz(t), \tag{14}$$

where $z(t) := [z_1(t,\cdot), z_2(t,\cdot)]^T$ and the operator A is defined by (4) and the sensing operator C is given by

$$C\varphi := \begin{bmatrix} \displaystyle\int_0^1 \delta(x-1)\varphi_1(x)dx \\ \displaystyle\int_0^1 \delta(x-1)\varphi_2(x)dx \end{bmatrix}, \quad \varphi = [\varphi_1, \varphi_2]^T.$$

In the above the symbol $\delta(\cdot - 1)$ denotes the Dirac delta function at $x = 1$.

As stated in Section 2, the operator A generates an analytic C_0-semigroup $T(t)$ on $X = [L^2(0,1)]^2$. By Lemma 2.2, the output $y(t)$ of system (13), (14) is given by

$$y(t) = CT(t)\varphi.$$

Definition 3.1. *System* (13), (14) *is said to be observable on $J = [t_0, t_1] \subset [0,\infty)$, $t_0 < t_1$ if and only if*

$$y(s) = CT(s)\varphi = 0, \ \ a.e.\ s \in J \implies \varphi = 0.$$

The observability condition on $J = [t_0, t_1]$ is equivalent to the condition $\mathrm{Ker}\, \mathcal{C}_J = \{[0,0]^T\}$, where the observability operator $\mathcal{C}_J : X \to L^2(t_0, t_1; \mathbf{R}^2)$ is defined by

$$\text{(15)} \qquad \mathcal{C}_J[\varphi_1, \varphi_2]^T(s) = CT(s)[\varphi_1, \varphi_2]^T, \quad a.e.\ s \in J = [t_0, t_1], \quad \forall [\varphi_1, \varphi_2]^T \in X.$$

Theorem 3.1. *System* (13)*,* (14) *is observable on any interval* $J = [t_0, t_1] \subset [0, \infty)$.

Proof. By (11), (12), (14) and (15), we can deduce that the condition $[\varphi_1, \varphi_2]^T \in \mathrm{Ker}\, \mathcal{C}_J$ is equivalent to the following two conditions

$$\text{(16)} \qquad \sum_{n=1}^{\infty} e^{-(\lambda_n^0 + h_0)s} \langle h_1(\varphi_1 - \varphi_2), \psi_n \rangle \phi_n(1) + \sum_{n=1}^{\infty} e^{-\lambda_n^0 s} \langle h_2\varphi_1 + h_1\varphi_2, \psi_n \rangle \phi_n(1) = 0,$$

$$\text{(17)} \qquad \sum_{n=1}^{\infty} e^{-(\lambda_n^0 + h_0)s} \langle -h_2(\varphi_1 - \varphi_2), \psi_n \rangle \phi_n(1) + \sum_{n=1}^{\infty} e^{-\lambda_n^0 s} \langle h_2\varphi_1 + h_1\varphi_2, \psi_n \rangle \phi_n(1) = 0, \quad a.e.\ s \in J = [t_0, t_1].$$

Since all series involved in (16), (17) are analytic in $s > 0$, then the conditions (16), (17) are equivalent to

$$\text{(18)} \qquad \sum_{n=1}^{\infty} e^{-(\lambda_n^0 + h_0)t} \langle h_1(\varphi_1 - \varphi_2), \psi_n \rangle \phi_n(1) + \sum_{n=1}^{\infty} e^{-\lambda_n^0 t} \langle h_2\varphi_1 + h_1\varphi_2, \psi_n \rangle \phi_n(1) = 0,$$

$$\text{(19)} \qquad \sum_{n=1}^{\infty} e^{-(\lambda_n^0 + h_0)t} \langle -h_2(\varphi_1 - \varphi_2), \psi_n \rangle \phi_n(1) + \sum_{n=1}^{\infty} e^{-\lambda_n^0 t} \langle h_2\varphi_1 + h_1\varphi_2, \psi_n \rangle \phi_n(1) = 0, \quad \forall t > 0.$$

Subtracting (18) from (19) and dividing by $h_1 + h_2$, we obtain

$$\text{(20)} \quad \sum_{n=1}^{\infty} e^{-(\lambda_n^0 + h_0)t} \langle \varphi_1 - \varphi_2, \psi_n \rangle \phi_n(1) = 0, \quad \forall t > 0.$$

So that by substituting (20) for (18), we have

$$\text{(21)} \quad \sum_{n=1}^{\infty} e^{-\lambda_n^0 t} \langle h_2\varphi_1 + h_1\varphi_2, \psi_n \rangle \phi_n(1) = 0, \quad \forall t > 0.$$

Here we prepare the following lemma on Dirichlet series (for a proof see [6]).

Lemma 3.1. *Let* $\{c_n\}_{n \geq 1}$ *be a bounded sequence and let* $\{\mu_n\}_{n \geq 1}$ *be a strictly increasing sequence given by* $\mu_n = \lambda_n^0$ *or* $\mu_n = \lambda_n^0 + h_0$*,* $\forall n \geq 1$*. If* $\sum_{n=1}^{\infty} c_n e^{-\mu_n t} = 0$*,* $\forall t > 0$*, then* $c_n = 0$*,* $\forall n \geq 1$*.*

Now we can give a Proof of Theorem 3.1. Let $[\varphi_1, \varphi_2]^T \in \mathrm{Ker}\, \mathcal{C}_J$. Then by Lemma 3.1, it follows from (20) and (21) that

$$\text{(22)} \qquad \langle \varphi_1 - \varphi_2, \psi_n \rangle \phi_n(1) = 0, \quad \forall n \geq 1,$$

$$\text{(23)} \qquad \langle h_2\varphi_1 + h_1\varphi_2, \psi_n \rangle \phi_n(1) = 0, \quad \forall n \geq 1.$$

Hence by (22), (23) and $\phi_n(1) \neq 0$, $\forall n \geq 1$, we have

$$\text{(24)} \qquad \langle \varphi_1, \psi_n \rangle = \langle \varphi_2, \psi_n \rangle = 0, \quad \forall n \geq 1.$$

Since the Riesz basis $\{\psi_n : \forall n \geq 1\}$ is complete in $L^2(0, 1)$, from (24) the conclusion $\varphi_1 = \varphi_2 = 0$ follows. Hence $\mathrm{Ker}\, \mathcal{C}_J = \{[0,0]^T\}$. This shows that system (13), (14) is observable on any interval J. □

Next, we study the case where only one output of (1) is observed. At first, we replace the system output equation (2) by

$$\text{(25)} \quad y(t) = \hat{C}z(t) = y_1(t) = z_1(t, 1), \quad t \in (0, \infty),$$

where $\hat{C} : [C[0,1]]^2 \subset X \to \mathbf{R}$ is defined by $\hat{C}[\varphi_1, \varphi_2]^T = \varphi_1(1)$, $\forall [\varphi_1, \varphi_2]^T \in [C[0,1]]^2$. In this case the observability operator $\hat{\mathcal{C}}_J : X \to L^2(t_0, t_1)$ on $J = [t_0, t_1]$ is given by

$$\hat{\mathcal{C}}_J[\varphi_1, \varphi_2]^T(s) = \hat{C}T(s)[\varphi_1, \varphi_2]^T, \quad a.e.\ s \in J.$$

Definition 3.2. (i) *System* (13)*,* (25) *is said to be observable on* $J = [t_0, t_1] \subset [0, \infty)$*,* $t_0 < t_1$ *if and only if*

$$y_1(s) = \hat{C}T(s)\varphi = 0, \ \ a.e.\ s \in J \implies \varphi = 0.$$

(ii) *Unobservable subspace of system* (13)*,* (25) *on* J *is defined by* $\hat{\mathcal{N}}_J = \mathrm{Ker}\, \hat{\mathcal{C}}_J$*.*

We shall give the results on the observability and the characterization of unobservable subspaces for the special cases of (13), (25). To this end, we define the set

$$E = \{(n, m) \in \mathbf{N} \times \mathbf{N} : s_n^2 - s_m^2 = 4D(h_1 + h_2),\ n > m\}$$

and introduce the following condition

(H): $E = \emptyset$ (empty set).

The set E depends only on the constants D, α, h_1, h_2. It is verified by the asymptotics of $\{s_n\}$ in (3) that E is an empty set or a finite set.

If the condition (H) is satisfied, then for any couple of natural numbers (n, m), $n > m$ we have

$$(\lambda_m^0 + h_0) - \lambda_n^0 = \frac{1}{4D}(s_m^2 - s_n^2) + (h_1 + h_2) \neq 0.$$

Remark 3.1. *By the asymptotics of* $\{s_n\}$, *there exists a sufficiently small* $\epsilon_0 > 0$ *such that* $s_{n+1}^2 - s_n^2 \geq \epsilon_0$, $\forall n \geq 1$. *From this fact, it follows that the condition* (H), *i.e.,* $E = \emptyset$ *is satisfied if* $4D(h_1 + h_2) < \epsilon_0$.

From Lemma 3.1, we can easily prove the following lemma.

Lemma 3.2. *Let both* $\{c_n^1\}_{n\geq 1}$ *and* $\{c_n^2\}_{n\geq 1}$ *be bounded sequences. If the assumption* (H) *is satisfied, then*

$$\sum_{n=1}^{\infty} \left(c_n^1 e^{-\lambda_n^0 t} + c_n^2 e^{-(\lambda_n^0 + h_0)t} \right) = 0, \quad \forall t > 0$$

implies $c_n^1 = c_n^2 = 0$, $\forall n \geq 1$, *where* $h_0 = h_1 + h_2 > 0$.

Theorem 3.2. *Assume* (H). *Then system* (13), (25) *is observable on any interval* $J = [t_0, t_1]$.

Proof. The observability of system (13), (25) on $J = [t_0, t_1]$ is equivalent to the condition that $\mathrm{Ker}\, \hat{\mathcal{C}}_J = \{[0, 0]^T\}$. Let $\varphi = [\varphi_1, \varphi_2]^T \in \mathrm{Ker}\, \hat{\mathcal{C}}_J$. Then by (11) and the definition of $\hat{\mathcal{C}}_J$, it follows by the analyticity of Dirichlet series that

$$\text{(26)} \quad \sum_{n=1}^{\infty} e^{-(\lambda_n^0 + h_0)t} \langle h_1(\varphi_1 - \varphi_2), \psi_n \rangle \phi_n(1) + \sum_{n=1}^{\infty} e^{-\lambda_n^0 t} \langle h_2\varphi_1 + h_1\varphi_2, \psi_n \rangle \phi_n(1) = 0, \quad \forall t > 0.$$

Under the assumption (H), we see that $\{\lambda_n^0\}_{n\geq 1} \cap \{\lambda_n^0 + h_0\}_{n\geq 1} = \emptyset$, so that by using $\phi_n(1) \neq 0$, $\forall n \geq 1$ and applying Lemma 3.2 to (26), we have

$$\text{(27)} \quad \langle h_1(\varphi_1 - \varphi_2), \psi_n \rangle = 0, \quad \langle h_2\varphi_1 + h_1\varphi_2, \psi_n \rangle = 0, \quad \forall n \geq 1.$$

Then from (27) we can conclude

$$\langle \varphi_1, \psi_n \rangle = \langle \varphi_2, \psi_n \rangle = 0, \quad \forall n \geq 1.$$

From this we have $\varphi_1 = \varphi_2 = 0$, i.e., $\mathrm{Ker}\, \hat{\mathcal{C}}_J = \{[0, 0]^T\}$. This completes the proof. □

Next we consider the case where the assumption (H) is not satisfied, i.e., E is given by

$$\text{(28)} \quad E = \{(n_1, m_1), \cdots, (n_l, m_l) : n_1 < \cdots < n_l,\ n_j > m_j,\ j = 1, \cdots, l\}.$$

In this case we introduce the following sets:

$$M_l = \{m_1, \cdots, m_l\}, \quad N_l = \{n_1, \cdots, n_l\}.$$

In the case of $E \neq \emptyset$, we need the following lemma (for a proof see [6]).

Lemma 3.3. *Let both* $\{c_n^1\}_{n\geq 1}$ *and* $\{c_n^2\}_{n\geq 1}$ *be bounded sequences. Assume that the set* E *is given by* (28). *Then the equality*

$$\sum_{n=1}^{\infty} \left(c_n^1 e^{-\lambda_n^0 t} + c_n^2 e^{-(\lambda_n^0 + h_0)t} \right) = 0, \quad \forall t > 0$$

implies

$$\text{(29)} \quad c_n^1 = 0, \quad n \in \mathbf{N} \backslash N_l,$$

$$\text{(30)} \quad c_n^2 = 0, \quad n \in \mathbf{N} \backslash M_l,$$

$$\text{(31)} \quad c_{n_j}^1 + c_{m_j}^2 = 0, \quad j = 1, \cdots, l.$$

Theorem 3.3. *Consider system* (13), (25). *Suppose that the condition* (H) *is not satisfied and the set* E *is given by* (28). *Then if the condition*

$$\text{(32)} \quad \{n_j :\ 1 \leq j \leq l\} \cap \{m_j :\ 1 \leq j \leq l\} = \emptyset$$

is satisfied, the unobservable subspace of system (13), (25) *on* $J = [t_0, t_1]$ *is independent of* t_0, t_1 *and is given by*

$$\hat{\mathcal{N}}_J = \mathrm{Span} \left\{ \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} \in X :\ \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} \phi_{n_j} - \dfrac{\phi_{n_j}(1)}{\phi_{m_j}(1)} \phi_{m_j} \\ \phi_{n_j} + \dfrac{h_2}{h_1} \dfrac{\phi_{n_j}(1)}{\phi_{m_j}(1)} \phi_{m_j} \end{bmatrix}, \quad 1 \leq j \leq l \right\}.$$

Proof. Let $[\varphi_1, \varphi_2]^T \in \hat{\mathcal{N}}_J = \mathrm{Ker}\, \hat{\mathcal{C}}_J$. Then as in the Proof of Theorem 3.2, we can verify the equality (26). Applying Lemma 3.3 to (26), we have from (29) and (30)

$$\text{(33)} \quad \langle \varphi_1, \psi_n \rangle - \langle \varphi_2, \psi_n \rangle = 0, \quad \forall n \in \mathbf{N} \backslash M_l,$$

$$\text{(34)} \quad h_2 \langle \varphi_1, \psi_n \rangle + h_1 \langle \varphi_2, \psi_n \rangle = 0, \quad \forall n \in \mathbf{N} \backslash N_l.$$

Hence from (33) and (34) it follows that

$$\langle \varphi_1, \psi_n \rangle = \langle \varphi_2, \psi_n \rangle = 0,\ \forall n \in \mathbf{N} \backslash (N_l \cup M_l).$$

This implies by (33), (34) and the condition (32) that φ_1 and φ_2 are represented as

$$\text{(35)} \quad \varphi_1 = \sum_{j=1}^{l} \Big(\langle \varphi_1, \psi_{n_j} \rangle \phi_{n_j} + \langle \varphi_1, \psi_{m_j} \rangle \phi_{m_j} \Big),$$

$$\text{(36)} \quad \varphi_2 = \sum_{j=1}^{l} \Big(\langle \varphi_1, \psi_{n_j} \rangle \phi_{n_j} - \frac{h_2}{h_1} \langle \varphi_1, \psi_{m_j} \rangle \phi_{m_j} \Big).$$

From the equality (31), we have

$$\langle h_2\varphi_1 + h_1\varphi_2, \psi_{n_j}\rangle\phi_{n_j}(1) + \langle h_1(\varphi_1 - \varphi_2), \psi_{m_j}\rangle\phi_{m_j}(1) = 0, \quad j = 1, \cdots, l,$$

so that again by (33) and (34),

$$\langle (h_2 + h_1)\varphi_1, \psi_{n_j}\rangle\phi_{n_j}(1) + \langle (h_1 + h_2)\varphi_1, \psi_{m_j}\rangle\phi_{m_j}(1) = 0, \quad j = 1, \cdots, l.$$

Therefore, by deviding $h_1 + h_2 \neq 0$ we have

$$\text{(37)} \quad \langle \varphi_1, \psi_{m_j}\rangle = -\frac{\phi_{n_j}(1)}{\phi_{m_j}(1)}\langle \varphi_1, \psi_{n_j}\rangle, \quad j = 1, \cdots, l.$$

Since the constants $C_j = \langle \varphi_1, \psi_{n_j}\rangle$ can be chosen arbitrarily, by the above equations (35), (36) and (37), we can conclude that any $[\varphi_1, \varphi_2]^T \in \operatorname{Ker} \hat{\mathcal{C}}_J$ can be represented as a linear combination of the following l numbers of independent vectors

$$\begin{bmatrix} \phi_{n_j} - \dfrac{\phi_{n_j}(1)}{\phi_{m_j}(1)}\phi_{m_j} \\ \phi_{n_j} + \dfrac{h_2}{h_1}\dfrac{\phi_{n_j}(1)}{\phi_{m_j}(1)}\phi_{m_j} \end{bmatrix}, \quad 1 \leq j \leq l.$$

This completes the Proof of Theorem 3.3. □

We can give the representation of unobservable subspace $\hat{\mathcal{N}}_J$ for the case of $N_l \cap M_l \neq \emptyset$. However, such the representation becomes much complicated compared with the case of $N_l \cap M_l = \emptyset$ (cf. Sano and Nakagiri [6]). We can verify that $\dim \hat{\mathcal{N}}_J = l$ even for the exceptional case, however, for the economy of pages we omit to give such the characterization for the exceptional case.

Secondly, we study the case where the system output equation (2) is replaced by

$$\text{(38)} \quad y(t) = \tilde{C}z(t) = y_2(t) = z_2(t, 1), \quad t \in (0, \infty),$$

where $\tilde{C} : [C[0,1]]^2 \subset X \to \mathbf{R}$ is defined by $\tilde{C}[\varphi_1, \varphi_2]^T = \varphi_2(1)$, $\forall[\varphi_1, \varphi_2]^T \in [C[0,1]]^2$. In this case the observability operator $\tilde{\mathcal{C}}_J : X \to L^2(t_0, t_1)$ on $J = [t_0, t_1]$ is given by

$$\tilde{\mathcal{C}}_J[\varphi_1, \varphi_2]^T(s) = \tilde{C}T(s)[\varphi_1, \varphi_2]^T, \quad a.e.\ s \in J.$$

Definition 3.3. (i) *System* (13), (38) *is said to be observable on* $J = [t_0, t_1] \subset [0, \infty)$, $t_0 < t_1$ *if and only if*

$$y_2(s) = \tilde{C}T(s)\varphi = 0, \ \ a.e.\ s \in J \implies \varphi = 0.$$

(ii) *Unobservable subspace of system* (13), (38) *on* J *is defined by* $\tilde{\mathcal{N}}_J = \operatorname{Ker} \tilde{\mathcal{C}}_J$.

Similarly as in the Proofs of Theorem 3.2 and Theorem 3.3, we can deduce the following theorems.

Theorem 3.4. *If the condition* (H) *is satisfied, then system* (13), (38) *is observable on any interval* $J = [t_0, t_1]$.

Theorem 3.5. *Consider system* (13), (38). *Suppose that the condition* (H) *is not satisfied and the set* E *is given by* (28). *Then if the condition* (32) *is satisfied, the unobservable subspace of system* (13), (38) *on* $J = [t_0, t_1]$ *is independent of* t_0, t_1 *and is given by*

$$\tilde{\mathcal{N}}_J = \operatorname{Span}\left\{ \begin{bmatrix}\varphi_1 \\ \varphi_2\end{bmatrix} \in X : \begin{bmatrix}\varphi_1 \\ \varphi_2\end{bmatrix} = \begin{bmatrix} \phi_{n_j} + \dfrac{h_1}{h_2}\dfrac{\phi_{n_j}(1)}{\phi_{m_j}(1)}\phi_{m_j} \\ \phi_{n_j} - \dfrac{\phi_{n_j}(1)}{\phi_{m_j}(1)}\phi_{m_j} \end{bmatrix}, \quad 1 \leq j \leq l \right\}.$$

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