Dynamics of gradient flows in the half-transversal Morse theory

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(Communicated by Shigefumi MORI, M.J.A., Dec. 12, 2008)

Abstract: In this note we suggest a construction of the Morse-Novikov theory for a class of non-transversal gradients and generalize to this class the basic results of the classical Morse-Novikov theory including its non-abelian version.

Key words: Morse theory; gradient flow; torsion; Novikov homology; zeta function; the Seiberg-Witten equation.

1. Introduction. First off, we recall the notions of the classical transversal Morse theory (real-valued and circle-valued). Then we proceed to a new branch: the *half-transversal Morse theory* and explain our results; we discuss possible applications to the Seiberg-Witten equations. The detail will be given elsewhere.

2. Transversal Morse theory.

2.1. The real-valued Morse theory. In the classical Morse theory every real-valued Morse function $f : M \to \mathbf{R}$ on a closed manifold M induces a cellular decomposition of M. Namely, if we choose a generic gradient v of f then the stable manifolds of the zeros of v form this cellular decomposition, so that the number of cells of dimension k equals the number $\sharp Crit_k(f)$ of critical points of f of index k. The cellular chain complex $\mathcal{M}_*(f, v)$ (*the Morse complex*) computes the homology of M. Moreover, one can construct a natural chain equivalence

$$\mathcal{M}_*(f, v) \to \Delta_*(M)$$

where $\Delta_*(M)$ is the simplicial chain complex of M. From this point of view the classical Morse theory gives just another way of constructing cellular decompositions of manifolds. This construction can be refined so as to give the *universal Morse complex* $\tilde{\mathcal{M}}_*(f, v)$ which is freely generated over $\mathbf{Z}\pi_1(M)$ by the critical points of f and computes the homology of the universal covering $\tilde{M}$ of M. There is a chain equivalence

$$\tilde{\mathcal{M}}_*(f, v) \xrightarrow{\sim} \Delta_*(\tilde{M}) \qquad (1)$$

which is a simple homotopy equivalence (that is, its torsion vanishes in the group Wh $(\pi_1(M))$).

Remark 2.1. The description above is actually very schematic; we have omitted several technical details; see [15, Chapter 6].

2.2. The Novikov homology. The case of Morse functions $f : M \to S^1$ is quite different, and the schema above can not be applied directly. The stable manifolds of the critical points in general do not form a cellular decomposition of M; the closure of a stable manifold is not necessarily compact, and can be everywhere dense in M. The appropriate generalization of the Morse theory to this case was constructed by S. P. Novikov about 1980. In this theory one associates to each Morse map $f : M \to S^1$ and a transversal f-gradient v the *Novikov complex* $\mathcal{N}_*(f, v)$, which is freely generated over $\mathbf{Z}((t))$ by the set of critical points of f, and we have a canonical chain equivalence

$$\phi : \mathcal{N}_*(f, v) \xrightarrow{\sim} \mathbf{Z}((t)) \underset{\mathbf{Z}[t,t^{-1}]}{\otimes} \Delta_*(\bar{M}) \qquad (2)$$

where $\bar{M}$ is the infinite cyclic covering induced by f from the covering $\mathbf{R} \to S^1$.

2.3. The twisted Novikov homology. Similarly to the case of real-valued Morse functions one can also construct a universal Novikov complex (see [8,12]). The base ring of this complex is a rather complicated algebraic object, namely a special completion of the group ring of the fundamental group, and is very difficult to work with, although it has obvious theoretical advantages. In [5], we have introduced a twisted version of the Novikov homology, which is simpler to compute, and is sufficient for many purposes. Let G denote the fundamental group of M, and let $\rho : G \to GL(n, \mathbf{Z})$ be a right representation of G (that is, $\rho(ab) = \rho(b)\rho(a)$ for

2000 Mathematics Subject Classification. Primary 57M27; Secondary 58E05.

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every $a, b \in G$). The twisted Novikov complex $\mathcal{N}_*(f, v; \rho)$ is a free chain complex over $\widehat{\Lambda} = \mathbf{Z}((t))$ which has $n \cdot \sharp\text{Crit}_k(f)$ generators in the degree k and we have a canonical chain equivalence

(3) $$\phi_\rho : \mathcal{N}_*(f, v; \rho) \xrightarrow{\sim} \widehat{\Lambda}^n \underset{\bar\rho}{\otimes} \Delta_*(\tilde{M})$$

where the tensor product in the right hand side refers to the structure of a right $\mathbf{Z}G$-module on $\widehat{\Lambda}^n$ defined by the representation

$$\bar\rho = \rho \otimes \xi : G \to GL(n, \widehat{\Lambda}).$$

Here ξ is the homotopy class of f in $[M, S^1] \approx \text{Hom}(G, \mathbf{Z})$ which is considered as the following composition of group homomorphisms:

$$G \to \mathbf{Z} = (\mathbf{Z}[\mathbf{Z}])^\times = (\mathbf{Z}[t, t^{-1}])^\times \subset \widehat{\Lambda}^\times = GL(1, \widehat{\Lambda}).$$

2.4. Closed orbits of the gradient flow. The gradient flow of the circle-valued Morse function can have closed orbits (contrarily to the case of real-valued Morse functions). It turns out that the Lefschetz zeta function counting these orbits can be expressed in terms of invariants of the simple homotopy type of the Novikov complex. This was discovered in the work of M. Hutchings and Y. J. Lee [6]. To explain our results let us first recall the notion of *torsion*.

Definition 2.2.

(a) For an acyclic based chain complex C_* over $\widehat{\Lambda} = \mathbf{Z}((t))$ one defines an element

$$\tau(C_*) \in \mathbf{Z}[[t]] = \bar{K}_1(\widehat{\Lambda})/\{t^{\pm 1}\}$$

(the *torsion* of C_*). For the chain complex

$$C_* = \{0 \leftarrow C_1 \xleftarrow{\phi} C_2 \leftarrow 0\}$$

where both C_1, C_2 are isomorphic to $\widehat{\Lambda}$ and the homomorphism ϕ is the multiplication by a unit α of the ring $\mathbf{Z}[[t]]$ we have $\tau(C_*) = \alpha$.

(b) For a chain equivalence $\phi : C_* \to D_*$ of two based chain complexes over $\widehat{\Lambda}$ one defines the torsion $\tau(\phi)$ as the torsion of the chain cone $Cone_*(\phi)$.

We have the following result [13,14] concerning the torsion of the canonical chain equivalence from Subsection 2.2 (valid for generic gradients v of the circle-valued function $f : M \to S^1$):

(4) $$\tau(\phi) = \prod_{PrCl(-v)} \left(1 - \nu(\gamma) t^{n(\gamma)}\right)^{-\mu(\gamma)} \in \mathbf{Z}[[t]].$$

The right hand side is known as *the Lefschetz zeta function of* $(-v)$ and denoted by $\zeta_{(-v)}$; here the product is over the set $PrCl(-v)$ of all prime closed orbits γ of $(-v)$, and $\nu(\gamma), \mu(\gamma) = \pm 1, n(\gamma) \in \mathbf{Z}$ are certain dynamical invariants of a closed orbit γ (see [15], Ch. 13 for details). For the case when the chain complex $\mathcal{N}_*(f, v) \otimes \mathbf{Q}$ is acyclic, this formula is equivalent to the Hutchings-Lee formula from [6]. If the Morse function f has no critical points the formula above is equivalent to the classical Milnor's formula [11] relating the torsion of the mapping torus of a continuous map and the Lefschetz zeta function of the map.

There is a generalization of the formula (4) to the case of the twisted Novikov homology. This is the first main result of this note:

Theorem 2.3. *Let $f : M \to S^1$ be a Morse-Novikov map on a closed manifold M, and v a transversal f-gradient. Then*

(5) $$\tau(\phi_\rho) = \prod_{PrCl(-v)} \det(1 - \nu(\gamma)\bar\rho([\gamma]))^{-\mu(\gamma)} \in \mathbf{Z}[[t]].$$

Here $\tau(\phi_\rho)$ is the torsion of the canonical chain equivalence (3). The right hand side of this formula is known as the *twisted zeta function* of the flow; we will denote it by $\zeta_{(-v,\rho)}$. It was introduced in the work of B. Jiang and S. Wang [7], where the authors studied the twisted invariants of maps of manifolds. Any such map gives rise to a circle-valued Morse function without critical points on the mapping torus of the map, so that the formula above can be considered as a generalization of their result.

3. Seiberg-Witten invariants and symmetric flows. Let M be a closed oriented 3-manifold. The Meng-Taubes theory [10] implies that if $b_1(M) > 0$, then the Seiberg-Witten invariant of M is essentially the Milnor-Turaev torsion of M. M. Hutchings and Y. J. Lee developed another approach to the Meng-Taubes theorem, which is based on the Morse-Novikov theory. To explain briefly their idea, recall that the Seiberg-Witten theory associates to every $Spin^{\mathbf{C}}$- structure α an integer $SW(\alpha)$, which is defined via counting of solutions of the Seiberg-Witten equations (see [16]). The Meng-Taubes theorem implies

(6) $$\sum_\alpha SW(\alpha) t^{\langle \alpha, \xi \rangle} = \tau(\bar{M}_\xi)$$

where ξ is a non-zero element of $H^1(M, \mathbf{Z})$, $\bar{M}_\xi$ is the infinite cyclic covering corresponding to this cohomology class, and $\langle \alpha, \xi \rangle$ stands for $\frac{1}{2} \langle c_1(\det \alpha) \cdot \xi, M \rangle$.

The Hutchings-Lee theorem says that

$$\tau(\mathcal{N}_*(f,v))\cdot\zeta_{(-v)}=\tau(\bar{M}_\xi) \tag{7}$$

where $\mathcal{N}_*(f,v)$ is the Novikov complex corresponding to any Morse function $f:M\to S^1$ in the homotopy class ξ and an f-gradient v. An alternative proof of the equality

$$\tau(\mathcal{N}_*(f,v))\cdot\zeta_{(-v)}=\sum_\alpha SW(\alpha)t^{\langle\alpha,\xi\rangle} \tag{8}$$

would imply the Meng-Taubes theorem. This project was realized in the paper [9], where T. Mark gives the proof of (8) along the lines of the TQFT for Seiberg-Witten equations developed by S. Donaldson (see [1]).

For the proof T. Mark introduces a new class of gradients of circle-valued Morse functions: the *symmetric gradients*. Let $f:W\to[-1,1]$ be a Morse function on a 3-dimensional cobordism between ∂_0W and ∂_1W, and v be an f-gradient. The pair (f,v) is called *symmetric* if f does not have critical points of indices 0 and 3, all the critical points of a given index i lie on the same critical level a_i, and there is an involution $I:W\to W$ such that

$$I(\partial_1W)=\partial_0W,\quad I\circ f=-f,\quad I_*(v)=-v.$$

(This definition is equivalent to T. Mark's one although differs slightly from it.) Observe that such gradients are necessarily non-transversal. A pair (f,v) where $f:M\to S^1$ and v is an f-gradient is called *symmetric* if there is a regular value λ of f such that cutting M along the regular surface $f^{-1}(\lambda)$ gives a symmetric Morse pair on the resulting cobordism W. If the descending discs of critical points of f of index 2 are transversal to the ascending discs of critical points of index 1, then the usual Morse-Novikov procedure of counting the flow lines joining the critical points determines a homomorphism

$$\mathcal{N}_2\xrightarrow{d}\mathcal{N}_1$$

where $\mathcal{N}_i$ is the free module over $\mathbf{Z}((t))$ generated by the critical points of f of index i. T. Mark proves the following theorem:

$$\det(d)\cdot\zeta_{(-v)}=\tau(\bar{M}_\xi). \tag{9}$$

4. Half-transversal Morse theory. The results of Section 2 rely heavily on the transversality condition for the gradient flow. Section 3 suggests that the analogs of these results can still hold for some classes of non-transversal gradients, like the symmetric ones. This is indeed the case; in this and the following sections we will announce the basic results of the Morse theory for the half-transversal gradients.

We will consider only 3-dimensional manifolds here, although the theory can be extended to manifolds of any dimension. For a Morse function $f:W\to[a,b]$ on a 3-dimensional cobordism and an f-gradient v let $(-v)^{\leadsto}$ denote the gradient descent map from the upper boundary to the lower boundary. If there are no critical points of indices 0 and 3 this map is defined on an open and dense subset of ∂_1W and maps it diffeomorphically to an open and dense subset of ∂_0W.

Definition 4.1. Let $f:W\to[a,b]$ be a Morse function on a 3-dimensional cobordism W. We say that (f,v) is a *smooth descent Morse pair* if

(a) f does not have critical points of indices 0 or 3.
(b) We have $Crit_2(f)=\{q_1,\dots,q_k\}$, $Crit_1(f)=\{p_1,\dots,p_k\}$; with $f(p_i)>f(q_j)$ for every i, j, and there are two flow lines joining p_i and q_i for every i.
(c) The map $(-v)^{\leadsto}$ can be extended to a diffeomorphism of ∂_1W onto ∂_0W.

Observe that if v is a *transversal* gradient, and the set of critical points is non-empty, the map $(-v)^{\leadsto}$ can never be extended to a diffeomorphism $\partial_1W\to\partial_0W$.

Definition 4.2. Let $f:M\to S^1$ be a Morse function on a closed 3-dimensional manifold M, and v be an f-gradient. We say that v is *half-transversal* if

(a) There is a regular value λ of f such that cutting M along the regular level surface $f^{-1}(\lambda)$ we obtain a smooth descent Morse pair on the corresponding cobordism.
(b) All stable and unstable manifolds of dimension 2 of v are transversal to each other.

We shall also say that (f,v) is a *half-transversal Morse pair*. Observe that in the half-transversal case there is a well-defined monodromy map

$$H:f^{-1}(\lambda)\xrightarrow{\approx}f^{-1}(\lambda),$$

derived from the gradient descent $(-v)^{\leadsto}$. This simplifies the computations for the half-transversal case as compared with the transversal one.

One can define the Novikov complex for the half-transversal gradients. There are only two non-zero terms, namely $\mathcal{N}_1$ and $\mathcal{N}_2$ and the boundary

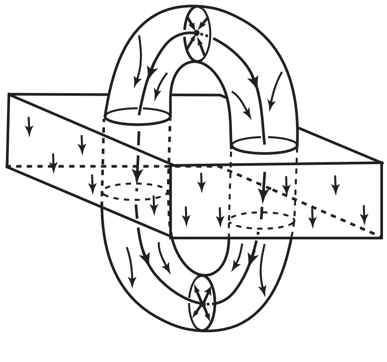

Fig. 1

operator from $\mathcal{N}_2$ to $\mathcal{N}_1$ is defined by the usual counting procedure of flow lines (which works here since the stable manifolds of the critical points of index 2 are transversal to the unstable manifolds of the critical points of index 1). The computational advantage of the half-transversal case over the transversal one, is that both the Novikov complex and the Lefschetz zeta function are computable in terms of the monodromy diffeomorphism H. We proved the next theorem in [4].

Theorem 4.3 [4]. *Let M be a 3-dimensional closed manifold, and (f, v) be a half-transversal Morse pair on M. Then there is a chain homotopy equivalence*

$$\phi : \mathcal{N}_*(f, v) \xrightarrow{\sim} \mathbf{Z}((t)) \underset{\mathbf{Z}[t,t^{-1}]}{\otimes} \Delta_*(\bar{M}) \tag{10}$$

such that its torsion equals the zeta function of the gradient flow:

$$\tau(\phi) = \prod_{PrCl(-v)} \left(1 - \nu(\gamma)t^{n(\gamma)}\right)^{-\mu(\gamma)} \in \mathbf{Z}[[t]]. \tag{11}$$

The proof proceeds by a direct construction of a cellular decomposition of the infinite cyclic covering of the manifold in question. Observe that T. Mark's approximation methods used in the proof of (9) are not sufficient here.

Our method allows also to generalize the results to the twisted case. This is the second main result of the present paper: the twisted Novikov complex for half-transversal gradients is chain equivalent to the chain complex computing the twisted Novikov homology, and the torsion of the chain equivalence.

Theorem 4.4. *In the assumptions of the theorem* 4.3*, let ρ be a right representation $\pi_1(M) \to GL(n, \mathbf{Z})$. Then there is a chain equivalence*

$$\phi_\rho : \mathcal{N}_*(f, v; \rho) \xrightarrow{\sim} \widehat{\Lambda}^n \underset{\bar{\rho}}{\otimes} \Delta_*(\tilde{M}) \tag{12}$$

such that its torsion equals the zeta function of the gradient flow:

$$\tau(\phi_\rho) = \prod_{PrCl(-v)} \det(1 - \nu(\gamma)\bar{\rho}([\gamma]))^{-\mu(\gamma)} \in \mathbf{Z}[[t]]. \tag{13}$$

5. Complements of knots. The corresponding analogs of the theorems of the previous sections hold also for the manifolds with boundary. One of the most interesting cases here is the case of complements of knots. For a knot K in the 3-sphere let $M = S^3 \setminus \operatorname{Int} N(K)$ be the complement to the open tubular neighborhood of the knot. Let $f : M \to S^1$ be a Morse function such that its restriction to the boundary of $N(K)$ is the fibration corresponding to the trivialization of the normal bundle to K. In this case the torsion of the infinite cyclic covering is easily computed from the Alexander polynomial of the knot; we have the following formula:

$$\tau(\mathcal{N}_*) \cdot \zeta_{(-v)} = \frac{A(t)}{1-t},$$

where $A(t)$ is the Alexander polynomial of the knot, and $\zeta_{(-v)}$ is the zeta function of the flow $(-v)$.

It is difficult to compute the Novikov complex and the zeta function for gradient flows, and in the transversal case there are not much examples available. In the half-transversal case the computation becomes possible using some techniques of Heegaard splittings of 3-manifolds [2,3]. For the twisted knots K_{2n-1} we have a half-transversal Morse pair (f, v) which satisfies:

$$\zeta_{(-v)} = (1-t)^3, \quad \tau(\mathcal{N}_*) = \frac{-n + (2n-1)t - nt^2}{(1-t)^4}.$$

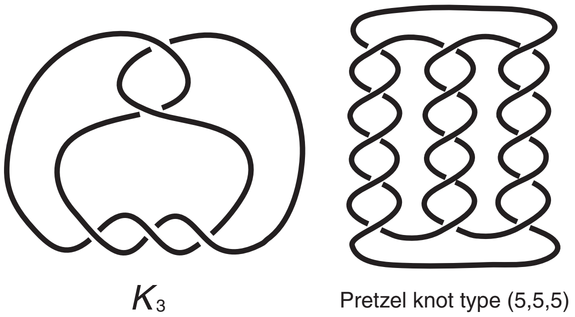

Fig. 2

Similarly, for the pretzel knot of type $(5,5,5)$ there exists a half-transversal pair (f,v) with

$$\zeta_{(-v)} = (1-t)^5, \quad \tau(\mathcal{N}_*) = \frac{19-37t+19t^2}{(1-t)^6}.$$

See [4] for the detail. We are computing the twisted invariants for these knots (in preparation).

6. Concluding remarks: relations with the Seiberg-Witten invariants. The work of T. Mark was motivated by the relation of the circle-valued Morse theory with the Seiberg-Witten invariants of 3-manifolds and the Meng-Taubes theorem [10].

Our results provide in particular the twisted version of (7) and indicate therefore that there should be a twisted version of the formula (6). We hope that further study of the twisted dynamical zeta functions of gradient flows will contribute to the realization of this project.

Acknowledgement. We thank the referee for valuable comments and suggestions.

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