A note on Newton's method for stochastic differential equations and its error estimate

By Kazuo AMANO

Department of Mathematics, Faculty of Engineering, Gunma University, Kiryu 376-8515, Japan

(Communicated by Heisuke HIRONAKA, M.J.A., Feb. 12, 2009)

Abstract: Kawabata and Yamada [3] proposed an *implicit* Newton method for stochastic differential equations and proved its convergence. They proved an error estimate in a sufficiently small time interval and extended it to a global convergence theorem by using open-closed method. In this note, the author gives an *explicit* Newton scheme which is equivalent to Kawabata-Yamada's *implicit* formulation (Remark 1) and prove its direct error estimate (Theorem 2.1). His result could provide a key to solve the open problem of second order convergence (see Remark of Theorem 2.1 and [2]).

Key words: Newton's method for stochastic differential equations; error estimate.

1. Prelinimaries. Let $a(t,x)$ and $b(t,x)$ be real-valued bounded C^1 smooth functions defined in $\mathbf{R}^2$. For the sake of simplicity, we assume that there exist nonnegative constants A_1 and B_1 satisfying

$$\left|\frac{\partial a}{\partial x}(t,x)\right| \le A_1,\ \left|\frac{\partial b}{\partial x}(t,x)\right| \le B_1$$

in $\mathbf{R}^2$. Let $\xi(t)$, $t \ge 0$ be a solution of the initial value problem for stochastic differential equation

$$(1) \quad d\xi(t) = a(t,\xi(t))\,dt + b(t,\xi(t))\,dw(t),\ \xi(0) = \xi_0,$$

where $w(t)$ denotes a standard Brownian motion and ξ_0 is a bounded random variable independent of $\mathcal{F}(w(s), s \ge 0)$. Without loss of generality, we assume that $\xi(t)$ is continuous with respect to $t \ge 0$. Our explicit Newton scheme for the problem (1) is formulated as follows: We define a sequence $\{\xi_n(t)\}$ by $\xi_0(t) = \xi_0$ and

$$\xi_{n+1}(t) = e^{\eta_n(t)}\left(\xi_0 + \int_0^t (a_{0,n}(s) - b_{0,n}(s)\,b_{1,n}(s))\,e^{-\eta_n(s)}ds + \int_0^t b_{0,n}(s)\,e^{-\eta_n(s)}dw(s)\right)$$

for $n = 0, 1, 2, \cdots$, where

$$\eta_n(t) = \int_0^t \left(a_{1,n}(s) - \frac{1}{2}\,(b_{1,n}(s))^2\right) ds + \int_0^t b_{1,n}(s)\,dw(s),$$

$$a_{0,n}(t) = a(t,\xi_n(t)) - \frac{\partial a}{\partial x}(t,\xi_n(t))\,\xi_n(t),$$

$$a_{1,n}(t) = \frac{\partial a}{\partial x}(t,\xi_n(t)),$$

$$b_{0,n}(t) = b(t,\xi_n(t)) - \frac{\partial b}{\partial x}(t,\xi_n(t))\,\xi_n(t),$$

$$b_{1,n}(t) = \frac{\partial b}{\partial x}(t,\xi_n(t)).$$

Though the above definition of $\{\xi_n(t)\}$ may look strange, it is actually a natural fomulation of Newton's method. In fact, each $\xi_n(t)$ is a solution of the linearized equation of (1) (see Lemma 1.1 and Remark 1).

Lemma 1.1. *Assume that* $a_i(t)$, $b_i(t)$, $i = 0, 1$ *are bounded continuous nonanticipative functions defined in* $[0,\infty)$. *Then, the initial value problem for linear stochastic differential equation*

$$d\xi(t) = (a_0(t) + a_1(t)\,\xi(t))\,dt + (b_0(t) + b_1(t)\,\xi(t))\,dw(t),\ \xi(0) = \xi_0$$

has an explicit solution

$$\xi(t) = e^{\eta(t)}\left(\xi_0 + \int_0^t (a_0(s) - b_0(s)b_1(s))\,e^{-\eta(s)}\,ds + \int_0^t b_0(s)\,e^{-\eta(s)}\,dw(s)\right),$$

where

2000 Mathematics Subject Classification. Primary 60H10, 60H35.

doi: 10.3792/pjaa.85.19

$$\eta(t) = \int_0^t \left(a_1(s) - \frac{1}{2}\, b_1^2(s) \right) ds + \int_0^t b_1(s)\, dw(s).$$

Lemma 1.1 follows immediately from Itô's formula.

Remark 1. *It follows immediately from the definition of $\xi_{n+1}(t)$ and Lemma 1.1 that $\xi_{n+1}(0) = \xi_0$ and*

$$\begin{aligned} d\xi_{n+1}(t) = {} & (a_{0,n}(t) + a_{1,n}(t)\, \xi_{n+1}(t))\, dt \\ & + (b_{0,n}(t) + b_{1,n}(t)\, \xi_{n+1}(t))\, dw(t) \end{aligned}$$

for $n = 0, 1, 2, \cdots$. Therefore, $\{\xi_n(t)\}$ is exactly the same sequence introduced by Kawabata and Yamada [3].

The approximation errors

$$\varepsilon_n(t) = \xi_n(t) - \xi(t), \ n = 0, 1, 2, \cdots$$

satisfy the following linear stochastic differential equations.

Lemma 1.2. *For any $n = 0, 1, 2, \cdots$, $\varepsilon_{n+1}(t)$ is a solution of the linear stochastic differential equation*

$$\begin{aligned} d\varepsilon_{n+1}(t) = {} & (\alpha_{0,n}(t) + a_{1,n}(t)\, \varepsilon_{n+1}(t))dt \\ & + (\beta_{0,n}(t) + b_{1,n}(t)\, \varepsilon_{n+1}(t))dw(t) \end{aligned}$$

satisfying the initial condition $\varepsilon_{n+1}(0) = 0$, where

$$\begin{aligned} \alpha_{0,n}(t) &= a(t, \xi_n(t)) - \frac{\partial a}{\partial x}(t, \xi_n(t))\, \varepsilon_n(t) \\ &\quad - a(t, \xi_n(t) - \varepsilon_n(t)), \\ \beta_{0,n}(t) &= b(t, \xi_n(t)) - \frac{\partial b}{\partial x}(t, \xi_n(t))\, \varepsilon_n(t) \\ &\quad - b(t, \xi_n(t) - \varepsilon_n(t)). \end{aligned}$$

Proof. Since, by Remark 1, $\xi_{n+1}(t)$ is a solution of the linear stochastic differential equation

$$\begin{aligned} d\xi_{n+1}(t) = {} & (a_{0,n}(t) + a_{1,n}(t)\, \xi_{n+1}(t))\, dt \\ & + (b_{0,n}(t) + b_{1,n}(t)\, \xi_{n+1}(t))\, dw(t), \end{aligned}$$

we have

$$\begin{aligned} & d\,\varepsilon_{n+1}(t) \\ & = d\,\xi_{n+1}(t) - d\,\xi(t) \\ & = \Big(a(t, \xi_n(t)) - \frac{\partial a}{\partial x}(t, \xi_n(t))\, \xi_n(t) \\ & \qquad - a(t, \xi(t)) + a_{1,n}(t)\, \xi_{n+1}(t) \Big)\, dt \\ & \quad + \Big(b(t, \xi_n(t)) - \frac{\partial b}{\partial x}(t, \xi_n(t))\, \xi_n(t) \\ & \qquad - b(t, \xi(t)) + b_{1,n}(t)\, \xi_{n+1}(t) \Big)\, dw(t) \\ & = \Big(a(t, \xi_n(t)) - \frac{\partial a}{\partial x}(t, \xi_n(t))\, \varepsilon_n(t) \\ & \qquad - a(t, \xi_n(t) - \varepsilon_n(t)) + a_{1,n}(t)\, \varepsilon_{n+1}(t) \Big)\, dt \\ & \quad + \Big(b(t, \xi_n(t)) - \frac{\partial b}{\partial x}(t, \xi_n(t))\, \varepsilon_n(t) \\ & \qquad - b(t, \xi_n(t) - \varepsilon_n(t)) + b_{1,n}(t)\, \varepsilon_{n+1}(t) \Big)\, dw(t) \\ & = (\alpha_{0,n}(t) + a_{1,n}(t)\, \varepsilon_{n+1}(t))\, dt \\ & \quad + (\beta_{0,n}(t) + b_{1,n}(t)\, \varepsilon_{n+1}(t))\, dw(t). \end{aligned}$$

□

A version of Gronwall's inequality plays an important role to estimate the error $\varepsilon_{n+1}(t)$ (see also [1]).

Lemma 1.3. *For any constant $T > 0$, if $f(t)$ and $g(t)$ are continuous nonnegative functions defined in a closed interval $[0, T]$ and if there exist nonnegative constants C_1 and C_2 such that*

$$f(t) \le C_1 \int_0^t g(s)\, ds + C_2 \int_0^t f(s)\, ds$$

for any $0 \le t \le T$, then we have

$$f(t) \le C_1 e^{C_2 t} \int_0^t g(s)\, ds$$

for all $0 \le t \le T$.

2. Theorem and its proof.

Theorem 2.1. *For any $T > 0$, there exists a nonnegative constant C depending only on T, A_1 and B_1 such that*

$$\sup_{0 \le t \le T} E\, \varepsilon_n^2(t) \le \frac{C^n}{n!} \sup_{0 \le t \le T} E\, \varepsilon_0^2(t) \tag{2}$$

for $n = 1, 2, 3, \cdots$.

Proof. For any $n = 0, 1, 2, \cdots$ and $0 \le t \le T$, Lemma 1.2 shows, by Minkowsky's and Hölder's inequalities and a fundamental property of stochastic integrals, that

$$\begin{aligned} & \{E\varepsilon_{n+1}^2(t)\}^{1/2} \\ & \le \left\{ E\left(\int_0^t (\alpha_{0,n}(s) + a_{1,n}(s)\, \varepsilon_{n+1}(s))\, ds \right)^2 \right\}^{1/2} \\ & \quad + \left\{ E\left(\int_0^t (\beta_{0,n}(s) + b_{1,n}(s)\, \varepsilon_{n+1}(s))\, dw(s) \right)^2 \right\}^{1/2} \end{aligned}$$

$$\le \left\{2E\left(\int_0^t \alpha_{0,n}(s)\,ds\right)^2 + 2E\left(\int_0^t a_{1,n}(s)\,\varepsilon_{n+1}(s)\,ds\right)^2\right\}^{1/2} + \left\{2E\left(\int_0^t \beta_{0,n}^2(s)\,ds\right) + 2E\left(\int_0^t b_{1,n}^2(s)\,\varepsilon_{n+1}^2(s)\,ds\right)\right\}^{1/2}$$
$$\le \left\{2TE\left(\int_0^t \alpha_{0,n}^2(s)\,ds\right) + 2E\left(\left(\int_0^t a_{1,n}^2(s)\,ds\right)\left(\int_0^t \varepsilon_{n+1}^2(s)\,ds\right)\right)\right\}^{1/2} + \left\{2E\left(\int_0^t \beta_{0,n}^2(s)\,ds\right) + 2E\left(\int_0^t b_{1,n}^2(s)\,\varepsilon_{n+1}^2(s)\,ds\right)\right\}^{1/2}.$$

Since

(3) $|\alpha_{0,n}(t)| \le 2A_1|\varepsilon_n(t)|,\ |\beta_{0,n}(t)| \le 2B_1|\varepsilon_n(t)|,$

the above result gives

$$\{E\varepsilon_{n+1}^2(t)\}^{1/2} \le \left\{8A_1^2TE\left(\int_0^t \varepsilon_n^2(s)\,ds\right) + 2A_1^2TE\left(\int_0^t \varepsilon_{n+1}^2(s)\,ds\right)\right\}^{1/2} + \left\{8B_1^2E\left(\int_0^t \varepsilon_n^2(s)\,ds\right) + 2B_1^2E\left(\int_0^t \varepsilon_{n+1}^2(s)\,ds\right)\right\}^{1/2}$$
$$\le 2\sqrt{2}(A_1\sqrt{T}+B_1)\left(\int_0^t E\varepsilon_n^2(s)\,ds\right)^{1/2} + \sqrt{2}(A_1\sqrt{T}+B_1)\left(\int_0^t E\varepsilon_{n+1}^2(s)\,ds\right)^{1/2}.$$

Consequently, we obtain

$$E\,\varepsilon_{n+1}^2(t) \le 16(A_1\sqrt{T}+B_1)^2\int_0^t E\,\varepsilon_n^2(s)\,ds + 4(A_1\sqrt{T}+B_1)^2\int_0^t E\,\varepsilon_{n+1}^2(s)\,ds;$$

this shows, by Lemma 1.3,

$$E\,\varepsilon_{n+1}^2(t) \le 16(A_1\sqrt{T}+B_1)^2 e^{4(A_1\sqrt{T}+B_1)^2T}\int_0^t E\,\varepsilon_n^2(s)\,ds$$

for all $0 \le t \le T$ and every $n = 0, 1, 2, \cdots$. The recursive use of the above estimate completes the proof. □

Remark 2. *Since* (2) *implies*

$$\lim_{n\to\infty}\sup_{0\le t\le T} E\,\varepsilon_n^2(t) = 0,$$

Theorem 2.1 *gives a simple proof of Kawabata and Yamada's convergence theorem. We easily have*

(4) $|\alpha_{0,n}(t)| \le A_2\,|\varepsilon_n(t)|^2,\ |\beta_{0,n}(t)| \le B_2|\varepsilon_n(t)|^2$

instead of (3) *if* $|a_{xx}| \le A_2$ *and* $|b_{xx}| \le B_2$ *for some constants* A_2 *and* B_2*, however we cannot prove the second order error estimate in the present note. For this purpose, our explicit formulation and a new type of approach would be necessary* [2].

References

[1] K. Amano, A stochastic Gronwall inequality and its applications, JIPAM. J. Inequal. Pure Appl. Math. **6** (2005), no. 1, Article 17. (electronic).

[2] K. Amano, Newton's method for stochastic differential equations and its probabilistic second order error estimate. (in preparation).

[3] S. Kawabata and T. Yamada, On Newton's method for stochastic differential equations, in *Séminaire de Probabilités*, XXV, 121–137, Lecture Notes in Math., 1485, Springer, Berlin, 1991.