Generic Torelli theorem for one-parameter mirror families to weighted hypersurfaces

Dedicated to Prof. Sampei Usui on his sixtieth birthday

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Abstract: One-parameter mirror families to weighted hypersurfaces are already constructed and well studied. A generic Torelli theorem for quintic-mirror family is proved by Sampei Usui. In this article, we give the proof of a generic Torelli theorem for the other families after Usui's proof for quintic-mirror family.

Key words: Mirror family; logarithmic Hodge theory; Torelli theorem.

1. Our objects. We consider four types one-parameter families of Calabi-Yau hypersurfaces in complex weighted projective four-space. For $\psi \in \mathbf{P}^1$, they are

$$\left\{ \begin{array}{l} (x_1, \cdots, x_5) \in \mathbf{P}^{(1,1,1,1,1)} \\ \left| \begin{array}{l} x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 \\ -5\psi x_1 x_2 x_3 x_4 x_5 = 0 \end{array} \right. \end{array} \right\}, \tag{1}$$

$$\left\{ \begin{array}{l} (x_1, \cdots, x_5) \in \mathbf{P}^{(2,1,1,1,1)} \\ \left| \begin{array}{l} 2x_1^3 + x_2^6 + x_3^6 + x_4^6 + x_5^6 \\ -6\psi x_1 x_2 x_3 x_4 x_5 = 0 \end{array} \right. \end{array} \right\}, \tag{2}$$

$$\left\{ \begin{array}{l} (x_1, \cdots, x_5) \in \mathbf{P}^{(4,1,1,1,1)} \\ \left| \begin{array}{l} 4x_1^2 + x_2^8 + x_3^8 + x_4^8 + x_5^8 \\ -8\psi x_1 x_2 x_3 x_4 x_5 = 0 \end{array} \right. \end{array} \right\}, \tag{3}$$

$$\left\{ \begin{array}{l} (x_1, \cdots, x_5) \in \mathbf{P}^{(5,2,1,1,1)} \\ \left| \begin{array}{l} 5x_1^2 + 2x_2^5 + x_3^{10} + x_4^{10} + x_5^{10} \\ -10\psi x_1 x_2 x_3 x_4 x_5 = 0 \end{array} \right. \end{array} \right\}. \tag{4}$$

Let μ_k be the group of k-th root of $1 \in \mathbf{C}$. When we divide these hypersurfaces (1), (2), (3), (4) by $(\mu_5)^3,\ \mu_3 \times (\mu_6)^2,\ (\mu_8)^3,\ (\mu_{10})^2$, canonical singularities appear. For $\psi \in \mathbf{C} \subset \mathbf{P}^1$, it is known that there is a simultaneous desingularization of these singularities, and we have four families $(W^i_\psi)_{\psi \in \mathbf{P}^1}$ $(i = 1, 2, 3, 4)$ of the mirrors to the above hypersurfaces in each case (1), (2), (3), (4).

Let

$$\nu_i = \begin{cases} \mu_5 & \text{if } i = 1, \\ \mu_6 & \text{if } i = 2, \\ \mu_8 & \text{if } i = 3, \\ \mu_{10} & \text{if } i = 4. \end{cases}$$

$(W^i_\psi)_\psi$ is parametrized by ψ. The singular fibres of $(W^i_\psi)_\psi$ are as follows:

- When ψ belongs to $\nu_i \subset \mathbf{C} \subset \mathbf{P}^1$, W^i_ψ has one ordinary double point.
- W^i_∞ is a normal crossing divisor in the total space.

The other fibres of $(W^i_\psi)_\psi$ are smooth with Hodge numbers $h^{p,q} = 1$ for $p + q = 3,\ p, q \geq 0$.

By the action of

$$\alpha \in \nu_i,\ (x_1, \cdots, x_5) \mapsto (x_1, \cdots, x_4, \alpha^{-1} x_5),$$

we have the isomorphism from the fibre over ψ to the fibre over $\alpha\psi$. Let λ be

$$\begin{cases} \psi^5 & \text{if } i = 1, \\ \psi^6 & \text{if } i = 2, \\ \psi^8 & \text{if } i = 3, \\ \psi^{10} & \text{if } i = 4, \end{cases}$$

and let

$$\begin{array}{ccc} (W^i_\lambda)_\lambda & = & ((W^i_\psi)_\psi)/\nu_i \\ \downarrow & & \downarrow \\ (\lambda\text{-plane}) & = & (\psi\text{-plane})/\nu_i. \end{array}$$

These families are our objects for which we will give the proof of a generic Torelli theorem. $(W^1_\lambda)_\lambda$ is

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so-called quintic-mirror family, and the theorem for this family is proved by Usui [U2]. (For more details of the above families, see e.g. [KT, M1, M2].)

2. Local monodromy. In this section, we review the local monodromy of $(W_\lambda^i)_{\lambda\in\mathbf{P}^1}$ $(i=1,2,3,4)$. For quintic-mirror family, Candelas, de la Ossa, Green and Parks gave the matrix representation of the local monodromy for a symplectic basis in [COGP]. For the other 3 families, Klemm and Theisen gave it in [KT]. We recall their results.

Let $(W_\lambda)_\lambda = (W_\lambda^i)_\lambda$ $(i=1,2,3,4)$, and fix $b\in\mathbf{P}^1-\{0,1,\infty\}$. Then, there is a symplectic basis of $H^3(W_b,\mathbf{Z})$ and the matrix representations A, T, T_∞ of local monodromies around $\lambda=0,1,\infty$ for this basis are listed as follows:

In the case of $i=1$,

$$A=\begin{pmatrix}11&8&-5&0\\5&-4&-3&1\\20&15&-9&0\\5&-5&-3&1\end{pmatrix},\ T=\begin{pmatrix}1&0&0&0\\0&1&0&-1\\0&0&1&0\\0&0&0&1\end{pmatrix},$$

$$T_\infty=\begin{pmatrix}-9&-3&5&0\\0&1&0&0\\-20&-5&11&0\\-15&5&8&1\end{pmatrix}.$$

In the case of $i=2$,

$$A=\begin{pmatrix}1&1&0&0\\3&-3&-1&1\\3&6&1&0\\3&-4&-1&1\end{pmatrix},\ T=\begin{pmatrix}1&0&0&0\\0&1&0&-1\\0&0&1&0\\0&0&0&1\end{pmatrix},$$

$$T_\infty=\begin{pmatrix}1&-1&0&0\\0&1&0&0\\-3&-3&1&0\\-6&4&1&1\end{pmatrix}.$$

In the case of $i=3$,

$$A=\begin{pmatrix}1&1&0&0\\2&-3&-1&1\\2&4&1&0\\2&-4&-1&1\end{pmatrix},\ T=\begin{pmatrix}1&0&0&0\\0&1&0&-1\\0&0&1&0\\0&0&0&1\end{pmatrix},$$

$$T_\infty=\begin{pmatrix}1&-1&0&0\\0&1&0&0\\-2&-2&1&0\\-4&4&1&1\end{pmatrix}.$$

In the case of $i=4$,

$$A=\begin{pmatrix}1&1&-1&0\\-1&-2&0&1\\0&-1&1&0\\-1&-3&0&1\end{pmatrix},\ T=\begin{pmatrix}1&0&0&0\\0&1&0&-1\\0&0&1&0\\0&0&0&1\end{pmatrix},$$

$$T_\infty=\begin{pmatrix}1&0&1&0\\0&1&0&0\\0&1&1&0\\1&3&1&1\end{pmatrix}.$$

We obtain these matrixes from the calculations of the local monodromies for the periods in [COGP, KT] and the relation between the symplectic basis of $H^3(W_b,\mathbf{Z})$ and the periods in [M2, Appendix C]. In particular, the above A and T are the inverse matrixes of the matrixes A and T in the lists of [COGP, KT] respectively.

3. Extended classifying space and period map. In this section, we recall some facts in [KU] that are necessary in the present article.

Let $w=3$, and $h^{p,q}=1$ $(p+q=3,\ p,q\geq 0)$. Let $H_0=\bigoplus_{j=1}^4 \mathbf{Z}e_j$, $\langle e_3,e_1\rangle_0=\langle e_4,e_2\rangle_0=1$, and $G_{\mathbf{Z}}=\mathrm{Aut}(H_0,\langle\ ,\ \rangle_0)$. Let D be the corresponding classifying space of polarized Hodge structures, and $\check{D}$ be the compact dual.

Let $(W_\lambda)_\lambda=(W_\lambda^i)_\lambda$ $(i=1,2,3,4)$. Fix a point $b\in\mathbf{P}^1-\{0,1,\infty\}$ on the λ-plane, identify $H^3(W_b,\mathbf{Z}) = H_0$, and let

$$\Gamma=\mathrm{Image}\big(\pi_1(\mathbf{P}^1-\{0,1,\infty\})\to G_{\mathbf{Z}}\big). \tag{1}$$

This Γ is not neat. In fact, the order of the local monodromy around 0 is

$$\begin{cases}5 & \text{if}\quad i=1,\\ 6 & \text{if}\quad i=2,\\ 8 & \text{if}\quad i=3,\\ 10 & \text{if}\quad i=4.\end{cases}$$

When we use some results in [KU], we need that Γ is neat. It is known that there exists a neat subgroup of $G_{\mathbf{Z}}$ of finite index (cf. [B]). Therefore, we have a neat subgroup of Γ of finite index. Taking this fact into consideration, we can use their results in the present case.

Let $N_1=\log T$, $N_\infty=\log T_\infty\in\mathrm{End}(\mathbf{Q}\otimes H_0, \langle\ ,\ \rangle_0)$. Here $\langle\ ,\ \rangle_0$ is regarded as the natural extension as $\mathbf{Q}$-bilinear form. They are illustrated as follows:

$$0:\quad \underset{(3,0)}{\bullet}\qquad \underset{(2,1)}{\bullet}\qquad \underset{(1,2)}{\bullet}\qquad \underset{(0,3)}{\bullet}$$

$$N_1 : \quad \begin{tikzcd}[row sep=small] & (2,2) \arrow[d] & \\ (3,0) & (1,1) & (0,3) \end{tikzcd}$$

$$N_\infty : \quad \begin{tikzcd} (3,3) \arrow[d] \\ (2,2) \arrow[d] \\ (1,1) \arrow[d] \\ (0,0) \end{tikzcd}$$

Let $\sigma_1 = \mathbf{R}_{\geq 0} N_1$, $\sigma_\infty = \mathbf{R}_{\geq 0} N_\infty$ and $\Xi = \{\mathrm{Ad}(g)\sigma \mid \sigma = \{0\}, \sigma_1, \sigma_\infty, g \in \Gamma\}$. Then Γ is strongly compatible with Ξ.

In [KU], the extended classifying space D_Ξ is constructed. As a set, $D_\Xi = \{(\sigma, Z) :$ nilpotent orbit $|\sigma \in \Xi,\ Z \subset \check{D}\}$ and $D \simeq \{(\{0\}, F) \mid F \in D\} \subset D_\Xi$. Since Γ is compatible with Ξ, Γ acts on D_Ξ. Kato and Usui endowed $\Gamma\backslash D_\Xi$ with the structure of logarithmic ringed space in [KU]. We denote the structure sheaf and the logarithmic structure of $\Gamma\backslash D_\Xi$ by $\mathcal{O}_{\Gamma\backslash D_\Xi}$ and $M_{\Gamma\backslash D_\Xi}$ respectively. The construction of $\mathcal{O}_{\Gamma\backslash D_\Xi}, M_{\Gamma\backslash D_\Xi}$ and the topology of $\Gamma\backslash D_\Xi$ for the present case is described concisely in [U2].

Next, we recall extended period map.

Let

$$\mathbf{P}^1 - \{0, 1, \infty\} \to \Gamma\backslash D \tag{2}$$

be the period map. Since the canonical bundle of W^i_λ is trivial, the differential of (2) is injective everywhere. Endow $\mathbf{P}^1$ with the logarithmic structure associated to the divisor $\{1, \infty\}$. Then, by [KU, 4.3.1. (i)], (2) extends to a morphism

$$\varphi : \mathbf{P}^1 \to \Gamma\backslash D_\Xi \tag{3}$$

of logarithmic ringed spaces. By [KU, 3.4.4. (i)] and the nilpotent orbit theorem of Schmid, we have

$$\begin{aligned} &\varphi(0) = (\text{point mod } \Gamma) \in \Gamma\backslash D, \\ &\varphi(1) = (\sigma_1\text{-nilpotent orbit mod } \Gamma), \\ &\varphi(\infty) = (\sigma_\infty\text{-nilpotent orbit mod } \Gamma). \end{aligned} \tag{4}$$

The image of the extended period map φ is an analytic curve (cf. [U1]).

Let $X = \Gamma\backslash D_\Xi$. Let $P_1 = 1, P_\infty = \infty \in \mathbf{P}^1$, and let $Q_1 = \varphi(P_1), Q_\infty = \varphi(P_\infty) \in X$. Then, by the observation of local monodromy and holomolphic 3-form basing on the descriptions in §1, §2 and this section, we have

$$\varphi^{-1}(Q_\lambda) = \{P_\lambda\} \quad \text{for} \quad \lambda = 1, \infty. \tag{5}$$

4. Generic Torelli theorem. We use the notation in the previous sections. In this section, we give the proof of a generic Torelli theorem for $(W^i_\lambda)_\lambda$ $(i = 2, 3, 4)$. The proof for $(W^1_\lambda)_\lambda$ is already given by Usui in [U2], and the proofs for the other 3 families are similar to that in [U2].

Theorem. *For each $i = 2, 3, 4$, the period map φ in §3 (3) is the normalization of analytic spaces over its image.*

The argument by using the fs logarithmic points P_1 and Q_1 at the boundaries for $i = 1$ in [U2, §4] works also well for $i = 2, 3, 4$, and gives the above theorem.

We give another proof of the above theorem by using the fs logarithmic points P_∞ and Q_∞ at the boundaries.

Proof. The method of the proof is similar to that for $i = 1$ given in [U2, §5]. We give the full proof in each case $i = 2, 3, 4$.

Since $\varphi^{-1}(Q_\infty) = \{P_\infty\}$ (§3, (5)), it is enough to show the following

Claim. $(M_X/\mathcal{O}^\times_X)_{Q_\infty} \to (M_{\mathbf{P}^1}/\mathcal{O}^\times_{\mathbf{P}^1})_{P_\infty}$ *is surjective.*

Let N be the logarithm of the local monodromy at $\lambda = \infty$. Let $\beta^1, \beta^2, \alpha_1, \alpha_2$ be a symplectic basis of H_0 which gives the matrix representation of the local monodromy in §2 in each case.

Before the proof of Claim, we prepare a Lemma.

Lemma. *In each case, there exists a symplectic basis g_3, g_2, g_1, g_0 of H_0 for which the matrix representation of N is listed as follows*:

In the case of $i = 2$,

$$(N(g_3), N(g_2), N(g_1), N(g_0))$$

$$= (g_3, g_2, g_1, g_0) \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & -9/2 & 0 & 0 \\ -9/2 & 7/2 & 1 & 0 \end{pmatrix}.$$

In the case of $i = 3$,

$$(N(g_3), N(g_2), N(g_1), N(g_0)) = (g_3, g_2, g_1, g_0)\begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & -3 & 0 & 0 \\ -3 & 11/3 & 1 & 0 \end{pmatrix}.$$

In the case of $i = 4$,

$$(N(g_3), N(g_2), N(g_1), N(g_0)) = (g_3, g_2, g_1, g_0)\begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1/2 & 0 & 0 \\ -1/2 & 17/6 & 1 & 0 \end{pmatrix}.$$

Proof of Lemma. The basis g_3, g_2, g_1, g_0 is given as follows:

$$\begin{cases} \text{In the case of } i = 2, 3, \\ g_3 = \beta^1,\ g_2 = \beta^2,\ g_1 = \alpha_1,\ g_0 = \alpha_2. \\ \text{In the case of } i = 4, \\ g_3 = -\alpha_1,\ g_2 = \beta^2,\ g_1 = \beta^1,\ g_0 = \alpha_2. \end{cases} \qquad \square$$

Proof of Claim. Let $\tilde{q}$ be a local coordinate on a neighborhood U of $P_\infty = \infty$ in $\mathbf{P}^1$, and let $z = (2\pi i)^{-1} \log \tilde{q}$ be a branch over $U - \{P_\infty\}$. Then $\exp(-zN)g_1 = g_1 - zg_0$ is single-valued. Let $\omega(\tilde{q})$ be a local frame of the locally free $\mathcal{O}_{\mathbf{P}^1}$-module F^3. Write $\omega(\tilde{q}) = \sum_{j=0}^{3} b_j(\tilde{q})g_j$, and define $t = b_3(\tilde{q})/b_2(\tilde{q})$. Then

$$\begin{aligned} t &= \frac{\langle g_1, \omega(\tilde{q})\rangle_0}{\langle g_0, \omega(\tilde{q})\rangle_0} \\ &= \frac{\langle \exp(-zN)g_1, \omega(\tilde{q})\rangle_0 + z\langle g_0, \omega(\tilde{q})\rangle_0}{\langle g_0, \omega(\tilde{q})\rangle_0} \\ &= z + (\text{single-valued holomorphic function in } \tilde{q}). \end{aligned}$$

Let $q = e^{2\pi i t}$. Then $q = u\tilde{q}$ for some $u \in \mathcal{O}^\times_{\mathbf{P}^1, P_\infty}$. Let V be a neighborhood of Q_∞ in $X = \Gamma\backslash D_\Xi$. Endow $\mathbf{C}$ with the logarithmic structure associated to the divisor $\{0\}$. Then we have a composite morphism of fs logarithmic local ringed spaces

$$U \to V \to \mathbf{C}, \quad \tilde{q} \mapsto q = e^{2\pi i (b_3/b_2)}\ (= u\tilde{q}).$$

Hence the composite morphism $(M_{\mathbf{P}^1}/\mathcal{O}^\times_{\mathbf{P}^1})_{P_\infty} \leftarrow (M_X/\mathcal{O}^\times_X)_{Q_\infty} \leftarrow (M_{\mathbf{C}}/\mathcal{O}^\times_{\mathbf{C}})_0$ of reduced logarithmic structures is an isomorphism. The claim follows. In fact, the morphism in the claim is an isomorphism since the rank of $(M_X/\mathcal{O}^\times_X)_{Q_\infty}$ is one in the present case. $\square$

We thus have proven the theorem in this section. $\square$

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