On the cohomology of the mod p Steenrod algebra

By Xiugui Liu*) and He Wang**)

(Communicated by Heisuke Hironaka, M.J.A., Oct. 13, 2009)

Abstract: Let p be an odd prime greater than seven and A the mod p Steenrod algebra. In this paper we prove that in the cohomology of A the product $h_1h_n\tilde{\delta}_{s+4} \in \mathrm{Ext}_A^{s+6,t(s,n)+s}(\mathbf{Z}_p, \mathbf{Z}_p)$ is nontrivial for $n \geqslant 5$, and trivial for $n = 3, 4$, where $\tilde{\delta}_{s+4}$ is actually $\tilde{\alpha}_{s+4}^{(4)}$ described by X. Wang and Q. Zheng, $0 \leqslant s < p-4$, $t(s,n) = 2(p-1)[(s+1)+(s+3)p+(s+3)p^2+(s+4)p^3+p^n]$. We show our results by explicit combinatorial analysis of the (modified) May spectral sequence. The method of proof is very elementary.

Key words: Steenrod algebra; cohomology; Adams spectral sequence; May spectral sequence.

1. Introduction and statement of results. In this paper, p always denotes an odd prime and $q = 2(p-1)$.

To determine the stable homotopy groups of spheres is one of the most important problems in algebraic topology. So far, several methods have been found to determine the stable homotopy groups of spheres. For example, we have the classical Adams spectral sequence (ASS) (cf. [1]) based on the Eilenberg-MacLane spectrum $K\mathbf{Z}_p$, whose E_2-term is the cohomology of A-$\mathrm{Ext}_A^{s,t}(\mathbf{Z}_p, \mathbf{Z}_p)$, where A denotes the mod p Steenrod algebra. So, for computing the stable homotopy groups of spheres with the classical ASS, we must compute the E_2-term of the ASS, $\mathrm{Ext}_A^{*,*}(\mathbf{Z}_p, \mathbf{Z}_p)$.

The known results on $\mathrm{Ext}_A^{*,*}(\mathbf{Z}_p, \mathbf{Z}_p)$ are as follows: $\mathrm{Ext}_A^{0,*}(\mathbf{Z}_p, \mathbf{Z}_p) = \mathbf{Z}_p$ by its definition. From [2], $\mathrm{Ext}_A^{1,*}(\mathbf{Z}_p, \mathbf{Z}_p)$ has $\mathbf{Z}_p$-basis consisting of $a_0 \in \mathrm{Ext}_A^{1,1}(\mathbf{Z}_p, \mathbf{Z}_p)$, $h_i \in \mathrm{Ext}_A^{1,p^iq}(\mathbf{Z}_p, \mathbf{Z}_p)$ for all $i \geqslant 0$, and $\mathrm{Ext}_A^{2,*}(\mathbf{Z}_p, \mathbf{Z}_p)$ has $\mathbf{Z}_p$-basis consisting of α_2, a_0^2, $a_0h_i (i > 0)$, $g_i (i \geqslant 0)$, $k_i (i \geqslant 0)$, $b_i (i \geqslant 0)$, and $h_ih_j (j \geqslant i+2, i \geqslant 0)$ whose internal degrees are $2q+1$, 2, $p^iq+1, p^{i+1}q+2p^iq$, $2p^{i+1}q+p^iq$, $p^{i+1}q$ and p^iq+p^jq respectively. In 1980, Aikawa [3] determined $\mathrm{Ext}_A^{3,*}(\mathbf{Z}_p, \mathbf{Z}_p)$ by λ-algebra.

In [4], X. Wang and Q. Zheng proved the following theorem.

Theorem 1.1 [4]. *For $p \geqslant 11$ and $0 \leqslant s < p-4$, there exists the fourth Greek letter family element $\tilde{\delta}_{s+4} \neq 0 \in \mathrm{Ext}_A^{s+4,t_1(s)+s}(\mathbf{Z}_p, \mathbf{Z}_p)$, where $t_1(s) = q[(s+4)p^3 + (s+3)p^2 + (s+2)p + (s+1)]$. Here we write $\tilde{\delta}_{s+4}$ for $\tilde{\alpha}_{s+4}^{(4)}$ which is described in* [4].

In [5], X. Liu and H. Zhao showed the following result.

Theorem 1.2 [5, Theorem 1.2]. *For $p \geqslant 11$ and $4 \leqslant s < p$, the product $h_0b_0\tilde{\delta}_s \neq 0$ in the classical Adams spectral sequence.*

The method of proof of Theorem 1.2 above is by explicit combinatorial analysis of the May spectral sequence (MSS), and very elementary. In the ASS h_0 and b_0 detect the α-element α_1 and β-element β_1 respectively, and $\tilde{\delta}_s$ is the element of lowest filtration which could detect the element δ_s arising from the existence of a self-map on Toda-Smith spectrum $V(3)$ inducing multiplication by v_4^s in BP-homology of $V(3)$. It follows that $h_0b_0\tilde{\delta}_s$ could detect the composite $\alpha_1\beta_1\delta_s$ in the ASS.

In this note, our main results can be stated as follows:

Theorem 1.3. *For $p \geqslant 11$, $0 \leqslant s < p-4$ and $t(s,n) = q[(s+1)+(s+3)p+(s+3)p^2+(s+4)p^3+p^n]$. Then in the cohomology of the mod p Steenrod algebra A $\mathrm{Ext}_A^{s+6,t(s,n)+s}(\mathbf{Z}_p, \mathbf{Z}_p)$,*

(1) *the product $h_1h_n\tilde{\delta}_{s+4}$ is nontrivial for $n \geqslant 5$.*

(2) *the product $h_1h_n\tilde{\delta}_{s+4}$ is trivial for $n = 3, 4$.*

The paper is arranged as follows: after introducing a method of detecting generators of the E_1-term $E_1^{*,*,*}$ of the MSS in Section 2. Section 3 is devoted to showing Theorem 1.3.

2. A method of determining generators of the May E_1-term $E_1^{*,*,*}$. In this section, we

2000 Mathematics Subject Classification. Primary 55S10; Secondary 55T15.

*) School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, P. R. China.

**) School of Mathematical Sciences, Nankai University, Tianjin 300071, P. R. China.

doi: 10.3792/pjaa.85.143

will give a new method used to determine generators of the May E_1-term $E_1^{*,*,*}$. We refer the reader to Section 2 of [5] for the MSS, and denote a_i, $h_{i,j}$ and $b_{i,j}$ by x, y and z respectively. By the graded commutativity of $E_1^{*,*,*}$, we can suppose a generator $g = (x_1 \cdots x_u)\ (y_1 \cdots y_v)\ (z_1 \cdots z_l) \in E_1^{s,t+b,*}$, where $t = (\bar{c}_0 + \bar{c}_1 + \cdots + \bar{c}_n p^n)q$ with $0 \leqslant \bar{c}_i < p\ (0 \leqslant i < n)$, $0 < \bar{c}_n < p$, $0 \leqslant b < q$.

Assertion. *If $s < b + q$, u must equal b.*

Otherwise, by the characteristics of deg a_i, deg $b_{i,j}$, deg $h_{i,j}$ and deg g, we would get $u = b + wq$ for some integer $w > 0$. It follows that dim $g \geqslant b + wq > s =$ dim g which is a contradiction. The assertion is proved.

So we have $g = (x_1 \cdots x_b)(y_1 \cdots y_v)(z_1 \cdots z_l) \in E_1^{s,t+b,*}$. By [5, (2.5)], the degrees of x_i, y_i and z_i can be expressed uniquely as:

$$\begin{cases} \deg x_i = (x_{i,0} + x_{i,1}p + \cdots + x_{i,n}p^n)q + 1, \\ \deg y_i = (y_{i,0} + y_{i,1}p + \cdots + y_{i,n}p^n)q, \\ \deg z_i = (0 + z_{i,1}p + \cdots + z_{i,n}p^n)q, \end{cases}$$

and

(a) $(x_{i,0}, x_{i,1}, \cdots, x_{i,n})$ is of the form $(1, \cdots, 1, 0, \cdots, 0)$,

(b) $(y_{i,0}, y_{i,1}, \cdots, y_{i,n})$ is of the form $(0, \cdots, 0, 1, \cdots, 1, 0, \cdots, 0)$,

(c) $(0, z_{i,1}, \cdots, z_{i,n})$ is of the form $(0, \cdots, 0, 1, \cdots, 1, 0, \cdots, 0)$.

By the graded commutativity of $E_1^{*,*,*}$, the generator $g = (x_1 \cdots x_b)(y_1 \cdots y_v)(z_1 \cdots z_l) \in E_1^{s,t+b,*}$ can be arranged in the following way:

$$\begin{cases} \text{(i)} \quad \text{If } i > j, \text{we put } a_i \text{ on the left side of } a_j, \\ \text{(ii)} \quad \text{If } j < k, \text{we put } h_{i,j} \text{ on the left side of } h_{w,k}, \\ \text{(iii)} \quad \text{If } i > w, \text{we put } h_{i,j} \text{ on the left side of } h_{w,j}, \\ \text{(iv)} \quad \text{Apply the same rules (ii) and (iii) to } b_{i,j}. \end{cases}$$

Then from (a)–(c) and (i)–(iv), the factors $x_{i,j}$, $y_{i,j}$ and $z_{i,j}$ in g must satisfy the following conditions:

$$(2.1) \quad \begin{cases} (1)\ x_{1,j} \geqslant x_{2,j} \geqslant \cdots \geqslant x_{b,j}; \\ (2)\ x_{i,0} \geqslant x_{i,1} \geqslant \cdots \geqslant x_{i,n}; \\ (3)\ \text{If } y_{i,j-1} = 0 \text{ and } y_{i,j} = 1, \\ \text{then for all } k < j\ y_{i,k} = 0; \\ (4)\ \text{If } y_{i,j} = 1 \text{ and } y_{i,j+1} = 0, \\ \text{then for all } k > j\ y_{i,k} = 0; \\ (5)\ y_{1,0} \geqslant y_{2,0} \geqslant \cdots \geqslant y_{v,0}; \\ (6)\ \text{If } y_{i,0} = y_{i+1,0}, y_{i,1} = y_{i+1,1}, \cdots, \\ y_{i,j} = y_{i+1,j}, \text{ then } y_{i,j+1} \geqslant y_{i+1,j+1}; \\ (7)\ \text{Apply the same rules } (3) \sim (6) \text{ to } z_{i,j}. \end{cases}$$

From $\deg g = \sum_{i=1}^{b} \deg x_i + \sum_{i=1}^{v} \deg y_i + \sum_{i=1}^{l} \deg z_i$, by the properties of the p-adic number we get the following group of equations

$$(2.2) \quad \begin{cases} x_{1,0} + \cdots + x_{b,0} + y_{1,0} + \cdots + y_{v,0} + 0 + \cdots + 0 \\ = \bar{c}_0 + k_0 p, \\ x_{1,1} + \cdots + x_{b,1} + y_{1,1} + \cdots + y_{v,1} + z_{1,1} + \cdots + \\ z_{l,1} = \bar{c}_1 + k_1 p - k_0, \\ \cdots \\ x_{1,n-1} + \cdots + x_{b,n-1} + y_{1,n-1} + \cdots + y_{v,n-1} + \\ z_{1,n-1} + \cdots + z_{l,n-1} = \bar{c}_{n-1} + k_{n-1}p - k_{n-2}, \\ x_{1,n} + \cdots + x_{b,n} + y_{1,n} + \cdots + y_{v,n} + z_{1,n} + \cdots + \\ z_{l,n} = \bar{c}_n - k_{n-1}, \end{cases}$$

where $k_i \geqslant 0$ for $0 \leqslant i \leqslant n-1$.

In the above group of equations, we get two integer sequences $K = (k_0, k_1, \cdots, k_{n-1})$ and $S = (\bar{c}_0 + k_0 p, \bar{c}_1 + k_1 p - k_0, \cdots, \bar{c}_n - k_{n-1})$ denoted by $(c_0, c_1, \cdots, c_n)$ which is determined by $(k_0, k_1, \cdots, k_{n-1})$ and $(\bar{c}_0, \bar{c}_1, \cdots, \bar{c}_n)$. We want to get the solutions of the group of equations (2.2) which satisfy the conditions (2.1).

Remark. Since the values of $x_{i,j}$, $y_{i,j}$ and $z_{i,j}$ must be 0 or 1, to solve the group of equations (2.2) will be mechanical. Since we want to get the solutions of the group of equations (2.2) which satisfy the conditions (2.1), we can use the conditions (2.1) in solving the group of equations. For example, if $x_{1,0} = 0$ for x_1, using the conditions (1)–(2) of (2.1), we will get all $x_{i,j} = 0$. By the method, to determine the solutions of the group of equations (2.2) which satisfy the conditions (2.1) will not be too difficult.

Notice that the elements x_i, y_i and z_i are uniquely determined by their degrees. A solution of (2.2) which satisfies (2.1) determines a generator g by setting deg x_i (respectively y_i and z_i) to be $(x_{i,0} + x_{i,1}p + \cdots + x_{i,n}p^n)q + 1$ (respectively $(y_{i,0} + y_{i,1}p + \cdots + y_{i,n}p^n)q$ and $(0 + z_{i,1}p + \cdots + z_{i,n})q$). Thus for the $E_1^{s,t+b,*}$-term where $t = (\bar{c}_0 + \bar{c}_1 p + \cdots + \bar{c}_n p^n)q$ with $0 \leqslant \bar{c}_i < p\ (0 \leqslant i < n)$, $0 < \bar{c}_n < p$, $0 \leqslant b < q$, the determination of $E_1^{s,t+b,*}$ is deduced to the following steps:

(1) List up all the possible (b, v, l) such that $b + v + 2l = s$.

(2) For each given (b, v, l), list all the sequences $K = (k_0, k_1, \cdots, k_{n-1})$ and $S = (c_0, c_1, \cdots, c_n)$ such that $c_i \leqslant b + v + l$ for all $0 \leqslant i \leqslant n$.

(3) For each given (b, v, l), the sequences $K = (k_0, k_1, \cdots, k_{n-1})$ and $S = (c_0, c_1, \cdots, c_n)$, solve the group of equations (2.2) by virtue of (2.1), then determine all the generators of $E_1^{s,t+b,*}$ by setting the corresponding second degrees.

3. Proof of Theorem 1.3. In this section we first give two lemmas which are needed in the proof of Theorem 1.3.

Lemma 3.1 [5, Lemma 3.1]. *For $p \geqslant 11$ and $0 \leqslant s < p-4$. Then the fourth Greek letter family element $\tilde{\delta}_{s+4} \in \mathrm{Ext}_A^{s+4,t_1(s)+s}(\mathbf{Z}_p, \mathbf{Z}_p)$ is represented by $a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3} \in E_1^{s+4,t_1(s)+s,*}$ in the E_1-term of the MSS, where $t_1(s) = [(s+1) + (s+2)p + (s+3)p^2 + (s+4)p^3]q$.*

Lemma 3.2. *Let $p \geqslant 11$, $n \geqslant 4$, $0 \leqslant s < p-4$. Then the May E_1-term satisfies*

$$E_1^{s+5,t(s,n)+s,*} = \begin{cases} 0 & n \geqslant 5 \text{ and } 0 \leqslant s < p-5, \\ M & n = 4, \\ K & n \geqslant 5 \text{ and } s = p-5. \end{cases}$$

Here, $t(s,n) = [(s+1) + (s+3)p + (s+3)p^2 + (s+4)p^3 + p^n]q$, M is the $\mathbf{Z}_p$-module generated by fourteen elements

$$\mathbf{g}1 = a_4^s h_{4,0} h_{3,1} h_{1,3} b_{4,0},$$
$$\mathbf{g}2 = a_4^s h_{4,0} h_{4,1} h_{3,1} b_{1,2},$$
$$\mathbf{g}3 = a_4^s h_{5,0} h_{3,1} h_{1,3} b_{3,0},$$
$$\mathbf{g}4 = a_4^s h_{4,0} h_{4,1} h_{1,3} b_{3,0},$$
$$\mathbf{g}5 = a_4^s h_{4,0} h_{3,1} h_{2,3} b_{3,0},$$
$$\mathbf{g}6 = a_5 a_4^{s-1} h_{4,0} h_{3,1} h_{1,3} b_{3,0},$$
$$\mathbf{g}7 = a_4^s h_{4,0} h_{3,1} h_{2,1} h_{2,3} h_{1,3},$$
$$\mathbf{g}8 = a_4^{s-1} a_2 h_{4,0} h_{4,1} h_{3,1} h_{2,2} h_{1,3},$$
$$\mathbf{g}9 = a_4^s h_{2,0} h_{4,1} h_{3,1} h_{2,2} h_{1,3},$$
$$\mathbf{g}10 = a_5 a_4^{s-1} h_{4,0} h_{3,1} h_{1,1} h_{2,2} h_{1,3},$$
$$\mathbf{g}11 = a_4^s h_{5,0} h_{3,1} h_{1,1} h_{2,2} h_{1,3},$$
$$\mathbf{g}12 = a_4^s h_{4,0} h_{4,1} h_{1,1} h_{2,2} h_{1,3},$$
$$\mathbf{g}13 = a_4^s h_{4,0} h_{3,1} h_{1,1} h_{3,2} h_{1,3},$$
$$\mathbf{g}14 = a_4^s h_{4,0} h_{3,1} h_{1,1} h_{2,2} h_{2,3},$$

where
$\mathbf{g}1 \in E_1^{s+5,t(s,4)+s,9s+7p+13}$,
$\mathbf{g}2 \in E_1^{s+5,t(s,4)+s,9s+p+19}$,
$\mathbf{g}i \in E_1^{s+5,t(s,4)+s,9s+5p+15} (3 \leqslant i \leqslant 6)$,
$\mathbf{g}i \in E_1^{s+5,t(s,4)+s,9s+19} (7 \leqslant i \leqslant 14)$,
and K is the $\mathbf{Z}_p$-module generated by one element

$$\mathbf{g}15 = a_n^{p-5} h_{n,0} h_{n-1,1} h_{4,1} h_{n-3,3} h_{n-4,4} \in E_1^{p,t(p-5,n)+p-5,(2n+1)(p-5)+8n-13}.$$

Proof. Consider $g \in E_1^{s+5,t(s,n)+s,*}$ where $t(s,n) = [(s+1) + (s+3)p + (s+3)p^2 + (s+4)p^3 + p^n]q$ with $(\bar{c}_0, \bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4, \cdots, \bar{c}_{n-1}, \bar{c}_n) = (s+1, s+3, s+3, s+4, 0, \cdots, 0, 1)$. Then $\dim g = s+5$ and $\deg g = t(s,n)+s$. Since $s+5 < s+q$, according to the assertion in Section 2, the number of x_i in g is s. By the reason of dimension, all the possibilities of g can be listed as $x_1 x_2 \cdots x_s y_1 z_1 z_2$, $x_1 x_2 \cdots x_s y_1 y_2 y_3 z_1$, $x_1 x_2 \cdots x_s y_1 y_2 y_3 y_4 y_5$.

Case 1. $g = x_1 x_2 \cdots x_s y_1 z_1 z_2$. Note that $s < p-4$. Then $\sum_{i=1}^{s} x_{i,j} + y_{1,j} + z_{1,j} + z_{2,j} \leqslant s+3 \leqslant s+3 < p$ for all $0 \leqslant j \leqslant n$. one easily gets the integer sequence $K = (k_0, k_1, \cdots, k_{n-1})$ in the corresponding group of equations (2.2) equals $(0, 0, \cdots, 0)$, and then $S = (c_0, c_1, c_2, c_3, c_4 \cdots, c_{n-1}, c_n) = (s+1, s+3, s+3, s+4, 0, \cdots, 0, 1)$. Since $\sum_{i=1}^{s} x_{i,3} + y_{1,3} + z_{1,3} + z_{2,3} \leqslant s+3 < s+4 = c_3$, the fourth equation of (2.2) has no solution. It follows that such g is impossible to exist.

Case 2. $g = x_1 x_2 \cdots x_s y_1 y_2 y_3 z_1$. Similar to Case 1, we can get that the integer sequence $K = (k_0, k_1, \cdots, k_{n-1})$ in the corresponding group of equations (2.2) is $(0, 0, \cdots, 0)$, and then $S = (c_0, c_1, c_2, c_3, c_4 \cdots, c_{n-1}, c_n) = (s+1, s+3, s+3, s+4, 0, \cdots, 0, 1)$.

Subcase 2.1. $n \geqslant 5$. Since $\sum_{i=1}^{s} x_{i,3} + y_{1,3} + y_{2,3} + y_{3,3} + z_{1,3} = s+4$, we get $x_{i,3} = y_{1,3} = y_{2,3} = y_{3,3} = z_{1,3} = 1$ for $1 \leqslant i \leqslant s$. Since $\sum_{i=1}^{s} x_{i,4} + y_{1,4} + y_{2,4} + y_{3,4} + z_{1,4} = 0$, we get $x_{i,4} = y_{1,4} = y_{2,4} = y_{3,4} = z_{1,4} = 0$ for $1 \leqslant i \leqslant s$. Then by the conditions (2), (4) and (7) in (2.1), we get $x_{i,j} = y_{1,j} = y_{2,j} = y_{3,j} = z_{1,j} = 0$ for $1 \leqslant i \leqslant s$ and $5 \leqslant j \leqslant n$, which contradicts $\sum_{i=1}^{s} x_{i,n} + y_{1,n} + y_{2,n} + y_{3,n} + z_{1,n} = 1$. So the corresponding group of equations (2.2) has no solution. It follows that g is impossible to exist.

Subcase 2.2. $n = 4$. We solve the corresponding group of equations (2.2) by virtue of (2.1), and get six nontrivial generators as follows:

$$\mathbf{g}1 = a_4^s h_{4,0} h_{3,1} h_{1,3} b_{4,0}, \ \mathbf{g}2 = a_4^s h_{4,0} h_{4,1} h_{3,1} b_{1,2},$$
$$\mathbf{g}3 = a_4^s h_{5,0} h_{3,1} h_{1,3} b_{3,0}, \ \mathbf{g}4 = a_4^s h_{4,0} h_{4,1} h_{1,3} b_{3,0},$$
$$\mathbf{g}5 = a_4^s h_{4,0} h_{3,1} h_{2,3} b_{3,0}, \ \mathbf{g}6 = a_5 a_4^{s-1} h_{4,0} h_{3,1} h_{1,3} b_{3,0},$$

where

$\mathbf{g}1 \in E_1^{s+5,t(s,4)+s,9s+7p+13}$, $\mathbf{g}2 \in E_1^{s+5,t(s,4)+s,9s+p+19}$ and $\mathbf{g}i \in E_1^{s+5,t(s,4)+s,9s+5p+15}$ for $3 \leqslant i \leqslant 6$.

Case 3. $g = x_1x_2\cdots x_sy_1y_2y_3y_4y_5$.

Subcase 3.1. $n = 4$. Similar to Case 2, we easily get that $S = (c_0, c_1, c_2, c_3, c_4) = (s+1, s+3, s+3, s+4, 1)$. One can solve the corresponding group of equations (2.2) by virtue of (2.1), and get eight nontrivial generators as follows:

$$\begin{aligned}
\mathbf{g}7 &= a_4^s h_{4,0}h_{3,1}h_{2,1}h_{2,3}h_{1,3},\\
\mathbf{g}8 &= a_4^{s-1}a_2h_{4,0}h_{4,1}h_{3,1}h_{2,2}h_{1,3},\\
\mathbf{g}9 &= a_4^s h_{2,0}h_{4,1}h_{3,1}h_{2,2}h_{1,3},\\
\mathbf{g}10 &= a_5a_4^{s-1}h_{4,0}h_{3,1}h_{1,1}h_{2,2}h_{1,3},\\
\mathbf{g}11 &= a_4^s h_{5,0}h_{3,1}h_{1,1}h_{2,2}h_{1,3},\\
\mathbf{g}12 &= a_4^s h_{4,0}h_{4,1}h_{1,1}h_{2,2}h_{1,3},\\
\mathbf{g}13 &= a_4^s h_{4,0}h_{3,1}h_{1,1}h_{3,2}h_{1,3},\\
\mathbf{g}14 &= a_4^s h_{4,0}h_{3,1}h_{1,1}h_{2,2}h_{2,3},
\end{aligned}$$

where $\mathbf{g}i \in E_1^{s+5,t(s,4)+s,9s+19}$ for $7 \leqslant i \leqslant 14$.

Subcase 3.2. $n \geqslant 5$ and $0 \leqslant s < p-5$. Similar to Case 2, one can get that $S = (c_0, c_1, c_2, c_3, c_4, \cdots, c_{n-1}, c_n) = (s+1, s+3, s+3, s+4, 0, \cdots, 0, 1)$. We solve the corresponding group of equations (2.2) by virtue of (2.1), and get a generator $a_4^s h_{4,0}h_{3,1}^2h_{1,3}h_{1,n}$ which is trivial by $h_{3,1}^2 = 0$.

Subcase 3.3. $n \geqslant 5$ and $s = p-5$. Since $\sum_{i=1}^{s} x_{i,j} + y_{1,j} + y_{2,j} + y_{3,j} + y_{4,j} + y_{5,j} \leqslant s+5 = p$ $(0 \leqslant j \leqslant n)$, all possibilities of the integer sequence $K = (k_0, k_1, \cdots, k_{n-1})$ in the corresponding group of equations (2.2) are $K_1 = (0, 0, \cdots, 0)$ and $K_i = (0, 0, 0, 0, 0, \cdots, 0, 1^{(i)}, 1, \cdots, 1)$ $(5 \leqslant i \leqslant n)$, where $1^{(i)}$ means that 1 is the i-th term of the sequence K_i. Then the corresponding sequence $S = (c_0, c_1, c_2, c_3, c_4, \cdots, c_{n-1}, c_n)$ are listed as follows:

$S_1 = (p-4, p-2, p-2, p-1, 0, \cdots, 0, 1)$,

$S_i = (p-4,\ p-2,\ p-2,\ p-1,\ 0, \cdots,\ 0,\ p^{(i)}, p-1, \cdots, p-1, 0)$ $(5 \leqslant i \leqslant n)$.

For S_1, we solve the corresponding group of equations (2.2) by virtue of (2.1), and get a generator $a_4^{p-5}h_{4,0}h_{3,1}^2h_{1,3}h_{1,n}$ which is trivial by $h_{3,1}^2 = 0$. For S_5, one can solve the corresponding group of equations (2.2) by virtue of (2.1), and get a generator $\mathbf{g}15 = a_n^{p-5}h_{n,0}h_{n-1,1}h_{4,1}h_{n-3,3}h_{n-4,4} \in E_1^{p,t(p-5)+p-5,(2n+1)(p-5)+8n-13}$. For $S_i (6 \leqslant i \leqslant n)$, it is easy to get the corresponding group of equations (2.2) has no solution.

Combining Cases 1-3 gives the desired result. □

Proof of Theorem 1.3. (1) It is known that $h_{1,n} \in E_1^{1,p^nq,*}$ is a permanent cycle and represents $h_n \in \mathrm{Ext}_A^{1,p^nq}(\mathbf{Z}_p, \mathbf{Z}_p)$ in the MSS for $n \geqslant 0$. By Lemma 3.1, $\tilde{\delta}_{s+4}$ is represented by $a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3} \in E_1^{s+4,t_1(s)+s,*}$ in the MSS. Thus, $h_{1,1}h_{1,n}a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3} \in E_1^{s+6,t(s,n)+s,9s+18}$ is a permanent cycle in the MSS and represents $h_1h_n\tilde{\delta}_{s+4} \in \mathrm{Ext}_A^{*,*}(\mathbf{Z}_p, \mathbf{Z}_p)$.

Case 1. $0 \leqslant s < p-5$. From Lemma 3.2, the May E_1-term

$$E_1^{s+5,t(s,n)+s,*} = 0,$$

which implies that $E_r^{s+5,t(s,n)+s,*} = 0$ for $r \geqslant 1$. Consequently, $h_{1,1}h_{1,n}a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3}$ cannot be hit by any May differential in the MSS. Thus in this case, $h_1h_n\tilde{\delta}_{s+4} \neq 0$.

Case 2. $s = p-5$. By Lemma 3.2,

$$E_1^{s+5,t(s,n)+s,*} = E_1^{p,t(p-5,n),*} = \mathbf{Z}_p\{\mathbf{g}15\}.$$

Note that $M(\mathbf{g}15) = (2n+1)(p-5) + 8n - 13$ and $M(h_{1,1}h_{1,n}a_4^{p-5}h_{4,0}h_{3,1}h_{2,2}h_{1,3}) = 9(p-5) + 18$. By the reason of May filtration, we have that $h_{1,1}h_{1,n}a_4^{p-5}h_{4,0}h_{3,1}h_{2,2}h_{1,3}$ is not in $d_1(E_1^{p,t(p-5,n)+p-5,(2n+1)(p-5)+8n-13})$. At the same time, from

$$\begin{aligned}
d_1(\mathbf{g}15) &= d_1(a_n^{p-5}h_{n,0}h_{n-1,1}h_{4,1}h_{n-3,3}h_{n-4,4})\\
&= -\,a_n^{p-5}d_1(h_{n,0})h_{n-1,1}h_{4,1}h_{n-3,3}h_{n-4,4}\\
&\quad + \cdots\\
&= -\,a_n^{p-5}h_{n-2,2}h_{2,0}h_{n-1,1}h_{4,1}h_{n-3,3}h_{n-4,4}\\
&\quad + \cdots\\
&\neq 0,
\end{aligned}$$

we get $E_r^{p,t(p-5,n)+p-5,(2n+1)(p-5)+8n-13} = 0$ $(r \geqslant 2)$, showing that $h_{1,1}h_{1,n}a_4^{p-5}h_{4,0}h_{3,1}h_{2,2}h_{1,3}$ is not in $d_r(E_r^{p,t(p-5,n)+p-5,(2n+1)(p-5)+8n-13})$ for $r \geqslant 1$. Thus $h_{1,1}h_{1,n}a_4^{p-5}h_{4,0}h_{3,1}h_{2,2}h_{1,3}$ cannot be hit by any May differential, showing that $h_1h_n\tilde{\delta}_{p-1} \neq 0 \in \mathrm{Ext}_A^{p+1,t(p-5,n)+p-5}(\mathbf{Z}_p, \mathbf{Z}_p)$.

This completes the proof of Theorem 1.3(1).

(2) Since $h_1h_3\tilde{\delta}_{s+4}$ is represented in the MSS by $h_{1,1}h_{1,3}a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3}$ which is trivial by $h_{1,3}^2 = 0$, it follows that $h_1h_3\tilde{\delta}_{s+4} = 0$. To show $h_1h_4\tilde{\delta}_{s+4} = 0$, it suffices to prove $h_{1,1}h_{1,4}a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3} \in$

$E_1^{s+6,t(s,4)+s,9s+18}$ which represents $h_1h_4\tilde{\delta}_{s+4} \in \mathrm{Ext}_A^{*,*}(\mathbf{Z}_p, \mathbf{Z}_p)$ is in $d_1(E_1^{s+5,t(s,4)+s,9s+19})$. By Lemma 3.2 we get that

$$E_1^{s+5,t(s,4)+s,9s+19} = \mathbf{Z}_p\{\mathbf{g}7, \cdots, \mathbf{g}14\}.$$

By [5, (2.3)] and [5, (2.4)], we compute the first May differential of $\mathbf{g}i (7 \leqslant i \leqslant 14)$ as follows (Using the graded commutativity in the MSS, we arrange the factors of every term of $d_1(\mathbf{g}i)$ $(7 \leqslant i \leqslant 14)$ in the way of (i), (ii) and (iii) in Section 2):

$$\begin{aligned} d_1(\mathbf{g}7) = (-1)^s(&-s\underline{a_4^{s-1}a_2h_{4,0}h_{3,1}h_{2,1}h_{2,2}h_{2,3}h_{1,3}}_1 \\ &- \underline{a_4^sh_{2,0}h_{3,1}h_{2,1}h_{2,2}h_{2,3}h_{1,3}}_2 \\ &+ \underline{a_4^sh_{4,0}h_{2,1}h_{1,1}h_{2,2}h_{2,3}h_{1,3}}_3 \\ &- \underline{a_4^sh_{4,0}h_{3,1}h_{1,1}h_{1,2}h_{2,3}h_{1,3}}_4), \end{aligned}$$

$$\begin{aligned} d_1(\mathbf{g}8) = (-1)^s(&-\underline{a_4^{s-1}a_0h_{4,0}h_{2,0}h_{4,1}h_{3,1}h_{2,2}h_{1,3}}_5 \\ &- \underline{a_4^{s-1}a_1h_{4,0}h_{4,1}h_{3,1}h_{1,1}h_{2,2}h_{1,3}}_6 \\ &+ \underline{a_4^{s-1}a_2h_{4,0}h_{3,1}h_{1,1}h_{3,2}h_{2,2}h_{1,3}}_7 \\ &- \underline{a_4^{s-1}a_2h_{4,0}h_{3,1}h_{2,1}h_{2,2}h_{2,3}h_{1,3}}_1), \end{aligned}$$

$$\begin{aligned} d_1(\mathbf{g}9) = (-1)^s(&s\underline{a_4^{s-1}a_0h_{4,0}h_{2,0}h_{4,1}h_{3,1}h_{2,2}h_{1,3}}_5 \\ &- \underline{a_4^sh_{1,0}h_{4,1}h_{3,1}h_{1,1}h_{2,2}h_{1,3}}_8 \\ &+ \underline{a_4^sh_{2,0}h_{3,1}h_{1,1}h_{3,2}h_{2,2}h_{1,3}}_9 \\ &- \underline{a_4^sh_{2,0}h_{3,1}h_{2,1}h_{2,2}h_{2,3}h_{1,3}}_2), \end{aligned}$$

$$\begin{aligned} d_1(\mathbf{g}10) = (-1)^s(&\underline{a_4^{s-1}a_0h_{5,0}h_{4,0}h_{3,1}h_{1,1}h_{2,2}h_{1,3}}_{10} \\ &- \underline{a_4^{s-1}a_1h_{4,0}h_{4,1}h_{3,1}h_{1,1}h_{2,2}h_{1,3}}_6 \\ &- \underline{a_4^{s-1}a_2h_{4,0}h_{3,1}h_{1,1}h_{3,2}h_{2,2}h_{1,3}}_7 \\ &+ \underline{a_4^{s-1}a_3h_{4,0}h_{3,1}h_{1,1}h_{2,2}h_{2,3}h_{1,3}}_{11} \\ &- \underline{a_4^sh_{4,0}h_{3,1}h_{1,1}h_{2,2}h_{1,3}h_{1,4}}_{12}), \end{aligned}$$

$$\begin{aligned} d_1(\mathbf{g}11) = (-1)^s(&-s\underline{a_4^{s-1}a_0h_{5,0}h_{4,0}h_{3,1}h_{1,1}h_{2,2}h_{1,3}}_{10} \\ &- \underline{a_4^sh_{1,0}h_{4,1}h_{3,1}h_{1,1}h_{2,2}h_{1,3}}_8 \\ &- \underline{a_4^sh_{2,0}h_{3,1}h_{1,1}h_{3,2}h_{2,2}h_{1,3}}_9 \\ &+ \underline{a_4^sh_{3,0}h_{3,1}h_{1,1}h_{2,2}h_{2,3}h_{1,3}}_{13} \\ &- \underline{a_4^sh_{4,0}h_{3,1}h_{1,1}h_{2,2}h_{1,3}h_{1,4}}_{12}), \end{aligned}$$

$$\begin{aligned} d_1(\mathbf{g}12) = (-1)^s(&s\underline{a_4^{s-1}a_1h_{4,0}h_{4,1}h_{3,1}h_{1,1}h_{2,2}h_{1,3}}_6 \\ &+ \underline{a_4^sh_{1,0}h_{4,1}h_{3,1}h_{1,1}h_{2,2}h_{1,3}}_8 \\ &+ \underline{a_4^sh_{4,0}h_{2,1}h_{1,1}h_{2,2}h_{2,3}h_{1,3}}_3 \\ &- \underline{a_4^sh_{4,0}h_{3,1}h_{1,1}h_{2,2}h_{1,3}h_{1,4}}_{12}), \end{aligned}$$

$$\begin{aligned} d_1(\mathbf{g}13) = (-1)^s(&s\underline{a_4^{s-1}a_2h_{4,0}h_{3,1}h_{1,1}h_{3,2}h_{2,2}h_{1,3}}_7 \\ &+ \underline{a_4^sh_{2,0}h_{3,1}h_{1,1}h_{3,2}h_{2,2}h_{1,3}}_9 \\ &+ \underline{a_4^sh_{4,0}h_{3,1}h_{1,1}h_{1,2}h_{2,3}h_{1,3}}_4 \\ &- \underline{a_4^sh_{4,0}h_{3,1}h_{1,1}h_{2,2}h_{1,3}h_{1,4}}_{12}), \end{aligned}$$

$$\begin{aligned} d_1(\mathbf{g}14) = (-1)^s(&-s\underline{a_4^{s-1}a_3h_{4,0}h_{3,1}h_{1,1}h_{2,2}h_{2,3}h_{1,3}}_{11} \\ &- \underline{a_4^sh_{3,0}h_{3,1}h_{1,1}h_{2,2}h_{2,3}h_{1,3}}_{13} \\ &- \underline{a_4^sh_{4,0}h_{2,1}h_{1,1}h_{2,2}h_{2,3}h_{1,3}}_3 \\ &- \underline{a_4^sh_{4,0}h_{3,1}h_{1,1}h_{1,2}h_{2,3}h_{1,3}}_4 \\ &- \underline{a_4^sh_{4,0}h_{3,1}h_{1,1}h_{2,2}h_{1,3}h_{1,4}}_{12}). \end{aligned}$$

Without generality, we let s be even. Then we get the matrix of coefficients of $d_1(\mathbf{g}7)$, $\cdots$, $d_1(\mathbf{g}14)$ under the elements $\underline{\quad}_1$, $\underline{\quad}_2$, $\underline{\quad}_3$, $\cdots$, $\underline{\quad}_{13}$ which are linearly independent as follows:

$$\begin{pmatrix} -s & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & s & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -s & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & s & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & s & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -s & -1 & -1 \end{pmatrix}$$

By the knowledge of matrix, we can get the rank of the upper matrix is 7. We add a row $(0,0,0,0,0,0,0,0,0,0,0,1,0)$ in the upper matrix and get a new matrix whose rank is also 7. This implies that $(0,0,0,0,0,0,0,0,0,0,0,1,0)$ can be linearly represented by the other rows of the upper matrix. Consequently $\underline{a_4^sh_{4,0}h_{3,1}h_{1,1}h_{2,2}h_{1,3}h_{1,4}}_{12}$ can be linearly represented by the May differentials $d_1(\mathbf{g}7)$, $\cdots$, $d_1(\mathbf{g}14)$, showing that $h_{1,1}h_{1,4}a_4^sh_{4,0}h_{3,1}h_{2,2}h_{1,3} \in E_1^{s+6,t(s,4)+s,9s+18}$ is in $d_1(E_1^{s+5,t(s,4)+s,9s+19})$.

This finishes the proof of Theorem 1.3. □

Acknowledgments. The first author was partially supported by Program for New Century Excellent Talents in University (No. NCET-08-0288) and NSFC (Nos. 10771105, 10501045). The authors thank the referee for valuable comments and suggestions.

References

[1] J.F. Adams, *Stable homotopy and generalised homology*, Univ. Chicago Press, Chicago, Ill., 1974.

[2] A. Liulevicius, The factorization of cyclic reduced powers by secondary cohomology operations, Mem. Amer. Math. Soc. No. **42** (1962), 112 pp.

[3] T. Aikawa, 3-dimensional cohomology of the mod p Steenrod algebra, Math. Scand. **47** (1980), no. 1, 91–115.

[4] X. Wang and Q. Zheng, The convergence of $\tilde{\alpha}_s^{(n)} h_0 h_k$, Sci. China Ser. A **41** (1998), no. 6, 622–628.

[5] X. Liu and H. Zhao, On a product in the classical Adams spectral sequence, Proc. Amer. Math. Soc. **137** (2009), no. 7, 2489–2496.