Finite-type invariants for curves on surfaces

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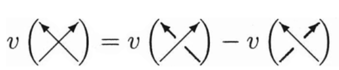
Abstract: In this note, we define a notion of finite-type for invariants of curves on surfaces as an analogue of the notion of finite-type for invariants of knots and 3-manifolds (Section 3). We also present a systematic construction for a large family of finite-type invariants SCI_n for curves on surfaces (Section 5). Arnold's invariants of plane isotopy classes of plane curves occur as invariants of order 1. Our theory of finite-type invariants of curves on surfaces is developed using the topological theory of words.

Key words: Finite-type invariants; immersed curves; topological theory of words; Arnold's invariants.

1. Introduction.

V. A. Vassiliev developed a method for knot classification by applying singularity theory to knots [10]. This method attempts to classify knots by using finite-type invariants (Vassiliev invariants). It remains unknown whether finite-type invariants can classify knots (the Vassiliev conjecture) [3, 10]. Every **C**-valued invariant v of oriented knots is extended inductively to singular knots, knots which may have double points, by resolving the double points using the formula:

(1) 

We say that v is a finite-type invariant of order less than or equal to n if v vanishes on every singular knot with at least $n+1$ double points, where v is extended by (1). It is known that there are many finite-type invariants of knots, though it is not easy to construct them.

V. I. Arnold introduced invariants of generic plane curves using a theory similar to that used by Vassiliev [1, 2]. M. Polyak and O. Viro gave a concrete construction of second and third order Vassiliev invariants by Gauss diagram formulae [5]; a Gauss diagram formula is a formula given by a sum over subdiagrams of a given diagram. Polyak also reconstructed Arnold's invariants by Gauss diagram formulae in a similar manner and combinatorially defined finite-type invariants of plane curves [4]. On the other hand, V. Turaev suggested that words be considered as generalisations of curves and knots and demonstrated that it is possible to classify words in the same manner as knots [7–9].

In this note, we define a notion of finite-type for invariants of curves on surfaces by replacing "double points" in the definition of Vassiliev invariants of knots with "self-tangency points and triple points" of curves on surfaces (Definition 1). Arnold's invariants of plane curves are finite-type invariants of first order in this sense. We also give a systematic construction of a large family of finite-type invariants SCI_n for curves on surfaces using the topological theory of words (Main Theorem). An idea of our construction of finite-type invariants is to consider a sum over subwords of a given word, like a Gauss diagram formula.

In this paper, we present the main results and the ideas of the proofs. The details and generalisations will be presented elsewhere.

2. Curves.

A *curve* is a smooth immersion of an oriented circle into an oriented surface. A curve is *generic* if it has only transversal double points of self-intersection. A curve is *singular* if it has only transversal double points, self-tangency points, and triple points of self-intersection. A pointed curve is a generic curve with a base point on the curve distinct from the self-intersections. Two curves are stably homeomorphic if there is a homeomorphism of their regular neighbourhoods in the ambient surfaces that maps the first curve onto the second one preserving the orientations of the curve and the surface. Simi-

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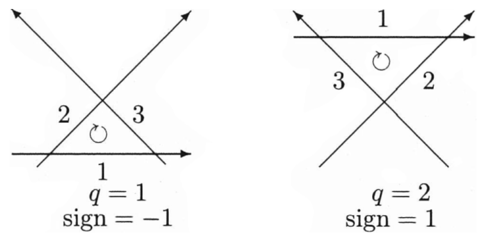


Fig. 1. The newborn triangle and the sign of the triangle.

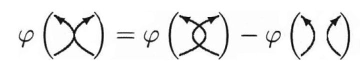
larly, two pointed curves are said to be *stably homeomorphic* if there is a homeomorphism as above preserving the base point.

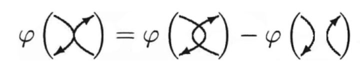
3. Definition of finite-type invariants. In this note, every local self-crossing is either a transversal double point or a point looking like the left hand side of the formulae (2)–(4) and the mirror image of the picture on the left hand side of (3). Note that each of these self-crossings is uniquely determined up to homeomorphism preserving orientation. Self-tangency points or triple points are called *singular points*. In particular, a self-tangency point is a *direct self-tangency* point if the two tangent branches are oriented in the same direction; otherwise, it is called an inverse self-tangency point. The direction of the resolution of a self-tangency point is *positive* if the resolution generates a curve with a larger number of double points.

We define the orientation of a resolution of a triple point following [1]. An arbitrary triple point gives rise to the *newborn triangle* which exists just after a resolution of a triple point as shown in Fig. 1. For the newborn triangle, we define the *sign* of a triangle. By the definition, each singular curve is a immersion of an oriented circle into an oriented surface. Every triple point of a curve has three preimages on the circle and their cyclic order is well defined. Then we obtain a well-defined orientation of the newborn triangle given by the order in which the immersed curve visits its sides. Each side of the immersed curve also has its own orientation. The orientation may coincide with the orientation defined by the cyclic order of the sides of the triangle or may be opposite to it. Let q be the number of the sides of the newborn triangle whose orientations coincide with that given by the cyclic order. The sign of the triangle is defined as $(-1)^q$ (Fig. 1).

The direction of the resolution of the triple point is *positive* if the sign of the newborn triangle is 1 after the resolution. The direction of the resolution of the singular point is negative if the direction is non-positive. It is possible to resolve singular points away from the base point.

Let G be some Abelian group. Every G-valued invariant φ of generic curves is extended inductively to singular curves by resolving the singular points using (2), (3), and (4):

(2) 

(3) 

(4) 

Definition 1. We say that φ is *a finite-type invariant of order less than or equal to* n if φ vanishes on every singular curve with at least $n+1$ singular points (self-tangency points or triple points), where φ is extended by (2), (3) and (4).

Examples of finite-type invariants will be given later in this paper (Sect. 5).

4. Signed words. In this section, we introduce a *signed word*. To define signed words, we review the definition of nanowords following [7–9].

An *alphabet* is a set and *letters* are its elements. A word of length $n \geq 1$ in an alphabet $\mathcal{A}$ is a mapping $w : \hat{n} \to \mathcal{A}$ where $\hat{n} = \{i \in \mathbf{Z} | 1 \leq i \leq n\}$. Such a word is encoded by the sequence $w(1)w(2)\cdots w(n)$. A word $w : \hat{n} \to \mathcal{A}$ is a *Gauss word* if the inverse image of each element of $\mathcal{A}$ consists of precisely two elements of $\hat{n}$.

For a set α, an α-alphabet is a set $\mathcal{A}$ endowed with a mapping $|\ | : \mathcal{A} \ni A \mapsto |A| \in \alpha$. A *nanoword* $(\mathcal{A}, w)$ over α is a pair (an α-alphabet $\mathcal{A}$, a Gauss word in the alphabet $\mathcal{A}$). By definition, there is a unique *empty nanoword* $\emptyset$ of length 0.

An *isomorphism* of α-alphabets $\mathcal{A}_1$, $\mathcal{A}_2$ is a bijection $f : \mathcal{A}_1 \to \mathcal{A}_2$ such that $|A| = |f(A)|$ for all $A \in \mathcal{A}_1$. Two nanowords $(\mathcal{A}_1, w_1)$ and $(\mathcal{A}_2, w_2)$ over α are *isomorphic* if there is an isomorphism of α-alphabets $f : \mathcal{A}_1 \to \mathcal{A}_2$ such that $w_2 = f w_1$. For an arbitrary nanoword $(\mathcal{A}, w)$ over α, a subnanoword of $(\mathcal{A}, w)$ is obtained by deleting a certain set of letters from both the α-alphabet $\mathcal{A}$ and Gauss word w.

Definition 2. Let $\alpha_0 = \{-1, 1\}$. A *signed word of length* $2n$ is a nanoword $(\mathcal{A}, w)$ over α_0 where the length of w is $2n$.

We consider a simple presentation for signed words. For every letter $A \in \mathcal{A}$ of a signed word $(\mathcal{A}, w)$, replace A with $\overline{A}$ if $|A| = -1$ and leave A

unchanged if $|A| = 1$. This allows us to encode a signed word by a sequence of letters $\overline{A}$ or A for $A \in \mathcal{A}$. The isomorphism of two signed words w and w' is written as $w \simeq w'$. For example, $\overline{A}BB\overline{A} \simeq \overline{B}CC\overline{B}$. However, $\overline{A}BB\overline{A}$ is not isomorphic to $A\overline{B}\,\overline{B}A$.

For an arbitrary signed word w, we define a *subword* of w as a sub-nanoword of $(\mathcal{A}, w)$ over α_0. For an arbitrary signed word w, $u \prec w$ means that u is a subword of w. For example, all the subwords of the signed word $A\overline{B}AC\overline{B}C$ are $\emptyset$, AA, $\overline{B}\,\overline{B}$, CC, $A\overline{B}A\overline{B}$, $AACC$, $\overline{B}C\overline{B}C$, and $A\overline{B}AC\overline{B}C$.

5. Construction of the invariant. For two arbitrary signed words u and w, define $\langle\ ,\rangle$ by

$$\langle u, w\rangle = \sum_{v \prec w} (u, v), \tag{5}$$

where (u, v) is 1 if $u \simeq v$ and is 0 otherwise. Let k be a field, $\mathbf{W}$ the k-linear space generated by all the isomorphism classes of the signed words, and $\mathbf{W}_n$ the k-linear space generated by the isomorphism classes of signed words of length $2n$. We denote by $\mathbf{U}^*$ the dual space of a given k-linear space $\mathbf{U}$. We extend $\langle\ ,\rangle$ linearly to $\langle\ ,\rangle : \mathbf{W} \times \mathbf{W} \to k$.

We associate with an arbitrary pointed curve Γ a nanoword $w(\Gamma)$ over α_0 following [8]. Let us label the double points of Γ by distinct letters A_1, $A_2, \ldots, A_m$ where m is the number of double points. Starting at the base point of Γ and following along Γ in the positive direction, we write down the labels of all double points until we return to the base point. Since every double point is traversed twice, this gives a Gauss word, $w(\Gamma)$, on the alphabet $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$. Let t_i^1 (resp., t_i^2) be the tangent vector to Γ at the double point labeled A_i appearing at the first (resp., second) time we pass through this double point. Set $|A_i| = -1$ if the pair (t_i^1, t_i^2) is positively oriented and $|A_i| = 1$ otherwise. This makes $\mathcal{A}$ into an α_0-alphabet and makes $w(\Gamma)$ into a nanoword over α_0. This nanoword is well defined up to isomorphism. This nanoword yields a signed word. For an arbitrary generic curve Γ with m double points, we denote by w_Γ a signed word: $\widehat{2m} \to \mathcal{A}$ that is determined by selecting an arbitrary base point as above.

Definition 3. The linear mapping ν on $\mathbf{W}_n$ is defined by $\nu(AxAy) = x\overline{A}y\overline{A}$ and $\nu(\overline{A}x\overline{A}y) = xAyA$ where x and y are words and A is a letter. For two arbitrary signed words w and w', the *cyclic equivalence* $\sim$ is defined as $w \sim w'$ if and only if there exists $l \in \mathbf{N}$ such that $\nu^l(w) = w'$. Let the subspace $\mathbf{W}_n^\nu$ of $\mathbf{W}_n$ be the linear space generated by $\{w \in \mathbf{W}_n | \nu(w) = w\}$. For a signed word v, $[v]$ denotes the sum of all the signed words which are cyclic equivalent to v. The formula $v \mapsto [v]$ extends to an endomorphism of $\mathbf{W}_n$ also denoted by the square brackets.

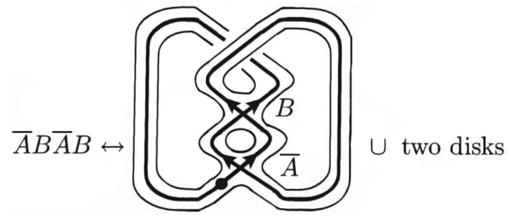


Fig. 2. Signed words and pointed curves.

For example, for $v = AA\overline{B}\,\overline{B}$, the cyclic equivalence class containing v is $\{AA\overline{B}\,\overline{B}, \overline{A}\,\overline{B}\,\overline{B}\,\overline{A}, \overline{B}\,\overline{B}AA, BAAB\}$, and $[v]$ is $AA\overline{B}\,\overline{B} + \overline{A}\,\overline{B}\,\overline{B}\,\overline{A} + \overline{B}\,\overline{B}AA + BAAB$. If $v = AABB$, $[v] = AABB + \overline{A}BB\overline{A}$ because $AABB \simeq BBAA$.

We denote the linear space generated by all stable homeomorphism classes of curves on oriented closed surfaces by $\mathcal{C}$.

Remark 1. We denote the linear space generated by all stable homeomorphism classes of curves with n double points on oriented closed surfaces by $\mathcal{C}_n$. There exists a bijective mapping from $\mathcal{C}_n$ to $\mathbf{W}_n^\nu$; this has been proved by V. Turaev [8]. Specifically, every signed word determines a regular neighbourhood of a curve Γ on a surface $\mathcal{S}$, where Γ gives the CW-decomposition of $\mathcal{S}$ (Fig. 2) [6]. In the rest of this paper, we identify $\mathcal{C}_n$ with $\mathbf{W}_n^\nu$ and $\mathcal{C}$ with $\mathbf{W}^\nu$ by $\Gamma \mapsto [w_\Gamma]$, where $\mathbf{W}^\nu$ is the linear space generated by $\{w \in \mathbf{W} \,|\, \nu(w) = w\}$.

For an arbitrary natural number n, we define a *signed curve invariant of order n*, $SCI_n : \mathbf{W}^\nu \to \mathbf{W}_n^*$, as follows: For a generic curve $\Gamma \in \mathbf{W}^\nu$, we define $SCI_n(\Gamma) : \mathbf{W}_n \to k$ by

$$SCI_n(\Gamma)(v) = \langle [v], w_\Gamma\rangle \quad (v \in \mathbf{W}_n), \tag{6}$$

where w_Γ is a signed word determined by Γ. We can verify, from the definitions of $[\ ]$ and $\langle\ ,\rangle$, that $SCI_n(\Gamma)$ is defined independently of the choice of base point of Γ.

Main theorem. *For any generic curve Γ on an oriented surface, $SCI_n(\Gamma)$ is a finite-type invariant of generic curves of order less than or equal to n.*

The proof of the main theorem is given in Sect. 7.

Let m_w be the number of representative elements of the cyclic equivalence class that contains a signed word w. Let $\overline{w} = \frac{1}{m_w}[w] \in \mathbf{W}^{\nu}$. Then, $SCI_n(\Gamma) = \langle [\cdot], \overline{w}_\Gamma \rangle$. For two arbitrary signed words v and u, v^* denotes a linear mapping such that $v^*(u) = (v, u)$. We extend $*$ linearly to a map $* : \mathbf{W} \to k$ by $(au + bv)^* = au^* + bv^*$ for a, $b \in k$. SCI_n restricted to $\mathbf{W}_n^{\nu}$ gives an isomorphism between $\mathbf{W}_n^{\nu}$ and $(\mathbf{W}_n^{\nu})^*$, sending $\overline{w}_\Gamma$ to $[w_\Gamma]^*$. We denote this isomorphism by ι_n. Let us consider some examples of SCI_n. Note that $SCI_n(\Gamma)(v) = \sum_u [u]^*(v)$ where $u \prec w_\Gamma$ and $u \in \mathbf{W}_n$. If Γ is a given curve and w_Γ is $\overline{A}\,\overline{B}C\overline{A}C\overline{B}$, then $SCI_1(\Gamma) = [\overline{A}\,\overline{A}]^* + [\overline{B}\,\overline{B}]^* + [CC]^*$, $SCI_2(\Gamma) = [\overline{A}\,\overline{B}\,\overline{A}\,\overline{B}]^* + [\overline{A}C\overline{A}C]^* + [\overline{B}CC\overline{B}]^*$, $SCI_3(\Gamma) = [\overline{A}\,\overline{B}C\overline{A}C\overline{B}]^*$, $SCI_n(\Gamma) = 0$ $(n \geq 4)$. $SCI_1(\Gamma)(XX) = [\overline{A}\,\overline{A}]^*(XX) + [\overline{B}\,\overline{B}]^*(XX) + [CC]^*(XX) = 3$.

Theorem 5.1. *For arbitrary k, l such that $1 \leq l \leq k \leq n$, the following holds*:

$$\binom{n-l}{k-l} SCI_l = SCI_l|_{\mathbf{W}_k^{\nu}} \circ \iota_k^{-1} \circ SCI_k. \tag{7}$$

Proof. The equality $(n-k+1) \cdot \binom{n}{k-1} = \binom{n}{k} \cdot \binom{k}{k-1}$ implies

$$(n-k+1) SCI_{k-1} = SCI_{k-1}|_{\mathbf{W}_k^{\nu}} \circ \iota_k^{-1} \circ SCI_k \tag{8}$$

for every k such that $2 \leq k \leq n$. By using (8), we see that $(n-l)!SCI_l = (n-k)!SCI_l|_{\mathbf{W}_{l+1}^{\nu}} \circ \iota_{l+1}^{-1} \circ SCI_{l+1}|_{\mathbf{W}_{l+2}^{\nu}} \circ \iota_{l+2}^{-1} \circ \cdots \circ SCI_{k-1}|_{\mathbf{W}_k^{\nu}} \circ \iota_k^{-1} \circ SCI_k = (n-k)!(k-l)!SCI_l|_{\mathbf{W}_k^{\nu}} \circ \iota_k^{-1} \circ SCI_k$. □

6. Arnold's invariants and *SCI*$_n$. In Sect. 6, Γ is used to denote an arbitrary generic plane curve. Let w_Γ be a signed word determined by selecting an arbitrary base point, and by using the definition in Sect. 5. Arnold's basic invariants (J^+, J^-, St) and SCI_n $(n \in \mathbf{N})$ are invariants for plane isotopy classes of curves on a plane. The definitions of J^+, J^- and St are provided in [1, 2].

Let $i(\Gamma)$ be the rotation number of a generic plane curve Γ. Then, $i(\Gamma)$ is a finite-type invariant of order 0 because the left-hand sides of (2), (3) and (4) should be equal to 0. We obtain the following equations:

$$J^+(\Gamma) - J^-(\Gamma) = SCI_1(\Gamma)(AA), \tag{9}$$

$$\begin{aligned} J^-(\Gamma) + 6St(\Gamma) = &-2SCI_2(\Gamma)(AABB \\ &- AA\overline{B}\,\overline{B} + \overline{A}\,\overline{A}\,\overline{B}\,\overline{B}) \\ &+ i^2(\Gamma) - 1. \end{aligned} \tag{10}$$

However, J^+, J^- and St cannot be represented by $i(\Gamma)$, SCI_1 and SCI_2 in a similar manner.

7. Proof of the main theorem. It is now shown that for an extended invariant I for singular curves with SCI_n, $I(\Gamma) = 0$ for an arbitrary singular curve Γ with m $(\geq n+1)$ singular points $(P_1, P_2, \ldots, P_m)$. We define σ by the mapping $P_i \mapsto \sigma_i$ and $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m)$, where $\sigma_i = \pm 1$. Let $\mathcal{J} = \{1, 2, \ldots, m\}$ and $\operatorname{sign}\sigma = \prod_{i \in \mathcal{J}} \sigma_i$. A generic curve Γ_σ is obtained from Γ by resolving at P_i, where $i = 1, 2, , m$, in the positive direction if $\sigma_i = 1$ and in the negative direction if $\sigma_i = -1$. Let w_{Γ_σ} be a signed word derived from Γ_σ by choosing an arbitrary base point.

By using the definitions of σ and Γ_σ in (2), (3) and (4), we obtain

$$I(\Gamma) = \sum_{\sigma \in \{-1,1\}^m} \operatorname{sign}\sigma \, SCI_n(\Gamma_\sigma). \tag{11}$$

Then, by using (5), (6), (11) and $[v] = \sum_u u$, where $v \sim u$, we obtain

$$\begin{aligned} (\Gamma)(v) &= \sum_{\sigma \in \{-1,1\}^m} \operatorname{sign}\sigma \sum_{u (\sim v)} \langle u, w_{\Gamma_\sigma} \rangle \\ &= \sum_{u (\sim v)} \sum_{\sigma \in \{-1,1\}^m} \operatorname{sign}\sigma \langle u, w_{\Gamma_\sigma} \rangle. \end{aligned} \tag{12}$$

In order to show that $\sum_\sigma \operatorname{sign}\sigma \, \langle u, w_{\Gamma_\sigma} \rangle$ vanishes, we consider the following conditions. For an arbitrary Γ, we can determine the set of generic curves $\{\Gamma_\sigma\}_\sigma$. Let w_{σ_0} be a signed word corresponding to Γ_{σ_0} with $\sigma_0 = (1, 1, \ldots, 1)$. By using w_{σ_0}, w_σ is determined as follows:

Let A, B, C, $\overline{A}$, $\overline{B}$ and $\overline{C}$ be the letters of signed words, and let x, y, z, and t be words. In the case of a negative resolution of a direct self-tangency point, it is necessary to consider two cases: if $w_{\Gamma_{\sigma_0}} = x\overline{A}By\overline{A}Bz$, then $w_{\Gamma_\sigma} = xyz$ (Fig. 3), and if $w_{\Gamma_{\sigma_0}} = xA\overline{B}yA\overline{B}z$, then $w_{\Gamma_\sigma} = xyz$. Even in the case of a negative resolution of an inverse self-tangency point, there are two cases: if $w_{\Gamma_{\sigma_0}} = x\overline{A}ByB\overline{A}z$, then $w_{\Gamma_\sigma} = xyz$ (Fig. 4), and if $w_{\Gamma_{\sigma_0}} = xA\overline{B}y\overline{B}Az$, then $w_{\Gamma_\sigma} = xyz$. In the case of a negative resolution of a triple point, there are eight cases: if $w_{\Gamma_{\sigma_0}} = xAByCBzCAt$, then $w_{\Gamma_\sigma} = xBAyBCzACt$; if $w_{\Gamma_{\sigma_0}} = x\overline{A}By\overline{A}CzCBt$, then $w_{\Gamma_\sigma} = xB\overline{A}yC\overline{A}zBCt$; if $w_{\Gamma_{\sigma_0}} = xA\overline{B}yCAz\overline{B}Ct$, $w_{\Gamma_\sigma} = x\overline{B}AyACzC\overline{B}t$ (Fig. 5); if $w_{\Gamma_{\sigma_0}} = xAByB\overline{C}zA\overline{C}t$, then $w_{\Gamma_\sigma} = xBAy\overline{C}Bz\overline{C}At$; if $w_{\Gamma_{\sigma_0}} = x\overline{A}\,\overline{B}y\overline{C}\,\overline{B}z\overline{C}\,\overline{A}t$, then $w_{\Gamma_\sigma} = x\overline{B}\,\overline{A}y\overline{B}\,\overline{C}z\overline{A}\,\overline{C}t$; if $w_{\Gamma_{\sigma_0}} = xA\overline{B}yA\overline{C}z\overline{C}\,\overline{B}t$, then $w_{\Gamma_\sigma} = x\overline{B}Ay\overline{C}Az\overline{B}\,\overline{C}t$; if $w_{\Gamma_{\sigma_0}} = x\overline{A}By\overline{C}\,\overline{A}zB\overline{C}$,

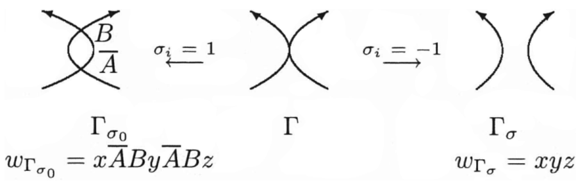


Fig. 3. An example of w_{Γ_σ} obtained by a negative resolution at a direct self-tangency point P_i.

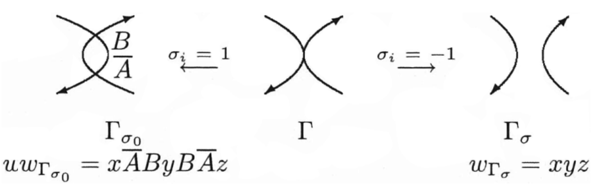


Fig. 4. An example of w_{Γ_σ} obtained by a negative resolution at an inverse self-tangency point P_i.

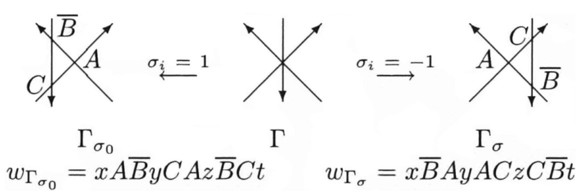


Fig. 5. An example of w_{Γ_σ} obtained by a negative resolution at a triple point P_i.

then $w_{\Gamma_\sigma} = xB\overline{A}y\overline{A}\,\overline{C}z\overline{C}Bt$; if $w_{\Gamma_{\sigma_0}} = x\overline{A}\,\overline{B}y\overline{B}Cz\overline{A}Ct$, then $w_{\Gamma_\sigma} = x\overline{B}\,\overline{A}yC\overline{B}zC\overline{A}t$.

Let u' be a subword of $w_{\Gamma_{\sigma_0}}$ and let $\sigma(u')$ be the subword of w_{Γ_σ} corresponding to u'. Note that $\sigma(u')$ may be $\emptyset$ for a negative resolution of a self-tangency point. The subword v' of w_{Γ_σ} is represented as $\sigma(u')$. By using (5) and the notion above,

$$\begin{aligned}
&\sum_{\sigma\in\{-1,1\}^m} \operatorname{sign}\sigma\langle u, w_{\Gamma_\sigma}\rangle \\
&= \sum_{\sigma\in\{-1,1\}^m} \operatorname{sign}\sigma \sum_{v'\prec w_{\Gamma_\sigma}} (u, v') \\
&= \sum_{\sigma\in\{-1,1\}^m} \operatorname{sign}\sigma \sum_{\sigma(u')\prec w_{\Gamma_\sigma}} (u, \sigma(u')) \\
&= \sum_{\sigma\in\{-1,1\}^m} \operatorname{sign}\sigma \sum_{u'\prec w_{\Gamma_{\sigma_0}}} (u, \sigma(u')) \\
&= \sum_{u'\prec w_{\Gamma_{\sigma_0}}} \sum_{\sigma\in\{-1,1\}^m} \operatorname{sign}\sigma(u, \sigma(u')).
\end{aligned} \tag{13}$$

Let $\mathcal{J}_{u'}$ be $\{\, i \in \mathcal{J} \mid$ a letter of $\sigma(u')$ is generated by a resolution at $P_i\}$. The sum over σ can be divided into one part with σ_i $(i \in \mathcal{J}_{u'})$ and another part with σ_i $(i \in \mathcal{J} \setminus \mathcal{J}_{u'})$. Hence, we have

$$\begin{aligned}
&\sum_{\sigma\in\{-1,1\}^m} \operatorname{sign}\sigma(u, \sigma(u')) \\
&= \sum_{\substack{\sigma_i=\pm1 \\ i\in\mathcal{J}_{u'}}} \prod_{i\in\mathcal{J}_{u'}} \sigma_i \sum_{\substack{\sigma_i=\pm1 \\ i\in\mathcal{J}\setminus\mathcal{J}_{u'}}} \prod_{i\in\mathcal{J}\setminus\mathcal{J}_{u'}} \sigma_i(u, \sigma(u')).
\end{aligned} \tag{14}$$

For an arbitrary u, where $\sigma(u') \in \mathbf{W}_n$, there exists at least one singular point P_i that is not related to $(u, \sigma(u'))$ because u and $u' \in \mathbf{W}_n$ and $m \geq n + 1$. In other words, $\mathcal{J} \setminus \mathcal{J}_{u'}$ is not empty. Then,

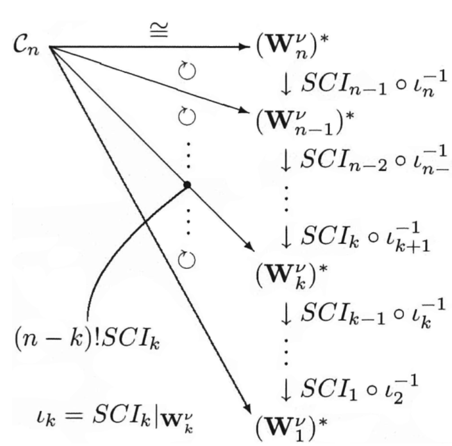
$$\begin{aligned}
&\sum_{\substack{\sigma_i=\pm1 \\ i\in\mathcal{J}\setminus\mathcal{J}_{u'}}} \prod_{i\in\mathcal{J}\setminus\mathcal{J}_{u'}} \sigma_i(u, \sigma(u')) \\
&= (u, \sigma(u')) \sum_{\substack{\sigma_i=\pm1 \\ i\in\mathcal{J}\setminus\mathcal{J}_{u'}}} \prod_{i\in\mathcal{J}\setminus\mathcal{J}_{u'}} \sigma_i \\
&= 0.
\end{aligned} \tag{15}$$

Here, (15) implies that $I(\Gamma)(v)$ in (12) vanishes, and $I(\Gamma) = 0$.

8. Summary. In this note, the author defines a notion of finite-type for invariants of curves on surfaces by using Arnold's idea related to the singularity theory among various definitions. The author is unable to determine the structure of the linear space $\mathbf{V}_n$ generated by finite-type invariants of order less than or equal to n.

Because of (6) and (8) we have the following relations between the subspaces $(\mathbf{W}^\nu_{\mathbf{n}})^*$ of $\mathbf{V}_n$.

(16)



This diagram shows how SCI_k $(1 \leq k \leq n)$ reduce information from $\mathcal{C}_n \simeq (\mathbf{W}^\nu_n)^*$ to $(\mathbf{W}^\nu_1)^*$.

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References

[1] V.I. Arnold, Plane curves, their invariants, perestroikas and classifications, in *Singularities and bifurcations*, 33–91, Amer. Math. Soc., Providence, RI., 1994.

[2] V.I. Arnold, *Topological invariants of plane curves and caustics*, Amer. Math. Soc., Providence, RI, 1994.

[3] J.S. Birman and X.-S. Lin, Knot polynomials and Vassiliev's invariants, Invent. Math. **111** (1993), no. 2, 225–270.

[4] M. Polyak, Invariants of curves and fronts via Gauss diagrams, Topology **37** (1998), no. 5, 989–1009.

[5] M. Polyak and O. Viro, Gauss diagram formulas for Vassiliev invariants, Internat. Math. Res. Notices **1994**, no. 11, 445ff., approx. 8 pp. (electronic).

[6] V. Turaev, Curves on surfaces, charts, and words, Geom. Dedicata **116** (2005), 203–236.

[7] V. Turaev, Topology of words, Proc. Lond. Math. Soc. (3) **95** (2007), no. 2, 360–412.

[8] V. Turaev, Knots and words, Int. Math. Res. Not. **2006**, Art. ID 84098, 23 pp.

[9] V. Turaev, Lectures on topology of words, Jpn. J. Math. **2** (2007), no. 1, 1–39.

[10] V.A. Vassiliev, Cohomology of knot spaces, in *Theory of singularities and its applications*, 23–69, Amer. Math. Soc., Providence, RI.