On new singular directions of some Schröder functions

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Abstract: In this paper, we study the Hayman T directions and the precise Borel directions of maximal kind of meromorphic solutions $f(z)$ of the Schröder equations $f(sz) = R(f(z))$, where $|s| > 1$ and $R(w)$ is a rational function with $\deg[R] \geq 2$. We will show that, if $arg[s]/2\pi \notin Q$, then $f(z)$ has any direction as Hayman T direction and maximus Borel direction as well. This is a continue work of [Ishizaki, K. and Yanaihara, N., Borel and Julia directions of meromorphic Schröder functions, Math. Proc. Camb. Phil. Soc. 139 (2005), 139–147.] and [Yuan, W.J., Qi, J.M. and Seiki Mori. Singular directions of meromorphic solutions of some non-autonomous Schröder equations, Complex Analysis and its Applications Proceedings of the 15th ICFIDCAA held in Osaka (Japan), July 30–August 3, 2007].

Key words: Hayman T direction; Borel direction of maximal kind; Schröder function.

1. Introduction and results. Let $R(w)$ be a rational function of degree $p \geq 2$ and s be a complex number with $|s| > 1$. Consider the Schröder equation

$$f(sz) = R(f(z)). \tag{1.1}$$

It is known that under some conditions (1.1) has a transcendental meromorphic solution $f(z)$, which is called Schröder function and its order is $\lambda = \log p/\log|s| > 0$. We suppose that the readers are familiar with the notations of the value distribution theory of meromorphic functions, such as $m(r, f)$, $N(r, f), T(r, f)$, for the detail see [12]. In order to make our statement clear, let us begin with some basic notations. Given an angular domain $\Omega = \{z : \alpha \leq \arg z \leq \beta\}$. Let $f(z)$ be a meromorphic function in Ω. Define

$$N(r, \Omega, f = a) = \int_1^r \frac{n(t, \Omega, f = a)}{t} dt$$

where $n(t, \Omega, f = a)$ is the number of the roots of $f(z) = a$ in $\Omega \cap \{1 < |z| < t\}$ counted according to multiplicity. Let $\overline{n}(t, \Omega, f = a)$ be the number of the roots of $f(z) = a$ in $\Omega \cap \{1 < |z| < t\}$ counted only once, then $\overline{N}(t, \Omega, f = a)$ can be defined in the same way.

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The singular directions of the meromorphic solutions of the Schröder equations are studied abroad (see [7–9]). Ishizaki, K. and Yanaihara, N. [7] investigated the Borel and Julia directions of some Schröder equations and got the following result.

Theorem A. *Let $f(z)$ be a meromorphic solution of the Schröder equations $f(sz) = R(f(z))$, where $|s| > 1$ and $R(w)$ is a rational function with* $\deg[R] \geq 2$. *If* $\arg[s]/2\pi \notin Q$, *then $f(z)$ has any direction as Borel direction.*

Zheng [14] introduced a new singular direction, which is called T direction. The existence of T direction was first confirmed by Guo, Zheng and Ng [4]. We recall its definition as follows:

Definition 1.1. *Let $f(z)$ be a meromorphic function in the complex plane. A direction* $\arg z = \theta$ *is called a T direction of $f(z)$, provided that given any $b \in \hat{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$ and any small $\varepsilon > 0$ we have*

$$\limsup_{r\to\infty} \frac{N(r, \Delta(\theta, \varepsilon), f = b)}{T(r, f)} > 0, \tag{1.2}$$

possibly with the exception of at most two values of b, where $\Delta(\theta, \varepsilon) = \{z : \theta - \varepsilon < \arg z < \theta + \varepsilon\}$. A direction $\arg z = \theta$ *is called a precise T direction of $f(z)$, if in (1.2), $N(r, \Delta(\theta, \varepsilon), f = b)$ is replaced by $\overline{N}(r, \Delta(\theta, \varepsilon), f = b)$.*

Later, Yuan, Qi and Seiki Mori [9] made some discussion on the T direction and Nevanlinna direction (see [9] for its definition) of some Schröder functions. And they got the following theorem.

Theorem B. *Let $f(z)$ be a meromorphic solution of the Schröder equations $f(sz) = R(f(z))$, where*

$|s| > 1$ and $R(w)$ is a rational function with $\deg[R] \geq 2$. If $\arg[s]/2\pi \notin Q$, then $f(z)$ has any direction as precise T direction and Nevanlinna direction as well.

According to the Hayman inequality (see [5]) on the estimation of $T(r, f)$ in terms of only two integrated counting functions for the roots of $f(z) = a$ and $f^{(k)}(z) = b$ with $b \neq 0$, Guo, Zheng and Ng proposed in [4] a singular direction named Hayman T direction as follows:

Definition 1.2. Let $f(z)$ be a transcendental meromorphic function. A direction $\arg z = \theta$ is called a Hayman T direction of $f(z)$ if for any small $\varepsilon > 0$, any positive integer k and any complex numbers a and $b \neq 0$, we have

$$\limsup_{r\to\infty} \frac{N(r, \Delta(\theta, \varepsilon), f = a) + N(r, \Delta(\theta, \varepsilon), f^{(k)} = b)}{T(r, f)} > 0.$$

Most recently, Zheng and Wu [15] discussed the existence of Hayman T directions of meromorphic functions and they proved the following

Theorem C. *Let $f(z)$ be a transcendental meromorphic function satisfying*

$$\limsup_{r\to\infty} \frac{T(r, f)}{(\log r)^3} = +\infty. \tag{1.3}$$

Then $f(z)$ has a Hayman T direction which is a T direction as well.

Remark 1. In the same paper, the authors gave an example to show the growth condition (1.3) is sharp. And they pointed out the Hayman T direction is different from the T direction.

It is interesting to investigate the Hayman T directions of the Schröder functions. We will obtain Theorem 1.1.

Theorem 1.1. *Let $f(z)$ be a meromorphic solution of the Schröder equations $f(sz) = R(f(z))$, where $|s| > 1$ and $R(w)$ is a rational function with $\deg[R] \geq 2$. If $\arg[s]/2\pi \notin Q$, then $f(z)$ has any direction as Hayman T direction.*

G. Valirion is the first one to introduce the concept of a proximate order $\lambda(r)$ for a meromorphic function f with finite positive order and the type function $U(r) = r^{\lambda(r)}$. And the following result is well known.

Proposition 1.1. *Let $T(r,f)$ be the Nevanlinna characteristic function of $f(z)$ with order*

$$\lambda(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r} < \infty.$$

Then there exists a function $\lambda(r)$ with the following properties

(1) *$\lambda(r)$ is monotone and piecewise continuous differentiable function for $r \geq r_0$, with $\lim_{r\to\infty} \lambda(r) = \lambda$;*

(2) $\lim_{r\to\infty} \lambda'(r) r \log r = 0$;

(3) $\limsup_{r\to\infty} \dfrac{T(r, f)}{U(r)} = 1$;

(4) *for each positive number d,*

$$\lim_{r\to\infty} \frac{U(dr)}{U(r)} = d^{\lambda}, \quad U(r) = r^{\lambda(r)}.$$

The proof can be found in Chuang [3]. In 1932, G. Valiron raised in terms of his type function the concept of one Borel direction of maximal kind, which is a direction such that for any small $\varepsilon > 0$, and any $a \in \hat{\mathbf{C}}$, possibly except at most two values of a, we have

$$\limsup_{r\to\infty} \frac{n(r, \Delta(\theta, \varepsilon), f = a)}{U(r)} > 0.$$

In 1983, Pang [11] studied the U type directions of meromorphic functions, and he obtained a theorem as follows:

Theorem D. *Let $f(z)$ be a meromorphic function with order $0 < \lambda < \infty$, $\lambda(r)$ be its proximate order, $U(r) = r^{\lambda(r)}$, then there exists an half line B : $\arg z = \theta$, for any meromorphic function $a(z)$, such that $T(r, a(z)) = o(U(r))$, for any $\varepsilon > 0$, we have*

$$\limsup_{r\to\infty} \frac{n(r, \Delta(\theta, \varepsilon), f = a(z))}{U(r)} > 0$$

with at most two exceptional functions.

From this, we should elicit the concept of the precise maximal Borel direction dealing with small functions.

Definition 1.3. Let $f(z)$ be a transcendental meromorphic function. A direction $\arg z = \theta$ is called a precise Borel direction of maximal kind of $f(z)$ if for any small $\varepsilon > 0$, for any small functions $a(z)$ such that $T(r, a(z)) = o(U(r))$ as $r \to \infty$, we have

$$\limsup_{r\to\infty} \frac{\overline{n}(r, \Delta(\theta, \varepsilon), f = a(z))}{U(r)} > 0,$$

with at most two exceptions.

What is the case for meromorphic solutions of the Schröder equations? Next, we will prove Theorem 1.2.

Theorem 1.2. *Let $f(z)$ be a meromorphic solution of the Schröder equations $f(sz) = R(f(z))$,* *where* $|s| > 1$ *and* $R(w)$ *is a rational function with* $\deg[R] \geq 2$. *If* $\arg[s]/2\pi \notin Q$, *then* $f(z)$ *has any direction as precise Borel direction of maximal kind dealing with small functions, here the small function* $a(z)$ *is defined as* $T(r,a) = o(U(r))$.

Remark 2. Obviously, the precise Borel direction of maximal kind must be the precise T direction, so we have obtained the existence of the precise T directions of the Schröder functions dealing with small functions, here the small function $a(z)$ is defined as $T(r,a) = o(T(r,f))$.

2. Some lemmas. First, let us recall Ahlfors-Shimizu characteristic in an angle (see [10]). Let $f(z)$ be a meromorphic function on an angle $\Omega = \{z : \alpha \leq arg\ z \leq \beta\}$. Set $\Omega(r) = \Omega \cap \{z : 1 < |z| < r\}$. Define

$$\mathcal{S}(r,\Omega,f) = \frac{1}{\pi}\iint_{\Omega(r)} \left(\frac{|f'(z)|}{1+|f(z)|^2}\right)^2 d\sigma,$$

and

$$\mathcal{T}(r,\Omega,f) = \int_1^r \frac{\mathcal{S}(t,\Omega,f)}{t}\,dt.$$

In order to prove our theorems, we need the following lemmas.

Lemma 2.1 [5]. *Let* $f(z)$ *be a meromorphic function in the whole complex plane* $\mathbf{C}$. *Then*

$$|T(r,f) - \mathcal{T}(r,\mathbf{C},f) - \log^+|f(0)|| \leq \frac{1}{2}\log 2.$$

Lemma 2.2 [8]. *Let* $f(z)$ *be a Schröder function of* (1.1) *with order* λ. *Then it holds that*

$$K_1 r^\lambda \leq \mathcal{T}(r,\mathbf{C},f) \leq K_2 r^\lambda$$

for some constants $0 < K_1 \leq K_2$, $\lambda = \log p/\log|s| > 0$.

Lemma 2.3 [8]. *Let* $R(w)$ *be a rational function, and* $f(z)$ *be a meromorphic function on* $\Omega(\alpha,\beta)$, *then for a constant* $L > 0$, *we have*

$$\mathcal{T}(r,\Omega,R(f)) \leq L\mathcal{T}(r,\Omega,f).$$

The following lemma is a theorem in [15], which is to control the term $\mathcal{T}(r,\Omega_\varepsilon)$ with the counting functions $N(r,\Omega,f=a)$ and $N(r,\Omega,f^{(k)}=b)$.

Lemma 2.4. *Let* $f(z)$ *be meromorphic in an angle* $\Omega = \{z : \alpha \leq \arg z \leq \beta\}$. *Then for any small* $\varepsilon > 0$, *any positive integer* k *and any two complex numbers* a *and* $b \neq 0$, *we have*

$$\mathcal{T}(r,\Omega_\varepsilon,f) \leq K\{N(2r,\Omega,f=a) + N(2r,\Omega,f^{(k)}=b)\} + O(\log^3 r)$$

for a positive constant K *depending only on* k, *where* $\Omega_\varepsilon = \{z : \alpha+\varepsilon < \arg z < \beta-\varepsilon\}$.

The following lemma is the second fundamental theorem for the case of a meromorphic function of slow growth, which is Theorem VII.3 in [10].

Lemma 2.5 [10]. *Let* $f(z)$ *be meromorphic in an angular domain* Ω. *Then for any small* $\varepsilon > 0$ *and three distinct points* a_j $(j=1,2,3)$ *on* $\hat{\mathbf{C}}$, *we have*

$$\mathcal{T}(r,\Omega_\varepsilon,f) \leq 3\sum_{j=1}^{3}\overline{N}(2r,\Omega,f=a_j) + O(\log^2 r)$$

for $r > 1$.

The following lemma is applicable in the discussion of angular distribution of a meromorphic function dealing with small functions, which is Theorem VIII in [10].

Lemma 2.6 [10]. *Let* $f(z)$ *and* $a_j(z)$ $(j=1,2,3,4)$ *be meromorphic functions in the complex plane and*

$$g(z) = \frac{a_1(z)f(z)+a_2(z)}{a_3(z)f(z)+a_4(z)}.$$

Consider an angle $\Omega(\alpha,\beta)$ *with* $0 < \beta-\alpha \leq 2\pi$, *then for any* $\varepsilon > 0$, *we have*

$$\mathcal{T}(r,\Omega_\varepsilon,g) \leq 27\mathcal{T}(64r,\Omega,f) + O\left(\int_1^r \frac{1}{t}\int_1^{128t}\frac{T(s,a)}{s}\,ds dt\right),$$

where $T(r,a) = \sum_{j=1}^4 T(r,a_j)$.

The following lemma is Lemma 1.1.2 in [13], which is useful for our study.

Lemma 2.7. *Let* $T(r)$ *be a non-negative and non-decreasing function in* $0 < r < \infty$. *If*

$$\liminf_{r\to\infty}\frac{T(dr)}{T(r)} \geq c > 1$$

for some $d > 1$, *then*

$$\int_1^r \frac{T(t)}{t}\,dt \leq \frac{2c\log d}{c-1}T(r) + O(1).$$

By Lemma 2.7, we can establish the following result.

Lemma 2.8. *Let* $f(z)$ *be a meomorphic function on the whole plane with order* λ, *and* $U(r)$ *be its type function, if there exists a direction* $\arg z = \theta$ *such that for any* $\varepsilon > 0$, *the following holds*

$$\limsup_{r\to\infty}\frac{\overline{N}(r,\Delta(\theta,\varepsilon),f=a)}{U(r)} > 0$$

with at most two values for exception, then $\arg z = \theta$ *must be a precise Borel direction of maximal kind of* $f(z)$.

Proof. If the result is not true, then there exist three complex numbers a_1, a_2, a_3 such that

$$\overline{n}(r, \Delta(\theta, \varepsilon), f = a_i) = o(U(r)),$$

then by Lemma 2.7, we have

$$\overline{N}(r, \Delta(\theta, \varepsilon), f = a_i) = o\left(\int_1^r \frac{U(t)}{t} dt\right) = o(U(r)).$$

This leads a contradiction. □

Lemma 2.9. *If* $a_i(z)$ $(i = 1, 2, 3, 4)$ *be four small functions such that* $T(r, a_i) = o(U(r))$, *then we have*

$$\int_1^r \frac{1}{t} \int_1^{128t} \frac{T(s, a)}{s} ds dt = o(U(r)),$$

where $T(r, a) = \sum_{j=1}^{4} T(r, a_j)$.

The proof of Lemma 2.9 is similar to Lemma 2.8, and we omit it.

3. Proof of Theorem 1.1.

Proof. We have known that the Schröder function $f(z)$ of (1.1) satisfies that

$$K_1 r^\lambda \le \mathcal{T}(r, \mathbf{C}, f) \le K_2 r^\lambda.$$

Dividing $\mathbf{C}$ into two sectors $\Omega_1 = \Delta(0, \frac{\pi}{2})$ and $\Omega_2 = \Delta(\pi, \frac{\pi}{2})$, we obtain that

$$\mathcal{T}(r, \Omega_j, f) \ge \frac{K_1}{2} r^\lambda, \quad \textit{for} \quad j = 1 \quad \textit{or} \quad j = 2.$$

When it holds for j (say $j = 1$), we divide Ω_1 into two sectors. Repeating this procedure, we get a direction $\arg z = \theta^*$ such that, for $\Delta_n^* = \Delta(\theta^*, \frac{2\pi}{2^n})$, and we have

$$\mathcal{T}(r, \Delta_n^*, f) \ge \frac{K_1}{2^n} r^\lambda$$

for any $n \in N$. Take any direction $\arg z = \theta_0$ and a sector $\Delta(\theta_0, \varepsilon)$. Choose a n_0 such that $\frac{2\pi}{2^{n_0}} < \frac{\varepsilon}{8}$. Thus there is a j_0 such that $j_0 > n_0$ and $|(\theta_0 + j_0 \arg s) - \theta^*| < \frac{\varepsilon}{8} (\mathrm{mod}\ 2\pi)$. By (1.1), we obtain that $f(z) = R^{j_0}(f(s^{-j_0} z))$, where $R^{j_0}(w)$ is the j_0-th iteration of $R(w)$. Thus by Lemma 2.3, with some constant $L(j_0)$,

$$\begin{aligned} \mathcal{T}(r, \Delta_{j_0}^*, f) &= \mathcal{T}(r, \Delta_{j_0}^*, R^{j_0}(f(s^{-j_0} z))) \\ &\le L(j_0) \mathcal{T}\left(|s|^{-j_0} r, \Delta\left(\theta_0, \frac{\varepsilon}{4}\right), f\right). \end{aligned} \tag{3.1}$$

Suppose that the direction $\arg z = \theta_0$ does not satisfy the property, then there exist $a \in \mathbf{C}$, and $b \neq 0$, $b \in \mathbf{C}$, such that

$$\begin{aligned} N(2r, \Delta(\theta_0, \varepsilon), f = a) + N(2r, \Delta(\theta_0, \varepsilon), f^{(k)} = b) \\ = o(\mathcal{T}(2r, \mathbf{C}, f)). \end{aligned}$$

By Lemma 2.4, we have

$$\mathcal{T}\left(r, \Delta\left(\theta_0, \frac{\varepsilon}{2}\right), f\right) = o(\mathcal{T}(2r, \mathbf{C}, f)) \quad as \quad r \to \infty.$$

Hence

$$\begin{aligned} \frac{K_1}{2^{j_0}} r^\lambda &\le \mathcal{T}(r, \Delta_{j_0}^*, f) \le L(j_0) \mathcal{T}\left(|s|^{-j_0} r, \Delta\left(\theta_0, \frac{\varepsilon}{4}\right), f\right) \\ &\le L(j_0) \mathcal{T}\left(|s|^{-j_0} r, \Delta\left(\theta_0, \frac{\varepsilon}{2}\right), f\right) \\ &= o(\mathcal{T}(2|s|^{-j_0} r, \mathbf{C}, f)) = o(K_2(2|s|^{-j_0} r)^\lambda), \\ &\quad as\ r \to \infty. \end{aligned}$$

This is impossible. Thus $f(z)$ has any direction as Hayman T direction. The proof is completed. □

4. Proof of Theorem 1.2.

Proof. We will use Lemma 2.5, Lemma 2.6, Lemma 2.8 and Lemma 2.9 to prove our theorem.

We have known that the Schröder function $f(z)$ of (1.1) satisfies that

$$K_1 r^\lambda \le \mathcal{T}(r, \mathbf{C}, f) \le K_2 r^\lambda.$$

Let $U(r)$ be the type function of $T(r, f)$, then we have

$$\limsup_{r \to \infty} \frac{\mathcal{T}(r, \mathbf{C}, f)}{U(r)} > 0.$$

Dividing $\mathbf{C}$ into two sectors $\Omega_1 = \Delta(0, \frac{\pi}{2})$ and $\Omega_2 = \Delta(\pi, \frac{\pi}{2})$, we obtain that

$$\limsup_{r \to \infty} \frac{\mathcal{T}(r, \Omega_j, f)}{U(r)} > 0 \quad \textit{for} \quad j = 1 \quad \textit{or} \quad j = 2.$$

When it holds for j (say $j = 1$), we divide Ω_1 into two sectors. Repeating this procedure, we get a direction $\arg z = \theta^*$ such that, for $\Delta_n^* = \Delta(\theta^*, \frac{2\pi}{2^n})$, and we have

$$\limsup_{r \to \infty} \frac{\mathcal{T}(r, \Delta_n^*, f)}{U(r)} > 0, \tag{4.1}$$

for any $n \in N$. Take any direction $\arg z = \theta_0$ and a sector $\Delta(\theta_0, \varepsilon)$. Choose a n_0 such that $\frac{2\pi}{2^{n_0}} < \frac{\varepsilon}{8}$. Thus there is a j_0 such that $j_0 > n_0$ and $|(\theta_0 + j_0 \arg s) - \theta^*| < \frac{\varepsilon}{8} (\mathrm{mod}\ 2\pi)$. By (1.1), we obtain that $f(z) = R^{j_0}(f(s^{-j_0} z))$, where $R^{j_0}(w)$ is the j_0-th iteration of $R(w)$. Thus by Lemma 2.3, with some constant $L(j_0)$,

$$\mathcal{T}(r, \Delta_{j_0}^*, f) \le L(j_0) \mathcal{T}\left(|s|^{-j_0} r, \Delta\left(\theta_0, \frac{\varepsilon}{4}\right), f\right).$$

Suppose that the direction $\arg z = \theta_0$ does not satisfy the property, then there exist three distinct small functions $a_i(z), i = 1, 2, 3$ such that

$$\overline{N}(r, \Delta(\theta_0, \varepsilon), f = a_i) = o(U(r)). \tag{4.2}$$

Set

$$g(z) = \frac{f(z) - a_1(z)}{f(z) - a_2(z)} \frac{a_3(z) - a_2(z)}{a_3(z) - a_1(z)},$$

then

$$f(z) = \frac{h_1(z)g(z) + h_2(z)}{h_3(z)g(z) + h_4(z)},$$

where

$$h_1 = a_3(a_2 - a_1), h_2 = a_1(a_3 - a_2),$$
$$h_3 = a_2 - a_1, h_4 = a_3 - a_2,$$

so that $T(r, h_i) = O(\sum_{j=1}^{3} T(r, a_j))$ $(i = 1, 2, 3, 4)$. Since $\sum_{j=1}^{3} T(r, a_j) = o(U(r))$, in view of Lemma 2.6 and Lemma 2.9, we have

$$\mathcal{T}\Big(r, \Delta\Big(\theta_0, \frac{\varepsilon}{2}\Big), f\Big) \le 27\mathcal{T}\Big(64r, \Delta\Big(\theta_0, \frac{2\varepsilon}{3}\Big), g\Big) + o(U(r)).$$

Then in view of Lemma 2.5, we have

$$\begin{aligned}
\mathcal{T}\Big(r, \Delta\Big(\theta_0, \frac{\varepsilon}{2}\Big), f\Big) &\le 27\mathcal{T}\Big(64r, \Delta\Big(\theta_0, \frac{2\varepsilon}{3}\Big), g\Big) + o(U(r)) \\
&\le 81\Big[\overline{N}\Big(128r, \Delta\Big(\theta_0, \frac{3\varepsilon}{4}\Big), g = 0\Big) \\
&\quad + \overline{N}\Big(128r, \Delta\Big(\theta_0, \frac{3\varepsilon}{4}\Big), g = 1\Big) \\
&\quad + \overline{N}\Big(128r, \Delta\Big(\theta_0, \frac{3\varepsilon}{4}\Big), g = \infty\Big)\Big] \\
&\quad + o(U(r)) \\
&= 81\sum_{j=1}^{3} \overline{N}\Big(128r, \Delta\Big(\theta_0, \frac{3\varepsilon}{4}\Big), f = a_j\Big) \\
&\quad + o(U(r)).
\end{aligned}$$

Notice (4.2) and (4) in Proposition 1.1, we have

$$\mathcal{T}\Big(r, \Delta\Big(\theta_0, \frac{\varepsilon}{2}\Big), f\Big) = o(U(r)) \quad as \quad r \to \infty.$$

Hence

$$\begin{aligned}
\mathcal{T}(r, \Delta^*_{j_0}, f) &\le L(j_0)\mathcal{T}\Big(|s|^{-j_0}r, \Delta\Big(\theta_0, \frac{\varepsilon}{4}\Big), f\Big) \\
&\le L(j_0)\mathcal{T}\Big(|s|^{-j_0}r, \Delta\Big(\theta_0, \frac{\varepsilon}{2}\Big), f\Big) \\
&= o(U(|s|^{-j_0}r)) = o(U(r)) \quad as \quad r \to \infty.
\end{aligned}$$

This contradicts (4.1). Thus the direction $\arg z = \theta_0$ is precise Borel direction of maximal kind dealing with small functions. □

5. Conclusion. We can see that the Schröder function is a good meromorphic function that if $\arg[s]/2\pi \notin Q$, then $f(z)$ has any direction as Borel direction, precise T direction, Nevanlinna direction, Hayman direction, Hayman T direction and precise Borel direction of maximal kind dealing with small functions. And we make a conclusion that any direction of $f(z)$ must be a Marty direction, here a half line $z = \theta$ is a Marty direction if and only if for any $\varepsilon > 0$ and any real number k, we have

$$\sup_{\Omega} \frac{|z|^{k+1}|f'(z)|}{1 + |z|^{2k}|f(z)|^2} = +\infty,$$

where $\Omega = \{z : 1 \le |z| < +\infty, \theta - \varepsilon < \arg z < \theta + \varepsilon\}$. The definition of Marty direction was posed in [1, 2, 6]. In [6], Jin and Song have a result that if $\arg z = \theta_0$ is not a Marty direction of $f(z)$, then there exists a positive number $\varepsilon > 0$ such that

$$\mathcal{T}(r, \Delta(\theta_0, \varepsilon), f) = O(\log^3 r).$$

However, this contradicts (3.1).

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